# Phase statistics of the soliton

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The characteristic function of soliton phase jitter is found analytically when the soliton is perturbed by amplifier noise. In addition to that from amplitude jitter, the nonlinear phase noise due to frequency and timing jitter is also analyzed. With nonlinear phase noise, the overall phase jitter is non-Gaussian distributed. For a fixed mean nonlinear phase shift, the contribution of nonlinear phase noise from frequency and timing jitter decreases with distance and signal-to-noise ratio. © 2004 Optical Society of America

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## 1. INTRODUCTION

The phase jitter of soliton due to amplifier noise, like Gordon–Haus timing jitter,[1] is usually assumed to be Gaussian distributed.[2–5] When the phase jitter of soliton was studied, the phase jitter variance was given or measured but the statistics of the soliton phase was not discussed.[2,3,6–8]

For nonsoliton systems, the statistics of nonlinear phase noise is found to be non-Gaussian distributed both experimentally[9] and theoretically.[10–13] However, those studies[9–13] just include the Gordon–Mollenauer effect[14] that is the nonlinear phase noise induced by the conversion of amplitude-to-phase jitter due to the fiber Kerr effect, mostly self-phase modulation. On the basis of the well-developed perturbation theory of the soliton,[5,15–17] phase jitter can also be induced by the interaction of frequency and timing jitter. In this paper I derive statistics of the soliton phase including the contribution of timing and frequency jitter-induced nonlinear phase noise. The characteristic function of soliton phase jitter is derived analytically, to my knowledge, for the first time. The probability density function (pdf) is simply the inverse Fourier transform of the corresponding characteristic function.

Most optical communication systems use the intensity of the optical signal to transmit information. Direct-detection differential phase-shift keying signaling has renewed attention recently,[18–27] mostly by use of return-to-zero (RZ) pulses for long-haul transmission to encode information in the phase difference between two consecutive pulses. To a certain extent, a soliton differential phase-shift keying system may be a good approximation to a phase-modulated dispersion-managed soliton[8] or RZ signal. With well-developed perturbation theory, the distribution of the soliton phase jitter can be derived analytically.

The error probability of a differential phase-shift keying soliton signal was calculated in Ref. 28 with the method of Refs. 29 and 30 without taking into account the effect of phase jitter. If the phase jitter is Gaussian distributed, the system can be analyzed by the formulas of Ref. 31. The phase jitter may be indeed Gaussian distributed in certain regimes around the center of the distribution,[3,32] especially if the pdf is plotted in linear scale. The tail probability less than, for example, $10^{-9}$ is certainly not Gaussian distributed. As optical communication systems aim for low error probability, a careful study of the statistics of the soliton phase is necessary to characterize the performance of the system.

The remaining sections of this paper are organized as follows: In Section 2, I give the stochastic equations of the phase jitter according to the first-order soliton perturbation theory; in Section 3, I derive the characteristic function of soliton phase jitter; in Section 4, I present the numerical results; and in Sections 5 and 6, I provide the discussion and conclusion of the paper, respectively.

## 2. STOCHASTIC EQUATIONS FROM SOLITON PERTURBATION

From the first-order perturbation theory, with amplifier noise, the soliton parameters evolve according to the following equations[5,15–17]:

$$\frac{\mathrm{d}A}{\mathrm{d}\zeta} = \Im\left\{ \int \mathrm{d}\tau f_A n(\zeta, \tau) \right\}, \tag{1}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\zeta} = \Re\left\{ \int \mathrm{d}\tau f_\Omega n(\zeta, \tau) \right\}, \tag{2}$$

$$\frac{\mathrm{d}T}{\mathrm{d}\zeta} = -\Omega + \Im\left\{ \int \mathrm{d}\tau f_T n(\zeta, \tau) \right\}, \tag{3}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\zeta} = \frac{1}{2}(A^2 - \Omega^2) + T\frac{\mathrm{d}\Omega}{\mathrm{d}\zeta} + \Re\left\{ \int \mathrm{d}\tau f_\phi n(\zeta, \tau) \right\}, \tag{4}$$

where $\Re\{\ \}$ and $\Im\{\ \}$ denote the real and imaginary parts of a complex number, respectively; $n(\zeta, \tau)$ is the amplifier noise with the autocorrelation of

$$E\{n(\zeta_1, \tau_1)n(\zeta_2, \tau_2)\} = \sigma_n{}^2 \delta(\zeta_1 - \zeta_2)\delta(\tau_1 - \tau_2);$$

$$(5)$$

$A(\zeta)$, $\Omega(\zeta)$, $T(\zeta)$, and $\phi(\zeta)$ are the amplitude, frequency, timing, and phase parameters of the perturbed soliton of

$$q_0(\tau, \zeta) = A(\zeta)\text{sech}\{A(\zeta)[\tau - T(\zeta)]\}$$

$$\times \exp[-i\Omega(\zeta)\tau + i\phi(\zeta)] \qquad (6)$$

with initial values of $A(0) = A$ and $\Omega(0) = \phi(0) = T(0) = 0$. Functions related to soliton parameters are

$$f_A = q_0{}^*, \qquad (7)$$

$$f_\Omega = \tanh[A(\tau - T)]q_0{}^*, \qquad (8)$$

$$f_T = \frac{\tau - T}{A}q_0{}^*, \qquad (9)$$

$$f_\phi = -\frac{1}{A}\{1 - A(\tau - T)\tanh[A(\tau - T)]\}q_0{}^*. \qquad (10)$$

