

Probability density of nonlinear phase noise

Keang-Po Ho

StrataLight Communications, Campbell, California 95008, and Graduate Institute of Communications Engineering,
College of Electrical Engineering and Computer Science, National Taiwan University, Taipei, Taiwan

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The probability density of nonlinear phase noise, often called the Gordon–Mollenauer effect, is derived analytically. The nonlinear phase noise can be accurately modeled as the summation of a Gaussian random variable and a noncentral chi-square random variable with two degrees of freedom. Using the received intensity to correct for the phase noise, the residual nonlinear phase noise can be modeled as the summation of a Gaussian random variable and the difference of two noncentral chi-square random variables with two degrees of freedom. The residual nonlinear phase noise can be approximated by Gaussian distribution better than the nonlinear phase noise without correction. © 2003 Optical Society of America

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1. INTRODUCTION

Gordon and Mollenauer¹ showed that, when optical amplifiers are used to compensate for fiber loss, the interaction of amplifier noise and the fiber Kerr effect causes phase noise, often called the Gordon–Mollenauer effect or nonlinear phase noise. By broadening the signal linewidth,² nonlinear phase noise degrades both phase-shifted keying and differential phase-shift keying systems, which have renewed attention recently.^{3–5} Because the nonlinear phase noise is correlated with the received intensity, the received intensity can be used to correct the nonlinear phase noise.^{6–8} The transmission distance can be doubled if the nonlinear phase noise is the dominant impairment.^{6,8}

Usually, the performance of the system is estimated based on the variance of the nonlinear phase noise.^{1,6–8} The probability-density function (pdf) is necessary to better understand the system and evaluates the system performance. This paper provides an analytical expression of the pdf for the nonlinear phase noise with^{6–8} and without¹ the correction by the received intensity. The characteristic functions are first derived analytically, and the pdf's are the inverse Fourier transform of the characteristic functions.

2. PROBABILITY-DENSITY FUNCTION

For simplicity and without loss of generality, assume that the total nonlinear phase noise is^{1,6,8}

$$\begin{aligned} \phi_{\text{NL}} = & |A + n_1|^2 + |A + n_1 + n_2|^2 \\ & + \cdots + |A + n_1 + \cdots + n_N|^2, \end{aligned} \quad (1)$$

where A is a real number representing the amplitude of the transmitted signal, $n_k = x_k + iy_k$, $k = 1, \dots, N$, are the optical amplifier noise introduced into the system at the k th fiber span, and n_k are independent identically distributed complex zero-mean circular Gaussian random variables with $E\{x_k^2\} = E\{y_k^2\} = E\{|n_k|^2\}/2 = \sigma^2$, where σ^2 is the noise variance per dimension per span and $E\{\cdot\}$ denotes expectation. The product of fiber nonlin-

ear coefficient and the effective length per span γL_{eff} is ignored in Eq. (1) for simplicity.^{1,6,8} The random variable of Eq. (1) is a quadratic form of complex random variables⁹; in order to find a simplified representation, we derive its characteristic function here.

First, we consider the random variable of

$$\begin{aligned} \varphi_1 = & |A + x_1|^2 + |A + x_1 + x_2|^2 \\ & + \cdots + |A + x_1 + \cdots + x_N|^2. \end{aligned} \quad (2)$$

The overall nonlinear phase noise of Eq. (1) is $\phi_{\text{NL}} = \varphi_1 + \varphi_2$, where

$$\varphi_2 = y_1^2 + |y_1 + y_2|^2 + \cdots + |y_1 + \cdots + y_N|^2 \quad (3)$$

is independent of φ_1 and has a pdf equal to that of φ_1 when $A = 0$. The random variable of Eq. (2) can be expressed as

$$\varphi_1 = NA^2 + 2A\mathbf{w}^T\mathbf{x} + \mathbf{x}^T\mathbf{C}\mathbf{x}, \quad (4)$$

where $\mathbf{w} = \{N, N-1, \dots, 2, 1\}^T$, $\mathbf{x} = \{x_1, x_2, \dots, x_N\}^T$, and the covariance matrix $\mathbf{C} = \mathcal{M}^T\mathcal{M}$ with

