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兩類路徑相關型衍生性金融商品之積分計價公式與高速計價演算法(2/3)

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一、中文摘要

衍生性金融商品為一種價值決定於某基本資產(如股票)的金融商品，其定價問題等於計算隨機變數之期望值。路徑相關型衍生性金商品的價值取決於基本資產價格之全部或部分歷史，它們的定價通常是複雜的計算問題。兩種常見路徑相關型特色為重設與均價，此重設均價選擇權曾經在臺灣的資本市場受到歡迎，但定價十分困難，其報酬與履約價均取決於基本資產價格之算術平均。本計劃研究重設﹝幾何﹞均價選擇權之訂價問題。我們的結果如下: (1) 簡單多維積分計價公式，(2) 一個適用一類路徑相關型衍生性金商品之計價公式，(3)更正文獻錯誤。

英文摘要

Financial derivatives are financial instruments whose payoff depends on some underlying asset such as stock. Mathematically speaking, pricing such instruments amounts to calculating the expected value of a random variable. Path-dependent derivatives are derivatives whose payoff depends on the *history* of the underlying asset's price. Pricing is computationally non-trivial for such derivatives. Two popular path-dependent features are resets and averaging. Average trigger reset options were popular in Taiwan's market and are very difficult to price. The payoffs and strike prices of such options depend on the arithmetic average of the asset prices. This proposal investigates the pricing of this type of options with geometric average. Our findings are: (1) a simple multi-dimensional integral closed-form pricing formula, (2) a general formula for pricing a class of path-dependant derivatives, and (3) correcting of errors in the literature.

二、計劃緣由與目的

Derivatives are financial instruments whose payoff depends on some **underlying asset** such as stock. Mathematically speaking, pricing such products amounts to calculating the expected value of a random variable. Such financial products represent a quantum leap for the global economy as they provide tools for manage financial risks. The notional market size of derivatives is in the trillions US dollars range. Research in this area therefore has important practical implications and has been extremely active. See Lyuu (2002) for a survey.

Path-dependent derivatives are derivatives whose payoff depends nontrivially on the *history* of the underlying asset's price. The price is usually described as a **stochastic process**. For our purpose, we will take the asset prices to be a time series $S_0, S_1, \ldots, S_n$, where the price S_i is a lognormal random variable with a known mean and variance. Pricing is usually computationally non-trivial for path-dependent derivatives. Research in this area represents the frontier in **computational finance**, **asset pricing**, and **financial economics**.

In the **analytical approach**, the goal is to come up with a closed-form or semi-closed-form formula. The mathematics involved is usually quite complex. In the **Monte Carlo simulation** approach, simulation of the asset price is conducted with the resulting payoffs then averaged. This method is quite general and when coupled with **variance-reduction techniques** and **quasi-Monte Carlo sequences** can be quite effective.

The purpose of our proposal is the pricing of the **average trigger reset option**, a variant of which was popular in Taiwan. The market derivatives used arithmetic average. But as arithmetic average is extremely hard to work with (see Kao and Lyuu (2003) for a relatively efficient algorithm), we shall work with geometric average.

The geometric average trigger reset option resets the strike price based on the geometric average of the underlying asset's prices over a monitoring window. There can also be *multiple* monitoring windows, thus multiple resets. This report derives an analytic formula and numerical methods for pricing this option with multiple resets. The analytic formula in fact is a corollary of a general formula that holds for a large class of path-dependent options: It prices any option whose payoff function can be written as $e^{b \cdot X^*} 1_{\{X \in A\}}$, where X is a multinormal random vector, b is some constant vector, and * denotes the transpose of a matrix. The reset call will not be exercised early if the underlying asset does not pay dividends; hence the same formula applies to American-style reset calls in this case.

We now state the general formula. Let μ be the mean vector of X and Σ be the covariance matrix of X. X is m-dimensional. Let $\Sigma' = C\Sigma C^*$. Then the price equals the discounted value of the following:

$$\frac{e^{\mu b^* + b\Sigma b^*/2}}{\left(\sqrt{2\pi}\right)^m (\det \Sigma')^{1/2}} \times \int e^{-\left[y-(\mu+b\Sigma)C^*\right]\Sigma'^{-1}\left[y-(\mu+b\Sigma)C^*\right]^*/2} dy \, .$$

The above is a complex formula, and its evaluation requires Monte Carlo simulation. When applied to our case with a single reset, it results in a formula that differs from that of Cheng and Zhang (2000). Numerical results will show that their formula does not give correct values.