The parameters of Eqs. (1)–(4) are normalized.[5]

From both Eqs. (1) and (2), we obtain

$$A(\zeta) = A + w_A(\zeta), \qquad (11)$$

$$\Omega(\zeta) = w_\Omega(\zeta), \qquad (12)$$

where $w_A$ and $w_\Omega$ are two independent zero-mean Wiener processes with autocorrelation functions of

$$E\{w_A(\zeta_1)w_A(\zeta_2)\} = \sigma_A{}^2 \min(\zeta_1, \zeta_2), \qquad (13)$$

$$E\{w_\Omega(\zeta_1)w_\Omega(\zeta_2)\} = \sigma_\Omega{}^2 \min(\zeta_1, \zeta_2), \qquad (14)$$

where $\sigma_A{}^2 = A\sigma_n{}^2$ and $\sigma_\Omega{}^2 = A\sigma_n{}^2/3$.[5,7,17] Defined for the amplitude, the signal-to-noise ratio (SNR) as a function of distance is

$$\frac{A^2}{\sigma_A{}^2\zeta} = \frac{A}{\sigma_n{}^2\zeta}. \qquad (15)$$

When we use Eqs. (3) and (12), the timing jitter is

$$T(\zeta) = -\int_0^\zeta w_\Omega(\zeta_1)d\zeta_1 + w_T(\zeta), \qquad (16)$$

where $w_T$ is a zero-mean Wiener process with an autocorrelation function of

$$E\{w_T(\zeta_1)w_T(\zeta_2)\} = \sigma_T{}^2 \min(\zeta_1, \zeta_2), \qquad (17)$$

with[5,7,17]

$$\sigma_T{}^2 = \frac{\pi^2}{12}\frac{\sigma_n{}^2}{A}. \qquad (18)$$

When we use Eqs. (3), (11), and (16), the phase jitter is

$$\phi(\zeta) = \frac{1}{2}\int_0^\zeta [A + w_A(\zeta_1)]^2 d\zeta_1 - \frac{1}{2}\int_0^\zeta w_\Omega{}^2(\zeta_1)d\zeta_1$$

$$+ \int_0^\zeta \left[-\int_0^{\zeta_1} w_\Omega(\zeta_2)d\zeta_2 + w_T(\zeta_1)\right]dw_\Omega(\zeta_1)$$

$$+ w_\phi(\zeta), \qquad (19)$$

where $w_\phi$ is a zero-mean Wiener process with an autocorrelation function of

$$E\{w_\phi(\zeta_1)w_\phi(\zeta_2)\} = \sigma_\phi{}^2 \min(\zeta_1, \zeta_2) \qquad (20)$$

with[5,7,17]

$$\sigma_\phi{}^2 = \frac{\sigma_n{}^2}{3A}\left(1 + \frac{\pi^2}{12}\right). \qquad (21)$$

The Wiener processes of $w_A$, $w_\Omega$, $w_T$, and $w_\phi$ are independent of each other. The amplitude [Eq. (11)], frequency [Eq. (12)], and timing [Eq. (16)] jitters are all Gaussian distributed. From Eq. (19), it is obvious that the phase jitter is not Gaussian distributed. If Eq. (4) is linearized or all higher-order terms of Eq. (19) are ignored, the phase jitter is Gaussian distributed and equal to $\phi(\zeta) \approx A\int_0^\zeta w_A(\zeta_1)d\zeta_1 + w_\phi(\zeta)$.[5] The characteristic function of the phase jitter of Eq. (19) is derived in Section 3 and compared with the Gaussian approximation.

## 3. CHARACTERISTIC FUNCTIONS OF PHASE JITTER

In the phase jitter of Eq. (19), there are three independent contributions from amplitude jitter (the first term), frequency and timing jitter (the second and third terms), and the projection of amplifier noise to phase jitter $w_\phi$. In this section, the characteristic functions of each individual component are derived, and the overall characteristic function of phase jitter is the product of the characteristic functions of each independent contribution.