$$\mathcal{M} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}. \quad (5)$$

The pdf of \mathbf{x} is $(2\pi\sigma^2)^{-N/2} \exp(-\mathbf{x}^T\mathbf{x}/2\sigma^2)$. The characteristic function of φ_1 , $\Psi_{\varphi_1}(\nu) = E\{\exp(j\nu\varphi_1)\}$, is

$$\Psi_{\varphi_1}(\nu) = \frac{\exp(j\nu NA^2)}{(2\pi\sigma^2)^{N/2}} \int \exp(2j\nu A\mathbf{w}^T\mathbf{x} - \mathbf{x}^T\Gamma\mathbf{x}) d\mathbf{x}, \quad (6)$$

where $\Gamma = \mathcal{I}/(2\sigma^2) - j\nu\mathbf{C}$ and \mathcal{I} is an $N \times N$ identity matrix. Using the relationship of

$$\begin{aligned} \mathbf{x}^T\Gamma\mathbf{x} - 2j\nu A\mathbf{w}^T\mathbf{x} = & (\mathbf{x} - j\nu A\Gamma^{-1}\mathbf{w})^T\Gamma(\mathbf{x} - j\nu A\Gamma^{-1}\mathbf{w}) \\ & + \nu^2 A^2 \mathbf{w}^T\Gamma^{-1}\mathbf{w}, \end{aligned} \quad (7)$$

with some algebra, the characteristic function of Eq. (6) is

$$\Psi_{\varphi_1}(\nu) = \frac{\exp(j\nu NA^2 - \nu^2 A^2 \mathbf{w}^T \Gamma^{-1} \mathbf{w})}{(2\sigma^2)^{N/2} \det(\Gamma)^{1/2}}, \quad (8)$$

where $\det[\]$ is the determinant of a matrix. The characteristic function of Eq. (8) is

$$\Psi_{\varphi_1}(\nu) = \frac{\exp[j\nu NA^2 - 2\sigma^2 \nu^2 A^2 \mathbf{w}^T (\mathcal{I} - 2j\nu\sigma^2 \mathcal{C})^{-1} \mathbf{w}]}{\det(\mathcal{I} - 2j\nu\sigma^2 \mathcal{C})^{1/2}}. \quad (9)$$

Substituting $A = 0$ into Eq. (9), the characteristic function of φ_2 is $\Psi_{\varphi_2}(\nu) = \det[\mathcal{I} - 2j\nu\sigma^2 \mathcal{C}]^{-1/2}$. The characteristic function of ϕ_{NL} is $\Psi_{\phi_{\text{NL}}}(\nu) = \Psi_{\varphi_1}(\nu) \Psi_{\varphi_2}(\nu)$, or

$$\Psi_{\phi_{\text{NL}}}(\nu) = \frac{\exp[j\nu NA^2 - 2\sigma^2 \nu^2 A^2 \mathbf{w}^T (\mathcal{I} - 2j\nu\sigma^2 \mathcal{C})^{-1} \mathbf{w}]}{\det(\mathcal{I} - 2j\nu\sigma^2 \mathcal{C})}. \quad (10)$$

If the covariance matrix \mathcal{C} has eigenvalues and eigenvectors of λ_k , \mathbf{v}_k , $k = 1, 2, \dots, N$, respectively, the characteristic function of Eq. (10) becomes

$$\Psi_{\phi_{\text{NL}}}(\nu) = \frac{\exp\left[j\nu NA^2 - 2\sigma^2 \nu^2 A^2 \sum_{k=1}^N \frac{(\mathbf{v}_k^T \mathbf{w})^2}{1 - 2j\nu\sigma^2 \lambda_k}\right]}{\prod_{k=1}^N (1 - 2j\nu\sigma^2 \lambda_k)} \quad (11)$$

and can be rewritten to

$$\Psi_{\phi_{\text{NL}}}(\nu) = \prod_{k=1}^N \frac{1}{1 - 2j\nu\sigma^2 \lambda_k} \exp\left[\frac{j\nu A^2 (\mathbf{v}_k^T \mathbf{w})^2 / \lambda_k}{1 - 2j\nu\sigma^2 \lambda_k}\right]. \quad (12)$$

From the characteristic function of Eq. (12), the random variable of ϕ_{NL} [Eq. (1)] is the summation of N independently distributed noncentral chi-square (χ^2) random variables with two degrees of freedom.¹⁰ The eigenvalues of the covariance matrix of \mathcal{C} are all positive and multiply to unity.

