National Chung Cheng University Admission Exam for Advanced Calculus, 1998

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1. Let R be the collection of all real numbers, and $R^3 = R \times R \times R$. Suppose $f(x, y, z) = (x + y^2 + 100z, x + 3y - 100z, e^{-z+100y^2})$, for $(x, y, z) \in R^3$.

   (1a) Find the determinent of Jacobian matrix of f at the point $(0, 0, 0)$.

   (1b) Could you find an open neighborhood of $(0, 0, 0)$, so that f is one-to-one in this open set? If your answer is yes, find it. If your answer is no, give the reason.

   Solution of (1a):

$$
\begin{aligned}
J_f(\mathbf{x}) &= \begin{bmatrix} D_1f_1(\mathbf{x}) & D_2f_1(\mathbf{x}) & D_3f_1(\mathbf{x}) \\ D_1f_2(\mathbf{x}) & D_2f_2(\mathbf{x}) & D_3f_2(\mathbf{x}) \\ D_1f_3(\mathbf{x}) & D_2f_3(\mathbf{x}) & D_3f_3(\mathbf{x}) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 2y & 3 & 200ye^{-z+100y^2} \\ 100 & -100 & -e^{-z+100y^2} \end{bmatrix}
\end{aligned}
$$

   Thus

$$
J_f(0, 0, 0) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 100 & -100 & -1 \end{bmatrix}
$$

   Solution of (1b): The answer is yes. Let

$$
\begin{aligned}
x + y^2 + 100z &= u \\
x + 3y - 100z &= v \\
e^{-z+100y^2} &= e^w \Leftrightarrow -z + 100y^2 = w
\end{aligned}
$$

and $\mathbf{p} = (0,0,0)$. Also we consider $(x,y,z) \in B_\epsilon(\mathbf{p})$ for some $\epsilon > 0$. Hence

$$-202\epsilon < 20001y^2 - 3y < 202\epsilon.$$

Note that $g(y) = 20001y^2 - 3y$ is strictly decreasing if $y < 10^{-5}$. Hence we take

$$\epsilon < 10^{-8}.$$

Hence we can solve a unique y from $g(y)$ if $(x,y,z) \in B_r(\mathbf{p})$ where $r = 10^{-8}$. Thus x and z can also solve uniquely. Therefore,

$$B_{10^{-8}}(0,0,0)$$

is our desired open neighborhood of (0,0,0).

2. Let (a,b) be a nonempty open set contained in R, and f be a function from (a,b) to R. We have the following two definition:

   (A1) Let $x_0 \in (a,b)$. We say f is continuous at x_0 iff for any $\epsilon > 0$, there exists $\delta > 0$ so that $(x_0-\delta, x_0+\delta) \subset (a,b)$ and for any $y \in (x_0-\delta, x_0+\delta)$, we have

   $$|f(y) - f(x_0)| < \epsilon.$$

   (A2) Let $x_0 \in (a,b)$. We say f is continuous at x_0 iff for any sequence $\{y_n\}_{n=1}^\infty \subset (a,b)$ satisfying $\lim_{n\to\infty} y_n = x_0$, we have

   $$\lim_{n\to\infty} f(y_n) = f(x_0).$$

   Show that (A1) is equivalent to (A2).

   Proof: (A1) $\Rightarrow$ (A2): Suppose each $\{y_n\}_{n=1}^\infty \subset (a,b)$ satisfying

   $$\lim_{n\to\infty} y_n = x_0.$$

Then for any $\epsilon' > 0$ there exists N such that $|y_n - x_0| < \epsilon'$ whenever $n \geq N$. Also, for any ϵ there exists $\delta > 0$ so that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ and for any $y \in (x_0 - \delta, x_0 + \delta)$, we have

$$|f(y) - f(x_0)| < \epsilon.$$

We take $\epsilon' = \delta$, and then $y_n \in (x_0 - \delta, x_0 + \delta)$ for all $n \geq N$. Thus