### A. Gordon–Mollenauer Effect
The first term of Eq. (19) is the Gordon–Mollenauer effect[14] of

$$\phi_{\text{GM}}(\zeta) = \frac{1}{2}\int_0^\zeta [A + w_A(\zeta_1)]^2 d\zeta_1 \qquad (22)$$

induced by the interaction of the fiber Kerr effect and amplifier noise, affecting phase-modulated non-RZ and RZ signal.[10,11,13]

The characteristic function of Gordon–Mollenauer nonlinear phase noise is given by[11,13]

$$\Psi_{\phi_{\text{GM}}(\zeta)}(\nu) = \sec^{1/2}(\zeta\sigma_A\sqrt{j\nu})\exp\left[\frac{A^2}{2\sigma_A}\sqrt{j\nu}\tan(\zeta\sigma_A\sqrt{j\nu})\right].$$

$$(23)$$

The above characteristic function of Eq. (23) can also be derived from Eq. (A7) of Appendix A.

The mean and variance of the phase jitter of Eq. (22) are

$$\langle \phi_{\text{GM}}(\zeta)\rangle = -j\frac{d}{d\nu}\Psi_{\phi_{\text{GM}}(\zeta)}(\nu)|_{\nu=0} = \frac{1}{2}A^2\zeta + \frac{1}{4}\sigma_A{}^2\zeta^2,$$

$$(24)$$

$$\sigma_{\phi_{\mathrm{GM}}(\zeta)}{}^2 = -\frac{\mathrm{d}^2}{\mathrm{d}\nu^2} \Psi_{\phi_{\mathrm{GM}}(\zeta)}(\nu)\big|_{\nu=0} - \langle \phi_{\mathrm{GM}}(\zeta)\rangle^2$$

$$= \frac{1}{3} A^2 \sigma_A{}^2 \zeta^3 + \frac{1}{12} \sigma_A{}^4 \zeta^4, \qquad (25)$$

respectively. The first term of Eq. (25) increases with $\zeta^3$, conforming to that of Ref. 14. Given a large fixed SNR of $A^2/(\sigma_A{}^2\zeta)$ [Eq. (15)], the second term of Eq. (25) is much smaller than the first term and also increases with $\zeta^3$. Note that the first term of the mean of Eq. (24) is also larger than the second term for large SNR.

The characteristic function of Eq. (23) depends on two parameters: the mean nonlinear phase shift of $A^2\zeta/2$ and the SNR of Eq. (15). Given a fixed mean nonlinear phase shift of $A^2\zeta/2$, the shape of the distribution depends only on the SNR.[11]

On the basis of Eq. (19), comparing Eq. (23) with the nonsoliton case of Ref. 11, the mean and standard deviation of the Gordon–Mollenauer phase noise of soliton are approximately half of that of the nonsoliton case with the same amplitude $A$ as the non-RZ or RZ level.[11]

## B. Frequency and Timing Effect

The frequency and timing jitter contributes to phase jitter by

$$\phi_{\Omega,T}(\zeta) = -\frac{1}{2}\int_0^\zeta w_\Omega{}^2(\zeta_1)\mathrm{d}\zeta_1 - \int_0^\zeta \int_0^{\zeta_1} w_\Omega(\zeta_2)\mathrm{d}\zeta_2 \mathrm{d}w_\Omega(\zeta_1)$$

$$+ \int_0^\zeta w_T(\zeta_1)\mathrm{d}w_\Omega(\zeta_1) \qquad (26)$$

as the second and third terms of Eq. (19).

By changing the order of integration for the second term of Eq. (26), we obtain

$$\phi_{\Omega,T}(\zeta) = \frac{1}{2}\int_0^\zeta w_\Omega{}^2(\zeta_1)\mathrm{d}\zeta_1 + \int_0^\zeta w_T(\zeta_1)\mathrm{d}w_\Omega(\zeta_1)$$

$$- w_\Omega(\zeta)\int_0^\zeta w_\Omega(\zeta_1)\mathrm{d}\zeta_1. \qquad (27)$$

From Eq. (A12) of Appendix A, the characteristic function of $\phi_{\Omega,T}(\zeta)$ is

$$\Psi_{\phi_{\Omega,T}(\zeta)}(\nu) = \Psi_{\varphi_1,\varphi_2,\varphi_3}\left(\frac{\nu}{2}, \nu, -\nu\right). \qquad (28)$$

The mean and variance of the phase jitter of Eq. (26) are

$$\langle \phi_{\Omega,T}(\zeta)\rangle = -j\frac{\mathrm{d}}{\mathrm{d}\nu}\Psi_{\phi_{\Omega,T}(\zeta)}(\nu)\big|_{\nu=0} = -\frac{1}{4}\sigma_\Omega{}^2\zeta^2, \quad (29)$$

$$\sigma_{\phi_{\Omega,T}(\zeta)}{}^2 = -\frac{\mathrm{d}^2}{\mathrm{d}\nu^2}\Psi_{\phi_{\Omega,T}(\zeta)}(\nu)\big|_{\nu=0} - \langle \phi_{\Omega,T}(\zeta)\rangle^2$$

$$= \frac{1}{2}\sigma_\Omega{}^2\sigma_T{}^2\zeta^2 + \frac{1}{4}\sigma_\Omega{}^4\zeta^4, \qquad (30)$$

respectively.