Without going into detail, the matrix

$$\mathcal{C}^{-1} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \quad (13)$$

is approximately a Toeplitz matrix for the series of 2, -1, 0, ... For large N , the eigenvalues of the covariance matrix of \mathcal{C} are asymptotically equal to¹¹

$$\begin{aligned} \frac{1}{\lambda_k} &\approx 2 \left\{ 1 - \cos \left[\frac{(2k+1)\pi}{2N} \right] \right\} \\ &= 4 \sin^2 \left[\frac{(2k-1)\pi}{4N} \right], \quad k = 1, \dots, N. \end{aligned} \quad (14)$$

The values of Eq. (14) are the discrete Fourier transform of each row of the matrix \mathcal{C}^{-1} .

With the correction of phase noise using received intensity,⁶⁻⁸ the residual nonlinear phase noise is

$$\begin{aligned} \phi_{\text{RES}} &= |A + n_1|^2 + |A + n_1 + n_2|^2 \\ &\quad + \cdots + |A + n_1 + \cdots + n_{N-1}|^2 \\ &\quad - (\alpha_{\text{opt}} - 1) |A + n_1 + \cdots + n_N|^2. \end{aligned} \quad (15)$$

As from the appendix, $\alpha_{\text{opt}} \approx (N+1)/2$ is the optimal scale factor to correct the nonlinear phase noise of Eq. (1) using the received intensity of $|A + n_1 + \cdots + n_N|^2$. The random variable corresponding to φ_1 [Eq. (4)] becomes

$$(N - \alpha_{\text{opt}})A^2 + 2A \mathbf{w}_r^T \mathbf{x} + \mathbf{x}^T \mathcal{C}_r \mathbf{x}, \quad (16)$$

where $\mathbf{w}_r = \mathbf{w} - \alpha_{\text{opt}} \times \{1, 1, \dots, 1\}^T$ and

$$\mathcal{C}_r = (\mathcal{M} - \mathcal{L})^T (\mathcal{M} - \mathcal{L}) - (\alpha_{\text{opt}} - 1) \mathcal{L}^T \mathcal{L}, \quad (17)$$

where

$$\mathcal{L} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}. \quad (18)$$

Following the procedure from Eqs. (4) to (10), the characteristic function of ϕ_{RES} is

$$\Psi_{\phi_{\text{RES}}}(\nu) = \frac{\exp[-j\nu(N - \alpha_{\text{opt}})A^2 - 2\sigma^2 \nu^2 A^2 \mathbf{w}_r^T (\mathcal{I} - 2j\nu\sigma^2 \mathcal{C}_r)^{-1} \mathbf{w}_r]}{\det(\mathcal{I} - 2j\nu\sigma^2 \mathcal{C}_r)}. \quad (19)$$

The characteristic functions of ϕ_{RES} in the form of eigenvalues and eigenvectors are similar to those of Eqs. (11) and (12). The characteristic functions of ϕ_{RES} have the same expression as Eq. (12) using a new set of eigenvalues and eigenvectors of the covariance matrix \mathcal{C}_r and the vector of \mathbf{w}_r .

Except for the first and last rows, the matrix \mathcal{C}_r^{-1} is also approximately a Toeplitz matrix for the series of 2, -1, 0, ... For large N , the eigenvalues of \mathcal{C}_r are asymptotically equal to

$$\frac{1}{\lambda_k} \approx 4 \sin^2 \left[\frac{(k-1.25)\pi}{2(N-1)} \right], \quad k = 2, \dots, N,$$

$$\lambda_1 \approx -\sum_{k=2}^N \lambda_k. \quad (20)$$

Other than the largest one in absolute value, the eigenvalues of C_r are all positive. All eigenvalues of the covariance C_r sum to approximately zero and multiple to $\alpha_{\text{opt}} - 1 \approx (N - 1)/2$.

