線上演算法之分析與搏奕論(2/2)

計劃編號：NSC 89-2213-E-002-134

執行期限：89/08/01—90/07/31

主持人：呂育道　臺灣大學資訊工程系教授

一、中文摘要

線上計算研究在資訊不完全情況下之演算法設計。此類演算法必須對每一個輸入做出決定，而且不能事後加以更改。競爭分析(competitive analysis)是分析線上演算法所採用的標準方法，它比較線上演算法之效益與最佳非線上演算法之效益。在使用競爭分析時，最大困難在於如何確實地算出最佳線上演算法的效益。前一計劃建立博奕論與競爭分析之間的對應關係，本計劃則應用該理論於一個重要的財務問題: 最佳購入持有策略。我們的結果如下: (1) 我們解出此問題之公式解，(2) 此解優於平均投資策略，(3) 姚期智的下限結果不是最佳，(4) 最後，我們證明最佳策略之唯一性。

英文摘要

On-line computation studies algorithm design under incomplete information. Such algorithms must respond to each piece of information and are not allowed to modify past decisions. Competitive analysis is the standard analytic tool for on-line algorithms. It compares the performance of on-line algorithm with that of the optimal off-line algorithm. The major difficulty in using it lies in solving the exact performance of optimal on-line algorithms. In a previous report, we establish the correspondence between game value and competitive analysis. Here, we proceed to solve an important problem in finance concerning the optimal buy-and-hold strategy. We solve that problem with a closed-form formula. Our optimal strategy dominates the standard dollar-averaging strategy. Furthermore, Yao's lower bound is not tight for our problem. Finally, we show that the optimal strategy is unique.

二、計劃緣由與目的

On-line computation has been receiving intense investigation by mainstream computer scientists [BEY98]. One of the reasons is its break from traditional paradigm because we no longer assume the input is provided in its entirety. Instead, the input data are revealed one piece at a time. The algorithm is asked to make decisions based on each piece without the option to change the decisions later. Such a framework fits many practical problems better than the traditional off-line setup, particular for problems that are *real-time* in nature.

The *competitive ratio* is the minimum ratio of the optimal off-line player's payoff to the on-line algorithm's expected payoff over all possible on-line algorithms in the worst case. A randomized on-line algorithm is *optimal* if it achieves this ratio.

Competitive analysis is the standard analytic tool for on-line algorithms. Although much work has been done in this regard, most bounds have been asymptotic in nature. *Exact* performance bounds for optimal on-line algorithms have been extremely rare. This goes to show we still don't understand the nature of on-line algorithms.

In this proposal's part one, we addressed this fundamental theoretical issue by deriving general results for this problem. We utilized game theory [PZ96] to formulate and tackle this issue. We established this general result:

Theorem. The competitive ratio equals the inverse of the game value.

We therefore conclude that the optimal mixed strategy of the on-line player is the optimal optmized on-line algorithm against the adversary, *if the game is finite.* From this result and the classic equivalence of finite zero-sum two-person game to linear program, we can deduce the following. Define a primal-dual problem using the payoff matrix defined above. Then the optimal on-line algorithm (the optimal mixed strategy, i.e.) is the optimal feasible solution times the game value.

Now we apply the theoretical result to a practical problem in finance called **buy-and-hold strategy**. Assume the exchange rate moves within a geometric envelope. In other words, if today's exchange rate is e, then tomorrow's exchange rate must be between e/θ and $e\theta$. There are n trading days. A trader starts with $1 and wants to exchange it into as many yen as possible in n trades. He is not allowed to trade yen back into dollars in the process. This setup is very general. It applies to any two financial securities whose relative prices are available.

Without knowing future exchange rates, the trader is in a dilemma. If he trades many dollars into yen in a trade, the future exchange rates may be more favorable. But if he trades few dollars into yen in a trade, the future exchange rates may be less favorable. We ask what is the competitive ratio of the problem and whether it can be achieved. There are many lower-bound results in the literature, the most famous being **Yao's lower bound** [Yao77]. We also want to know how good is the lower bound compared to the exact bound, if found.

三、實驗方法

By restricting ourselves to static strategies, we can reduce the number of pure strategies of our trader to a finite n. The idea is to identify a strategy with a probability density function over the pure strategies. But to apply our general theorem, we need to limit the number of strategies available to the adversary from an infinite number to a finite number. This is accomplished by eliminating *dominated* strategies of the adversary's. It turns out that we only need to consider n strategies for the adversary. These strategies take *extreme* up moves and then *extreme* down moves.

We have reduced the problem into a finite game. The payoff between the trader's strategies and the adversary's strategies has a clean formula θ^k where k is a nonnegative integer. Once the payoffs are derived, the payoff matrix is available. We then plug into the matrix into our general theorem for the optimal competitive ratio.

四、結論與討論

Here are our results. The optimal competitive ratio equals

$$\frac{n\theta-(n-2)}{\theta+1}.$$

The optimal trading strategy invests

$$\begin{cases} \dfrac{\theta}{n\theta-(n-2)} & i=1,n \\ \dfrac{\theta-1}{n\theta-(n-2)} & i=2,3,\ldots,n-1 \end{cases}$$

dollars on day i. This may be the first time in the literature where a *closed-form* formula is obtained.

The **dollar-averaging strategy** is very popular among investors and mutual funds. The strategy trades $1/n$ dollar each day regardless of the exchange rate. We show that the competitive ratio of this strategy equals

$$\frac{n\left(1-\theta^{-1}\right)}{1-\theta^{-n}}.$$

The dollar-averaging strategy, having a larger competitive ratio, is thus inferior to our optimal strategy.

Yao's lower bound for our problem is

$$\frac{n\left(1-\theta^{-1}\right)}{\left(1-\theta^{-\lceil n/2\rceil}\right)+\left(\theta^{-1}-\theta^{\lfloor n/2\rfloor-1}\right)}.$$

This bound is lower than the optimal ratio, hence is not tight.

The above strategy enjoys two interesting properties. First, it is the only optimal strategy given the restrictions. Second, even if we generalize the available strategies by use of real-time information in that we solve a primal-dual problem at each day, our strategy remains optimal.

五、參考文獻

[BEY98] A. Borodin and R. El-Yaniv. *Online Computation and Competitive Analysis*. Cambridge University Press, 1998.

[PZ96] L.A. Petrosjan and N.A. Zenkevich. *Game Theory*. Singapore: World Scientific, 1996.

[Yao77] A.C. Yao. "Probabilistic computation: towards a unified measure of complexity." In *Proc. 18th Annual Symposium on Foundations of Computer Science*, pp. 222–227, 1977.