Comparing the means of Eqs. (24) and (29), in terms of absolute value, the mean nonlinear phase shift due to the Gordon–Mollenauer effect is much larger than that due to the frequency and timing effect. Comparing the variances of Eqs. (25) and (30), the variance of nonlinear phase noise due to the Gordon–Mollenauer effect is also much larger than that due to the frequency and timing effect.

Unlike the Gordon–Mollenauer effect, the characteristic function of Eq. (28), from Appendix A, is not determined only on the SNR and the mean nonlinear phase shift [Eq. (29)].

## C. Linear Phase Noise

The last term of Eq. (19) gives the linear phase noise of

$$\phi_{\mathrm{LN}}(\zeta) = w_\phi(\zeta) \qquad (31)$$

with a characteristic function of

$$\Psi_{\phi_{\mathrm{LN}}(\zeta)}(\nu) = \exp\left(-\frac{1}{2}\sigma_\phi{}^2\zeta\nu^2\right). \qquad (32)$$

From the characteristic function of Eq. (32), the linear phase noise depends solely on the SNR [Eq. (15)].

The characteristic function of the overall phase jitter $\phi(\zeta)$ is the multiplication of the characteristic functions of Eqs. (23), (28), and (32).

Although the actual mean nonlinear phase shift is

$$\langle \phi(\zeta)\rangle = \langle \phi_{\Omega,T}(\zeta)\rangle + \langle \phi_{\mathrm{GM}}(\zeta)\rangle, \qquad (33)$$

we mostly call $A^2\zeta/2$ the mean nonlinear phase shift as a good approximation in high SNR.

## 4. NUMERICAL RESULTS

The pdf is the inverse Fourier transform of the corresponding characteristic function. Figures 1(a)–1(d) show the evolution of the distribution of the phase jitter [Eq. (19)] with distance. The system parameters are $A = 1$ and $\sigma_n{}^2 = 0.05$. Those parameters are chosen for typical distribution of the phase jitter.

Figures 1(a)–1(c) are the distribution of the Gordon–Mollenauer nonlinear phase noise [Eq. (23)], frequency and timing nonlinear phase noise [Eq. (28)], and the linear phase noise [Eq. (32)], respectively, as components of the overall phase jitter of Eq. (19). Figure 1(d) is the distribution of the overall phase jitter Eq. (19). The pdf's in Figs. 1(a)–1(d) are normalized to a unity peak value for illustration purpose. The $x$ axes do not have the same scale. From Fig. 1, the nonlinear phase noises from the Gordon–Mollenauer effect and frequency and timing effect are obviously not Gaussian distributed. With small mean and variance, the nonlinear phase noise from the frequency and timing effect has a long tail.

Figures 2(a) and 2(b) plot the pdf's of Fig. 1 in logarithmic scale for the cases of $\zeta = 1,2$. The Gaussian approximation is also plotted in Fig. 2 for the overall phase jitter $\phi(\zeta)$. In both cases of $\zeta = 1,2$, the Gaussian approximation is not close to the exact pdf's in the tails. However, if the pdf's are plotted in linear scale, the Gaussian approxi-
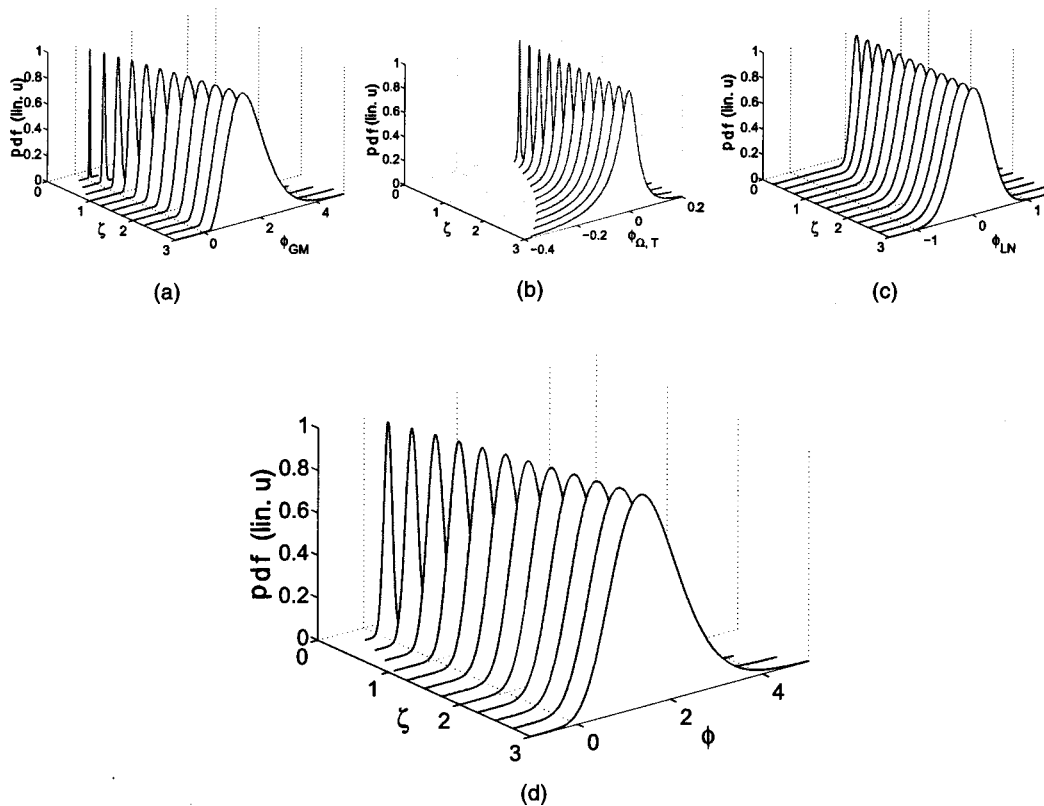
Fig. 1.  Distributions of soliton phase jitter for different distances for $A = 1$ and $\sigma_n^2 = 0.05$.  (a) $\phi_{\mathrm{GM}}(\zeta)$, (b) $\phi_{\Omega,T}(\zeta)$, (c) $\phi_{\mathrm{LN}}(\zeta)$, (d) $\phi(\zeta)$.  The distributions are normalized for a unity peak.  The $x$ axes are not in the same scale. The distribution is plotted in linear units.