3. NUMERICAL RESULTS AND RANDOM VARIABLE MODELS

The pdf's of both ϕ_{NL} [Eq. (1)] and ϕ_{RES} [Eq. (15)] can be calculated by taking the inverse Fourier transforms of the corresponding characteristic functions of $\Psi_{\phi_{\text{NL}}}(\nu)$ [Eq. (10)] and $\Psi_{\phi_{\text{RES}}}(\nu)$ [Eq. (19)], respectively. Figure 1 shows the pdf of ϕ_{NL} [Eq. (1)] and ϕ_{RES} [Eq. (15)]. Figure 1 is plotted for the case that the optical signal-to-noise ratio $\rho_O = A^2/(2N\sigma^2) = 18$, corresponding to an error probability of 10^{-9} if the amplifier noise is the only impairment. The number of spans is $N = 32$. The x axis is normalized with respect to NA^2 , approximately equal to the mean nonlinear phase noise from the appendix.

Figure 1 can confirm that, using the received intensity to correct for nonlinear phase noise, the standard deviation of nonlinear phase noise can be reduced by a factor of 2.⁶⁻⁸ The appendix shows that the variance of nonlinear phase noise can be reduced by a factor of ~ 4 .

From the characteristic function of Eq. (12), the random variables of both ϕ_{NL} and ϕ_{RES} can be modeled as the combination of $N = 32$ independently distributed noncentral χ^2 random variables with two degrees of freedom. Some studies^{1,6,7} implicitly assume a Gaussian distribution by using the Q factor to characterize the random variables. When many independently distributed random variables with more or less the same variance are summed (or subtracted) together, the summed random variable approaches the Gaussian distribution. For the characteristic function of Eq. (12), the Gaussian assumption is valid only if the eigenvalues λ_k are more or less the same. From Eq. (14), the largest eigenvalue λ_1 of the covariance matrix C is ~ 9 times larger than the second largest eigenvalue λ_2 . From Eq. (20), the two largest eigenvalues λ_1 and λ_2 of the covariance matrix C_r are ~ 5.5 times larger than the third largest eigenvalue λ_3 . The approximation of Eq. (14) is accurate within 3.2% for $N = 32$. The approximation of Eq. (20) is not as good as that for Eq. (14) and accurate within 10% for $N = 32$.

While the Gaussian assumption for both ϕ_{NL} and ϕ_{RES} may not be valid, other than the noncentral χ^2 random variables with two degrees of freedom corresponding to some large eigenvalues, the other random variables should sum to a Gaussian distribution. By modeling the summation of random variables with smaller eigenvalues as a Gaussian distribution, the nonlinear phase noise of Eq. (12) can be modeled as a summation of two or three instead of $N = 32$ independently distributed random variables.

Note that the variance of the noncentral χ^2 random variables with two degrees of freedom in Eq. (12) is $4\sigma^4\lambda_k^2 + 4A^2(\mathbf{v}_k^T \mathbf{w})^2$.¹⁰ While the above reasoning just takes into account the contribution from the eigenvalue of λ_k but ignores the contribution from the eigenvector \mathbf{v}_k , numerical results show that the variance of each individual noncentral χ^2 random variable increases with the

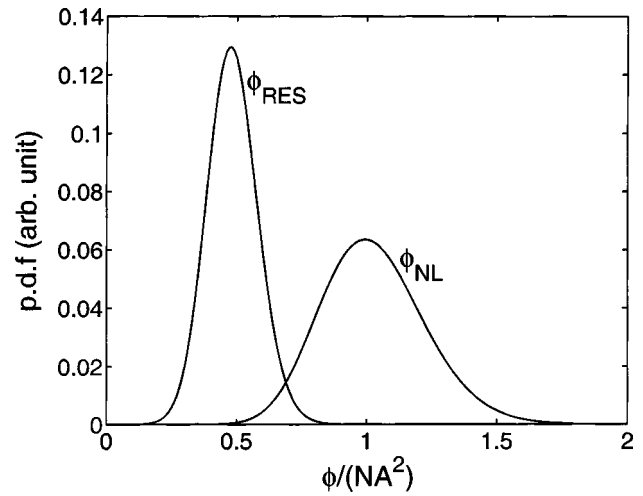


Fig. 1. pdf of both ϕ_{NL} and ϕ_{RES} .