Our general formula prices options such as vanilla options, geometric Asian options, rainbow options, quanto options, and many others. Discretely monitored barrier options also fall into this category, with the closed-form formula as a multiple integration. Cheng and Zhang (2000) derive an analytic formula for the option with a single monitoring window. However, we show that their formula is erroneous.

It can be proved that for average trigger reset calls, early exercise will not occur if the underlying stock does not pay dividends or if the option is dividend-protected (which is the case in Taiwan). Here is the proof. It is well-known that an American-style vanilla call will not be exercised early if the underlying asset does not pay dividends. In fact, at any time t before maturity, the continuation value exceeds the exercise payoff $S(t)$-$K(t)$, where $K(t)$ denotes the prevailing strike price at time t. Because the strike price of an otherwise identical reset call could only be reset to a level lower than $K(t)$, its continuation value must be at least as high as that of the vanilla call, thus higher than the exercise payoff too. As a result, a reset call will not be exercised early either.

For general American-style reset options, an $O\left(n^4 h^2\right)$-time algorithm on n-period binomial lattice is presented, where h is the length (in number of periods) of each monitoring window. A much more efficient $O\left(n^3 hm\right)$-time algorithm with a novel application of generating functions prices European-style reset options. Here m denotes the number of monitoring windows. Monte Carlo simulation suggests that the European-style geometric average trigger reset option and the arithmetic version have similar option values. This implies that results in this paper give tight prices for the difficult arithmetic version. Unlike the standard reset option, the geometric average trigger reset option does not have significant discontinuous deltas.

When the option is American-style and when early exercise is possible, we must resort to lattice methods. See Lyuu (2002). For this kind of method, we discretize the lognormal model and come up with a binomial lattice. Then use backward induction to price our options. When the option is European-style, the combinatorial method can lower the running time by at least an order of magnitude.

三、實驗方法

The price of a geometric average trigger reset call is higher than that of an arithmetic one, and vice versa for a put. Numerical results in the table below suggest that geometric and arithmetic average trigger reset options have close values even when the volatility is as high as 150%. This implies that our results also give tight prices for arithmetic average trigger reset options.

Volatility	50%	80%	100%	120%	150%
Geometric	26.165	37.781	44.971	51.562	60.460
Arithmetic	26.105	37.653	44.784	51.424	60.162
Difference	0.23%	0.34%	0.42%	0.27%	0.49%

Compared to Monte Carlo, the lattice method is extremely accurate. See the table below.

Reset Dates (Year)	Lattice (American)	Lattice (European)	MC	S.E.
1	8.3217	8.3018	8.3020	0.060755
1, 0.8	10.8541	10.4507	10.3667	0.058637
1, 0.8, 0.6	12.4521	11.9824	11.9054	0.057088
1, 0.8, 0.6, 0.4	13.7323	13.1883	13.1036	0.056627
1, 0.8, 0.6, 0.4,0.2	14.7350	14.1174	14.0317	0.057096

The combinatorial method is extremely fast. The table below documents this observation empirically.

Timing in seconds		
Lattice	1.24 s	29.12 s
Combinatorial	0.21 s	0.58 s

四、結論與討論

The geometric average trigger reset option resets the strike price based on the geometric average of the underlying asset's prices over monitoring windows. Similar contracts have been traded on exchanges in Asia. This paper derives an analytic formula for such options. It is proved that the reset call will not be exercised early if the underlying asset does not pay dividends; hence the same formula applies to American-style reset calls in this case. The formula is in fact a corollary of a much more general formula that is of independent interest as it is applicable to a large class of path-dependent options. An $O\left(n^4h^2\right)$-time algorithm for general American-style reset options is presented. A much more efficient $O\left(n^3hm\right)$-time algorithm exists for pricing European-style options. The correctness of these three approaches are verified by numerical experiments. Numerical evidence suggests that our pricing formula and algorithms give very tight upper (lower) bounds on arithmetic average trigger reset calls (puts, respectively). Finally, unlike the standard reset option, the geometric average trigger reset option does not have significant discontinuous deltas.

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