mation may be close to the actual distribution, especially for large phase jitter.[32]  The pdf's in Fig. 2 are not normalized to a unity peak.

From both Figs. 1 and 2, the nonlinear phase noises of $\phi_{\mathrm{GM}}$ and $\phi_{\Omega,T}$ are not symmetrical with respect to their corresponding means.  Whereas $\phi_{\mathrm{GM}}$ spreads further to positive phase, $\phi_{\Omega,T}$ spreads further to negative phase.  Plotted in the same scale, the nonlinear phase noise of $\phi_{\mathrm{GM}}$ due to the Gordon–Mollenauer effect is much larger than the nonlinear phase noise of $\phi_{\Omega,T}$ due to the frequency and timing effect.

The pdf's in Fig. 1 cannot cover all possible cases.  Although both the Gordon–Mollenauer and linear phase noises depend on the mean nonlinear phase shift $A^2\zeta/2$ and SNR, the nonlinear phase noise induced by the frequency and timing effect does not have a simple scaled relationship.

For a mean nonlinear phase shift of $\frac{1}{2}A^2\zeta = 1$ rad,[14] Figs. 3(a) and 3(b) plot the distribution of the overall phase jitter [Eq. (19)] for a SNR of 10 and 20 for $\zeta = 1, 10$.  After a scale factor, the distributions of both the Gordon–Mollenauer and the linear phase noise are the same as that in Fig. 2.  In addition to the overall phase jitter, Fig. 3 also plots the distribution of the nonlinear phase noise from the frequency and timing effect of $\phi_{\Omega,T}$.

For a fixed mean nonlinear phase shift and SNR, from Fig. 3, the nonlinear phase noise from the frequency and timing effect of $\phi_{\Omega,T}(\zeta)$ has less of an effect on the overall phase jitter for long distance than for short distance.  Figure 1 is plotted for a short distance of $\zeta \leq 3$ to show

the contribution of frequency and timing jitter to nonlinear phase noise.  The effect of $\phi_{\Omega,T}(\zeta)$ is smaller for a large SNR of 20 than a small SNR of 10.  The main contribution to the overall phase jitter is always the Gordon–Mollenauer effect and the linear phase noise.

## 5.  DISCUSSION

The phase jitter of Eq. (19) is derived on the basis of the first-order perturbation theory[5,15–17] of Eqs. (1)–(4).  The non-Gaussian distribution is induced by the higher-order terms of Eq. (19) or the nonlinear terms of Eq. (4).  Second- and higher-order soliton perturbation[33,34] may give further non-Gaussian characteristics to the phase jitter.  Currently, there is no comparison between contributions of the higher-order terms of Eq. (4) and higher-order soliton perturbation.

In this paper, as in much of the literature,[1,2,5,7,8,14–17] the effect of amplitude jitter to the noise variances of $\sigma_A^2$, $\sigma_\Omega^2$, $\sigma_T^2$, and $\sigma_\phi^2$ is ignored.  The noise variances of $\sigma_A^2$, $\sigma_\Omega^2$, $\sigma_T^2$, and $\sigma_\phi^2$ are assumed independent of distance.  If the amplitude noise variance is $\sigma_A^2 = A(\zeta)\sigma_n^2$ with dependence on the instantaneous amplitude jitter, amplitude, frequency, and timing jitters are all non-Gaussian distributed.[35]  As an example, amplitude jitter is noncentral chi-squared distributed.[35,36]  However, the statistics of phase jitter [Eq. (19)] does not have a simple analytical solution when the noise variance depends on amplitude jitter.  With a high SNR, the amplitude jitter is always much smaller than the amplitude $A(0) = A$.  Even in high SNR, the phase jitter is non-Gaussian on the basis of Eq. (19).