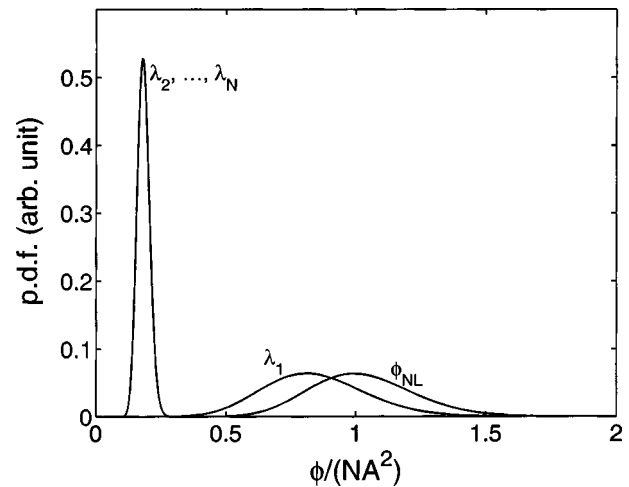


Fig. 2. pdf of ϕ_{NL} is the convolution of a Gaussian pdf and a noncentral χ^2 pdf with two degrees of freedom.

corresponding eigenvalue of λ_k . A later part of this paper also validates the argument.

From Fig. 1, the pdf of ϕ_{NL} has significant difference from that of a Gaussian distribution. Figure 2 divides the pdf of ϕ_{NL} into the convolution of two parts. The first part has no observable difference with a Gaussian pdf and corresponds to the second largest to the smallest eigenvalues, λ_k , $k = 2, \dots, N$, of the characteristic function of Eq. (12). The second part is a noncentral χ^2 pdf with two degrees of freedom and corresponds to the largest eigenvalue λ_1 , where $\sigma^2\lambda_1 \approx 2/(\pi^2\rho_O) \times NA^2$. The pdf of ϕ_{NL} in Fig. 1 is also plotted in Fig. 2 for comparison. The mean and variance of the first part of the Gaussian random variable are $\sum_{k=2}^N A^2(\mathbf{v}_k^T \mathbf{w})^2/\lambda_k + 2\sigma^2\lambda_k$ and $4\sum_{k=2}^N \sigma^4\lambda_k^2 + A^2(\mathbf{v}_k^T \mathbf{w})^2$, respectively. The second part of the noncentral χ^2 pdf with two degrees of freedom has a variance parameter of $\sigma^2\lambda_1$ and noncentrality parameter of $A^2(\mathbf{v}_1^T \mathbf{w})^2/\lambda_1$.¹⁰

To verify that the modeling in Fig. 2 is accurate, the cumulative tail probabilities are calculated by $\int_{-\infty}^x p(\xi)d\xi$ and $\int_x^{+\infty} p(\xi)d\xi$, where $p(\xi)$ is the pdf. Figure 3 shows the cumulative tail probabilities as a function of the Q factor for ϕ_{NL} , defined as $Q = (\phi_{\text{NL}} - \overline{\phi_{\text{NL}}})/\sigma_{\phi_{\text{NL}}}$, where $\overline{\phi_{\text{NL}}}$ and $\sigma_{\phi_{\text{NL}}}^2$ are the mean and variance of the nonlinear

phase noise given in the appendix. Using Gaussian approximation,^{1,6,7} this definition of the Q factor gives the tail probability or error probability of $(1/2)\text{erfc}(|Q|/\sqrt{2})$, where $\text{erfc}(\)$ is the complementary error function. Figure 3 shows the cumulative tail probabilities calculated by numerical integration according to Eq. (10) as circles, the model as the summation of a Gaussian and a noncentral χ^2 random variable with two degrees of freedom of Fig. 2 as solid lines, and the Gaussian assumption as dotted lines. From Fig. 3, the Gaussian approximation by the Q factor is not accurate, especially for the tail probability for less than the mean. From Fig. 3, the nonlinear phase noise can be modeled very accurately as the summation of a Gaussian random variable and a noncentral χ^2 random variable with two degrees of freedom. From Fig. 2, the noncentral χ^2 random variable with two degrees of freedom corresponding to λ_1 has a very large variance, such that the pdf of ϕ_{NL} in Fig. 1 has a significant difference with a Gaussian pdf.