$$|f(y_n) - f(x_0)| < \epsilon$$

for all $n \geq N$. Hence

$$\lim_{n \to \infty} f(y_n) = f(x_0).$$

(A2) $\Rightarrow$ (A1): Suppose (A1) does not hold. There exists ϵ_0 such that for every $\delta > 0$, $|f(x) - f(x_0)| \geq \epsilon_0$ holds if $|x - x_0| < \delta$ for some $x \in (a, b)$. Take $\delta = 1/n$ for all $n \in N$, then there exists $x_n \in (x_0 - 1/n, x_0 + 1/n)$ such that $|f(x_n) - f(x_0)| \geq 1/n$. Note that $\{x_n\} \to x_0$, but

$$\lim_{n \to \infty} |f(x_n) - f(x_0)| \geq \epsilon_0 > 0$$

if $\lim_{n \to \infty} |f(x_n) - f(x_0)|$ exists, a contradiction. Hence (A2) implies (A1).

3. We give the following axiom.

   (Axiom of completeness for real numbers): Let S be a nonempty subset contained in R. If S has an upper bound in R, then S has the least upper bound in R.

   By using this axiom, first show that

   (3a) If $T = \{a_n\}_{n=1}^{\infty}$ is an increasing sequence contained in R and T has an upper bound, then $\lim_{n \to \infty} a_n$ exists and is equal to the least upper bound of T.

   Then show that

(3b) Any Cauchy sequence contained in R is a convergent sequence.

Proof of (3a): By the axiom of completeness for real numbers, since T is nonempty and T has an upper bound, then T has the least upper bound in R, say L. Since L is the least upper bound of T, for any $\epsilon > 0$, there exists N such that $L - a_N > \epsilon$. Note that T is increasing, and thus

$$L - a_N \leq L - a_{N+1} \leq L - a_{N+2} \leq ...$$

Hence $L - a_n < \epsilon$ for all $n \geq N$. Since ϵ is arbitrary,

$$\lim_{n \to \infty} a_n = L.$$

Proof of (3b): Let $\{a_n\}$ be a Cauchy sequence contained in R. First we prove that $\{a_n\}$ is bounded. Take $\epsilon = 1$, there exists N such that

$$|a_n - a_m| < 1$$

whenever $n, m \geq N$. Put $m = N$, then $|a_n - a_N| < 1$ if $n \geq N$. Thus $a_N - 1 < a_n < a_N + 1$. Put

$$M = \max(a_1, a_2, ..., a_N, a_N + 1),$$

and thus $\{a_n\}$ is bounded by M. Let

$$S_n = \{a_k : k \in N, k \geq n\},$$

and $S_n \subset S_1$ is also bounded for all $n \in N$. Thus by the axiom of completeness for real numbers, S_n has a least upper bound b_n and a greatest lower bound c_n for all n.. Note that $c_n \leq a_n \leq b_n$ for all n. Also, since $S_{n+1} \subset S_n$ for all n, $\{b_n\}$ is decreasing and $\{c_n\}$ is increasing. Also, $b_n \geq b_{n+m} \geq c_{n+m} \geq c_m$ for all $m, n \in N$, that is, $\{b_n\}$ and $\{c_n\}$ are bounded. By (3a) we can let

$$b = \inf\{b_n : n \in N\}, c = \sup\{c_n : n \in N\}.$$

First we prove that $b = c$. For any $\epsilon > 0$ there is an integer N such that $a_n - \epsilon < a_m < a_n + \epsilon$ whenever $m, n \geq N$. Let