## 6. CONCLUSION

On the basis of the first-order soliton perturbation theory, the distribution of soliton phase jitter due to amplifier noise is derived analytically for the first time to my knowledge. In addition to the main contribution of the Gordon–Mollenauer effect, the nonlinear phase noise due to frequency and timing jitter is also considered. Induced by the Gordon–Mollenauer effect or frequency and timing jitter, neither the nonlinear phase noises nor the overall phase jitter are Gaussian distributed. For a fixed mean nonlinear phase shift, the contribution of nonlinear phase noise from frequency and timing jitter decreases with distance and SNR.

## APPENDIX A

Here we find the joint characteristic function of

$$\varphi_1 = \int_0^\zeta w_\Omega^2(\zeta_1)\mathrm{d}\zeta_1, \tag{A1}$$

$$\varphi_2 = \int_0^\zeta w_T(\zeta_1)\mathrm{d}w_\Omega(\zeta_1), \tag{A2}$$



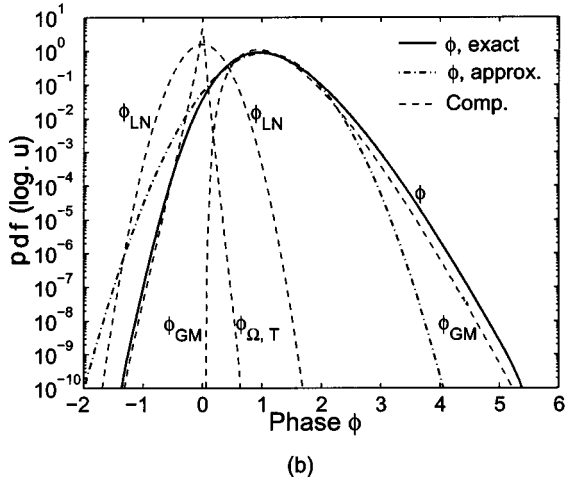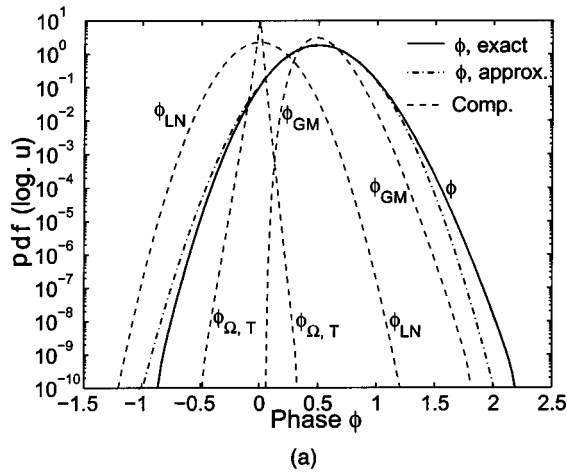Fig. 2. Distributions of soliton phase jitter for two distances of (a) $\zeta = 1$ and (b) $\zeta = 2$. Solid curves, exact overall phase jitter; dashed–dotted curves, Gaussian approximation of the overall phase jitter; dashed curves, components of phase jitter. The distribution is plotted in algorithmic units.
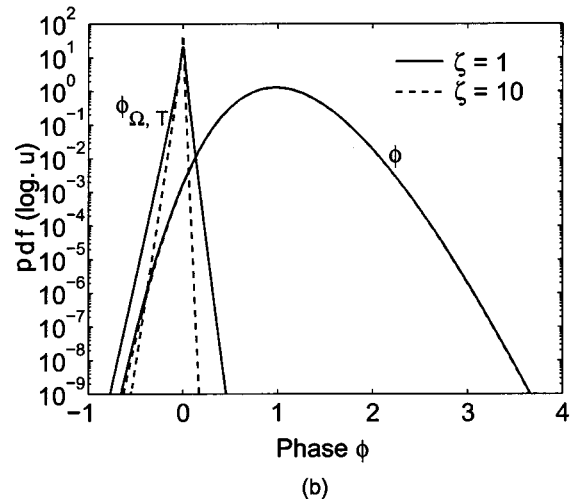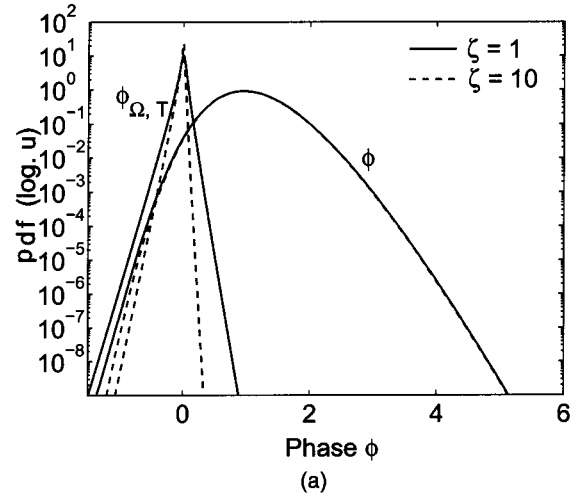