Instead of the combination of $N = 32$ noncentral χ^2 random variables with two degrees of freedom, similar to the decomposition of Fig. 2, the random variable of ϕ_{RES}

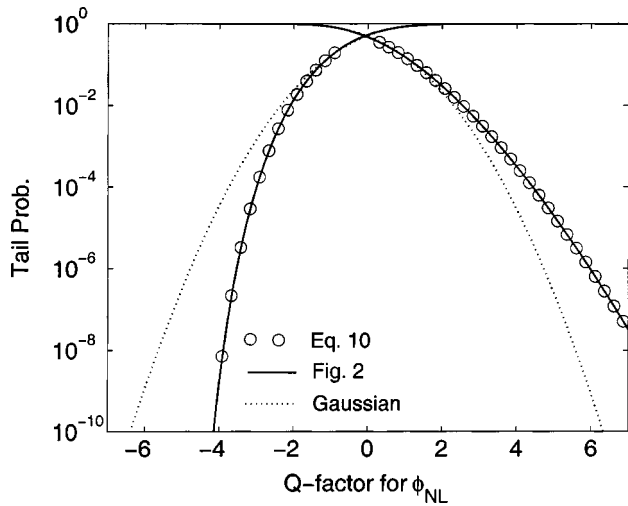


Fig. 3. Cumulative tail probability of ϕ_{NL} as compared with the model of Fig. 2 and a Gaussian approximation.

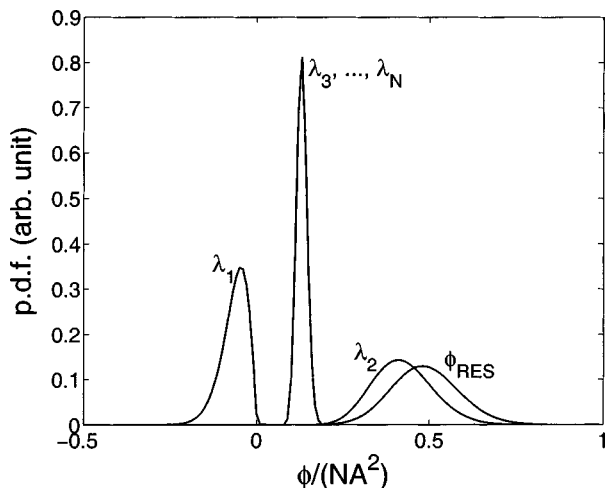


Fig. 4. pdf of ϕ_{RES} is the convolution of a Gaussian pdf and two noncentral χ^2 pdf's with two degrees of freedom.

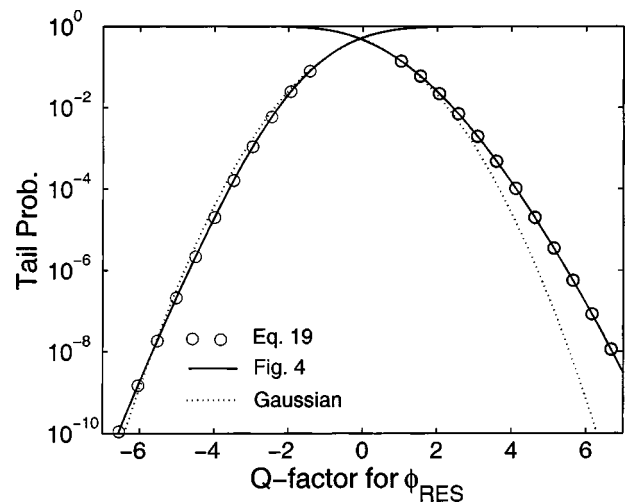


Fig. 5. Cumulative tail probability of ϕ_{RES} as compared with the model of Fig. 4 and a Gaussian approximation.