$$E_1 = \{a_m : m \geq N\}, E_2 = \{a_n + \epsilon : n \geq N\}.$$

Thus $\inf E_1 \leq \sup E_2$. Hence

$$b_N \leq c_N + \epsilon.$$

Since

$$0 \leq b - c \leq b_N - c_N \leq \epsilon$$

for any arbitrary $\epsilon > 0$, therefore $b = c$. Finally, we prove $\{a_n\}$ converges to b or c. For any $\epsilon > 0$, we can find b_{N_1} such that $b_{N_1} < b + \epsilon$ for some N_1. Also, we can find c_{N_2} such that $c_{N_2} > b - \epsilon$ for some N_2. Take $N = \max(N_1, N_2)$, then

$$\begin{aligned} b_n &\leq b_{N_1} < b + \epsilon \\ c_n &\geq c_{N_2} > b - \epsilon. \end{aligned}$$

Thus

$$b - \epsilon < c_n \leq a_n \leq b_n < b + \epsilon.$$

Since ϵ is arbitrary, $\{a_n\}$ is a convergent sequence.

Note: I did not use Bolzano-Weierstrass Theorem since I was very boring.

4. Let f be a continuous function from $[0, 1]$ to R, and

$$S_n = \frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n}),$$

$n = 1, 2, 3, \ldots.$ Show that

(4a) $\{S_n\}_{n=1}^{\infty}$ is a convergent sequence.

Then show that

(4b) Suppose $f(x) \geq 0$ for all $x \in [0,1]$, and $f(x_0) > 0$ for some $x_0 \in [0,1]$. Also, f is continuous on $[0,1]$. Then we have $\lim_{n\to\infty} S_n > 0$.

(To show (4a) and (4b), you may use the property that if f is continuous on $[0,1]$, then f is uniformly continuous on $[0,1]$.)

Proof of (4a): Let $\epsilon > 0$ be given. Choose $\eta > 0$ so that

$$(b-a)\eta < \epsilon.$$

Since f is continuous on the compact set $[0,1]$, then f is uniformly continuous on it. Hence there exists a $\delta > 0$ such that

$$|f(x) - f(t)| < \eta$$

if $x \in [0,1]$, and $|x-t| < \delta$. If P is any partition of $[0,1]$ such that $\Delta x_i \leq \delta$ for all i, then

$$M_i - m_i \leq \eta \ \ (i = 1, ..., n)$$

and therefore

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^{n}(M_i - m_i)\Delta x_i \\ &\leq \eta(b-a) \\ &< \epsilon. \end{aligned}$$

Hence f is Riemann-integrable. Note that

$$P = \{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$$

is a partition of $[0,1]$ such that $\Delta x_i = 1/n$ for all i. Thus $L(P,f) \leq S_n \leq U(P,f)$ for $n > 1/\delta$. Hence $L(P,f) \leq S_n \leq L(P,f) + \epsilon$. Hence

$$\sup L(P,f) \leq S_n \leq \sup L(P,f) + \epsilon$$

for large enough n. Hence $\{S_n\}_{n=1}^{\infty}$ is a convergent sequence.

Proof of (4b): We prove the following exercise equivalently.

Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with Exercise 1.)

Proof of the exercise: Suppose not, then there is $p \in [a, b]$ such that $f(p) > 0$. Since f is continuous at $x = p$, for $\epsilon = f(p)/2$, there exist $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ whenever $x \in (x - \delta, x + \delta) \bigcap [a, b]$, that is,

$$0 < \frac{1}{2}f(p) < f(x) < \frac{3}{2}f(p)$$

for $x \in B_r(p) \subset [a, b]$ where r is small enough. Next, consider a partition P of $[a, b]$ such that

$$P = \{a, p - \frac{r}{2}, p + \frac{r}{2}, b\}.$$

Thus

$$L(P, f) \geq r \cdot \frac{1}{2}f(p) = \frac{rf(p)}{2}.$$

Thus

$$\sup L(P, f) \geq L(P, f) \geq \frac{rf(p)}{2} > 0,$$

a contradition since $\int_a^b f(x)dx = \sup L(P, f) = 0$. Hence $f = 0$ for all $x \in [a, b]$.