Fig. 3. Distributions of soliton phase jitter for SNR of (a) 10 and (b) 20, including the pdf of the overall phase jitter and the contribution from the frequency and timing effect. The distribution is plotted in algorithmic units.

$$\varphi_3 = w_\Omega(\zeta)\int_0^\zeta w_\Omega(\zeta_1)\mathrm{d}\zeta_1. \tag{A3}$$

By changing the integration order, we obtain

$$\varphi_2 = \int_0^\zeta \int_0^{\zeta_1} \mathrm{d}w_T(\zeta_2)\mathrm{d}w_\Omega(\zeta_1)$$

$$= \int_0^\zeta [w_\Omega(\zeta) - w_\Omega(\zeta_2)]\mathrm{d}w_T(\zeta_2). \tag{A4}$$

The joint characteristic function of $\varphi_1$, $\varphi_2$, and $\varphi_3$ is

$$\Psi_{\varphi_1,\varphi_2,\varphi_3}(\nu_1,\ \nu_2,\ \nu_3) = E\{\exp(j\nu_1\varphi_1 + j\nu_2\varphi_2 + j\nu_3\varphi_3)\}, \tag{A5}$$

where $E\{\}$ denotes expectation. Similar to option pricing with stochastic volatility,[37] the expectation of Eq. (A5) can be evaluated in two steps, first over $w_T$ and then over $w_\Omega$. In the average over $w_T$, it is obvious that $\varphi_2$ is a zero-mean Gaussian random variable with a variance of $\sigma_T{}^2 \int_0^\zeta [w_\Omega(\zeta) - w_\Omega(\zeta_1)]^2 \mathrm{d}\zeta_1$, where we obtain

$$\Psi_{\varphi_1,\varphi_2,\varphi_3}(\nu_1,\,\nu_2,\,\nu_3) = E\left\{-\frac{\sigma_T^2\nu_2^2}{2}\int_0^\zeta [w_\Omega(\zeta) - w_\Omega(\zeta_1)]^2 \mathrm{d}\zeta_1 + j\nu_1\int_0^\zeta w_\Omega^2(\zeta_1)\mathrm{d}\zeta_1 + j\nu_3 w_\Omega(\zeta)\int_0^\zeta w_\Omega(\zeta_1)\mathrm{d}\zeta_1\right\}$$

$$= E\left[-\frac{\sigma_T^2\nu_2^2\zeta}{2}w_\Omega^2(\zeta) + (j\nu_3 + \sigma_T^2\nu_2^2)w_\Omega(\zeta)\int_0^\zeta w_\Omega(\zeta_1)\mathrm{d}\zeta_1 + \left(j\nu_1 - \frac{\sigma_T^2\nu_2^2}{2}\right)\int_0^\zeta w_\Omega^2(\zeta_1)\mathrm{d}\zeta_1\right].$$

$$(A6)$$

First, we have[10,13,38,39]

$$E\left[j\omega_1 w_\Omega(\zeta) + j\omega_2\int_0^\zeta w_\Omega(\zeta_1)\mathrm{d}\zeta_1 + \frac{j\omega_3}{2}\int_0^\zeta w_\Omega^2(\zeta_1)\mathrm{d}\zeta_1\right]$$

$$= \sec^{1/2}(\sqrt{j\omega_3}\sigma_\Omega\zeta)\exp\left\{-\frac{1}{2}\left(\omega_1^2\sigma_\Omega^2 + \frac{\omega_2^2}{j\omega_3}\right)\frac{\tan(\sqrt{j\omega_3}\sigma_\Omega\zeta)}{\sqrt{j\omega_3}\sigma_\Omega} + j\frac{\omega_1\omega_2}{\omega_3}[\sec(\sqrt{j\omega_3}\sigma_\Omega\zeta) - 1] - j\frac{\omega_2^2\zeta}{2\omega_3}\right\}$$

$$= \sec^{1/2}(\sqrt{j\omega_3}\sigma_\Omega\zeta)\exp\left[-\frac{1}{2}\boldsymbol{\omega}_{1,2}^T \mathcal{C}(j\omega_3)\boldsymbol{\omega}_{1,2}\right], \quad (A7)$$