can be modeled as the summation of a Gaussian random variable and the difference of two noncentral χ^2 random variables with two degrees of freedom. Figure 4 shows the pdf of ϕ_{RES} as the convolution of a Gaussian pdf and two noncentral χ^2 pdf's with two degrees of freedom. The two noncentral χ^2 random variables correspond to the two largest eigenvalues of the covariance matrix C_r with more or less the same magnitude but different signs. The Gaussian random variable corresponds to the summation of $N - 2$ noncentral χ^2 random variables with two degrees of freedom for the eigenvalues of $\lambda_3, \dots, \lambda_N$. Because the variance parameter of $\sigma^2\lambda_1$ is negative, the corresponding random variable in Eq. (12) is the negative of a noncentral χ^2 random variable with two degrees of freedom. The pdf corresponding to λ_1 in Fig. 4 is the mirror image of a noncentral χ^2 pdf with two degrees of freedom with respect to the y axis. The random variable corresponding to the combined term of both λ_1 and λ_2 in Eq. (12) is the difference of two noncentral χ^2 random variables with two degrees of freedom.

Figure 5 shows the cumulative tail probabilities as a function of the Q factor for ϕ_{RES} , defined as $Q = (\phi_{RES} - \overline{\phi_{RES}})/\sigma_{\phi_{RES}}$, where $\overline{\phi_{RES}}$ and $\sigma_{\phi_{RES}}^2$ are the mean and variance of the residual nonlinear phase noise shown in the appendix. The cumulative tail probabilities calculated by numerical integration according to Eq. (19) is shown as circles, the model as the summation of a Gaussian random variable and the difference of two noncentral χ^2 random variables with two degrees of freedom of Fig. 4 is shown as solid lines, and the Gaussian assumption^{1,6,7} is shown as dotted lines. From Figs. 1 and 4, the pdf of ϕ_{RES} resembles a Gaussian pdf with mean and variance from Ref. 8 and the appendix. The residual nonlinear phase noise of ϕ_{RES} can be modeled accurately as a Gaussian random variable, especially for the tail probabilities less than the mean. Even for the tail probabilities larger than the mean, the Gaussian model for ϕ_{RES} is better than that for ϕ_{NL} . If the tail probabilities for $>10^{-5}$ is of interest, Gaussian approximation for ϕ_{RES} can be used.

4. CONCLUSION

The characteristic functions of nonlinear phase noise, with and without the correction using the received intensity, are derived analytically as a product of N noncentral χ^2 characteristic functions with two degrees of freedom. The pdf's are calculated exactly as the inverse Fourier transform of the characteristic functions. The pdf of the nonlinear phase noise can be modeled as the convolution of a Gaussian pdf and a noncentral χ^2 pdf with two degrees of freedom. Using the received intensity to correct for the phase noise, the pdf of the residual nonlinear phase noise can be modeled accurately as the convolution of a Gaussian pdf and two noncentral χ^2 pdf's with two degrees of freedom. The Gaussian approximation of the residual nonlinear phase noise is much better than that for nonlinear phase noise.

APPENDIX A: OPTIMAL LINEAR COMPENSATOR

This appendix shows important results from Ref. 8. The optimal scale factor to minimize the variance of ϕ_{RES} is

$$\alpha_{\text{opt}} = \frac{N+1}{2} \frac{A^2 + (2N+1)\sigma^2/3}{A^2 + N\sigma^2} \approx \frac{N+1}{2}. \quad (21)$$

The variance of the residual nonlinear phase noise of Eq. (15) is reduced to

$$\sigma_{\phi_{\text{RES}}}^2 = (N-1)N(N+1)\sigma^2 \times \frac{A^4 + 2N\sigma^2 A^2 + (2N^2 + 1)\sigma^4/3}{3(A^2 + N\sigma^2)} \quad (22)$$

from that of the nonlinear phase noise of

$$\sigma_{\phi_{\text{NL}}}^2 = \frac{4}{3}N(N+1)\sigma^2 \left[\left(N + \frac{1}{2} \right) A^2 + (N^2 + N + 1)\sigma^2 \right]. \quad (23)$$

The mean of the nonlinear phase noise Eq. (1) is

$$\overline{\phi_{\text{NL}}} = N[A^2 + (N+1)\sigma^2]. \quad (24)$$

The mean of the residual nonlinear phase noise is

$$\overline{\phi_{\text{RES}}} = \overline{\phi_{\text{NL}}} - \alpha_{\text{opt}}(A^2 + 2N\sigma^2). \quad (25)$$

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