where $\boldsymbol{\omega}_{1,2} = (\omega_1,\,\omega_2)^T$ and

$$\mathcal{C}(j\omega_3) = \left\{\begin{matrix} \dfrac{\sigma_\Omega\tan(\sqrt{j\omega_3}\sigma_\Omega\zeta)}{\sqrt{j\omega_3}} & \dfrac{1}{j\omega_3}[\sec(\sqrt{j\omega_3}\sigma_\Omega\zeta) - 1] \\[2mm] \dfrac{1}{j\omega_3}[\sec(\sqrt{j\omega_3}\sigma_\Omega\zeta) - 1] & \dfrac{1}{j\omega_3}\left[\dfrac{\tan(\sqrt{j\omega_3}\sigma_\Omega\zeta)}{\sqrt{j\omega_3}\sigma_\Omega} - \zeta\right] \end{matrix}\right\}. \quad (A8)$$

As a verification, if $\omega_3$ approaches zero, the covariance matrix is

$$\lim_{\omega_3\to 0} C(j\omega_3) = \sigma_\Omega^2\begin{bmatrix} \zeta & \frac{1}{2}\zeta^2 \\ \frac{1}{2}\zeta^2 & \frac{1}{3}\zeta^3 \end{bmatrix}, \quad (A9)$$

that is the covariance matrix of the vector of

$$\mathbf{w}_\zeta = \left[w_\Omega(\zeta),\,\int_0^\zeta w_\Omega(\zeta_1)\mathrm{d}\zeta_1\right]^T \quad (A10)$$

without any dependence on the random variable $\varphi_1$. Note that the equation corresponding to Eq. (A7) in Refs. 10 and 39 does not have the limit of Eq. (A9).

The characteristic function of Eq. (A7) is that of a correlated two-dimensional Gaussian random variable of $\mathbf{w}_\zeta$ with dependence to $\varphi_1$. The first two terms of Eq. (A6) are a quadratic (or bilinear) function of $\mathbf{w}_\zeta$, i.e., $\frac{1}{2}\mathbf{w}_\zeta^T\mathcal{M}(j\nu_2,j\nu_3)\mathbf{w}_\zeta$, where

$$\mathcal{M}(j\nu_2,\,j\nu_3) = \begin{bmatrix} -\sigma_T^2\nu_2^2\zeta & j\nu_3 + \sigma_T^2\nu_2^2 \\ j\nu_3 + \sigma_T^2\nu_2^2 & 0 \end{bmatrix}. \quad (A11)$$

The characteristic function of the quadratic function of zero-mean Gaussian random variables is $\det[\mathcal{I} - \mathcal{C}\mathcal{M}]^{-1/2}$ (Ref. 12), where det[ ] denotes the determinant of a matrix.

The joint characteristic function is

$$\Psi_{\varphi_1,\varphi_2,\varphi_3}(\nu_1,\,\nu_2,\,\nu_3)$$

$$= \frac{\sec^{1/2}[(2j\nu_1 - \sigma_T^2\nu_2^2)^{1/2}\sigma_\Omega\zeta]}{\det[\mathcal{I} - \mathcal{C}(2j\nu_1 - \sigma_T^2\nu_2^2)\mathcal{M}(j\nu_2,j\nu_3)]^{1/2}}, \quad (A12)$$

where $\mathcal{I}$ is the identity matrix. The substitution of $j\omega_3$ by $2j\nu_1 - \sigma_T^2\nu_2^2$ is obvious when we compare Eqs. (A6) and (A7). We can obtain[11,38]

$$\Psi_{\varphi_1}(\nu_1) = \sec^{1/2}(\sqrt{2j\nu_1}\sigma_\Omega\zeta) \quad (A13)$$

and also[37]

$$\Psi_{\varphi_2}(\nu_2) = \mathrm{sech}^{1/2}(\sigma_T\sigma_\Omega\zeta\nu_2), \quad (A14)$$

respectively. The statistical properties of $\int_0^\zeta w_T(\zeta_1)\mathrm{d}w_\Omega(\zeta_1)$, $\int_0^\zeta w_\Omega(\zeta_1)\mathrm{d}w_T(\zeta_1)$, $\int_0^\zeta [w_\Omega(\zeta) - w_\Omega(\zeta_1)]\mathrm{d}w_T(\zeta_1)$, and $\int_0^\zeta [w_T(\zeta) - w_T(\zeta_1)]\mathrm{d}w_\Omega(\zeta_1)$ are the same.

We can also obtain

$$\Psi_{\varphi_3}(\nu_3) = \left[1 - j\nu_3\sigma_\Omega^2\zeta^2 + \frac{1}{12}\nu_3^2\sigma_\Omega^4\zeta^4\right]^{-1/2}. \quad (A15)$$

Whereas both random variables $\varphi_1$ and $\varphi_3$ are determined by $\sigma_\Omega\zeta$, the random variable of $\varphi_2$ is determined by $\sigma_T\sigma_\Omega\zeta$.

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