Some Extended Results on the Search for Good Convolutional Codes

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***Abstract*— We provide useful results on two classes of convolutional codes: binary codes and nonbinary codes. The best codes or the best known codes for these two classes of convolutional codes are found by computer search. Some of them are better than those found in the past. We specify these codes by their transfer function matrices, distance spectra, and information-weight spectra. Furthermore, we derive an upper bound on the free distances of binary-to-M-ary codes and q-ary-to-M-ary codes. Numerical values of this bound closely fit the computer-searched values.**

***Index Terms*— Codes, convolutional codes, distance spectra, information-weight spectra, linear codes.**

I. Introduction

In this correspondence, we provide useful results on two classes of convolutional codes: binary codes and nonbinary codes. Each code in the first class is a conventional *binary convolutional code* [1] with rate k/n where k is the number of message bits fed to the encoder each time and n is the number of code bits produced by the encoder each time. Each code in the second class is a *nonbinary convolutional code* for which each time the encoder takes $\log_q 2^k$ q-ary message symbols which are first converted into k bits as input and produces n output message bits which correspond to $\log_M 2^n$ M-ary code symbols. Some codes in this class have been discussed by Trumpis [2], Piret [3], [4], as well as Ryan and Wilson [5]. The q-ary code, which is a special case of codes in this class with $M = q$, has been investigated by Ryan and Wilson [5]. We assume that both q and M are powers of 2 which are useful in practical applications.

Binary convolutional codes are the most frequently used codes among the two classes of codes. For this class of codes, some powerful codes of rates $1/2$, $1/3$, and $1/4$ were first found by Odenwalder [6], Bahl and Jelinek [7], and Larsen [8], and those for rates $2/3$ and $3/4$ were found by Paaske [9] and Johannesson and Paaske [10]. Each of the codes issued in [6]–[10] has the largest possible free distance among convolutional codes of the associated code rate. These codes are also listed in several textbooks such as [12]–[14].

For nonbinary convolutional codes, there are two subclasses: binary-to-M-ary codes and q-ary-to-M-ary codes with $q > 2$. Binary-to-M-ary convolutional codes are useful for the M-ary channel in which orthogonal signaling is employed by the transmitter and noncoherent signal detection is used by the receiver. Hence, these codes can be applied to the independent Rayleigh channel or the partial band noise interference channel [5]. These codes can also be applied to design a class of multidimensional MPSK trellis codes [15].

The q-ary-to-M-ary codes with $q > 2$ are similar to binary-to-M-ary codes except that the input symbols are q-ary. These codes are useful if the input sequences are composed of nonbinary symbols.

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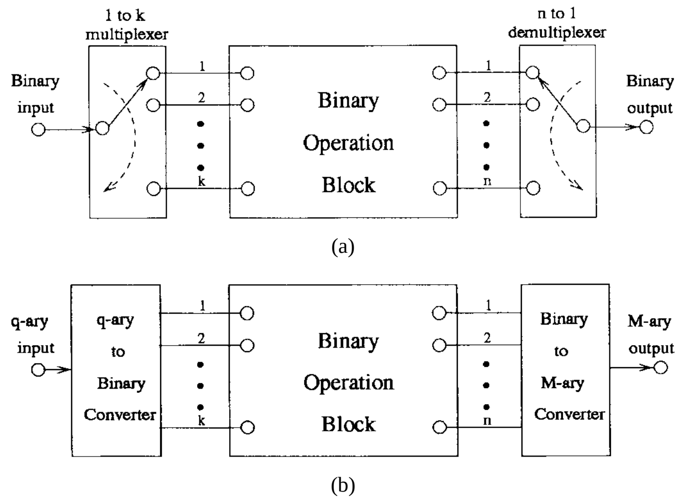

Fig. 1. The encoders of a binary convolutional code and a q-ary-to-M-ary convolutional code. (a) Binary convolutional encoder. (b) q-ary to M-ary convolutional encoder.

When maximum-likelihood trellis decoding is employed in a coded system, the error probability of information symbol (binary or q-ary) P_s for a symmetric and memoryless channel can be estimated by [12]

$$P_s \leq \frac{1}{\log_q 2^k} \sum_{d=d_\infty}^{\infty} N_{\mathrm{I}}(d) P_e(d) \tag{1}$$

where d_∞ is the free distance of the code, $P_e(d)$ is the pairwise error probability for a codeword pair separated by a distance d, and $N_{\mathrm{I}}(d)$ is the dth component of the information-weight spectrum of the code. Note that $P_e(d)$ is determined by the statistical behavior of the channel and $N_{\mathrm{I}}(d)$ is determined by the code. Besides the information-weight spectrum, the distance spectrum is another important distance property of a code because the first event error probability of a coded system P_f is upper-bounded by

$$P_f \leq \sum_{d=d_\infty}^{\infty} N(d) P_e(d) \tag{2}$$

where $N(d)$ is the dth component of the distance spectrum. Therefore, the two distance properties (information-weight and distance spectra) are important in evaluating the performance of a convolutional coded system. Algorithms for computer search to find the distance properties of convolutional codes have been proposed in [16] and [17]. The results provided by these proposed algorithms are only for $(n, 1)$ binary convolutional codes. Among them, the FAST algorithm proposed by Cedervall [17] is the fastest because it substantially limits the paths which need to be searched in a code tree by employing the distance profile.

In this correspondence, we use an exhaustive search to find the best codes of the two classes of convolutional codes. In the exhaustive search, the FAST algorithm is slightly modified to determine the distance properties of (n, k) codes. We find that some of the codes

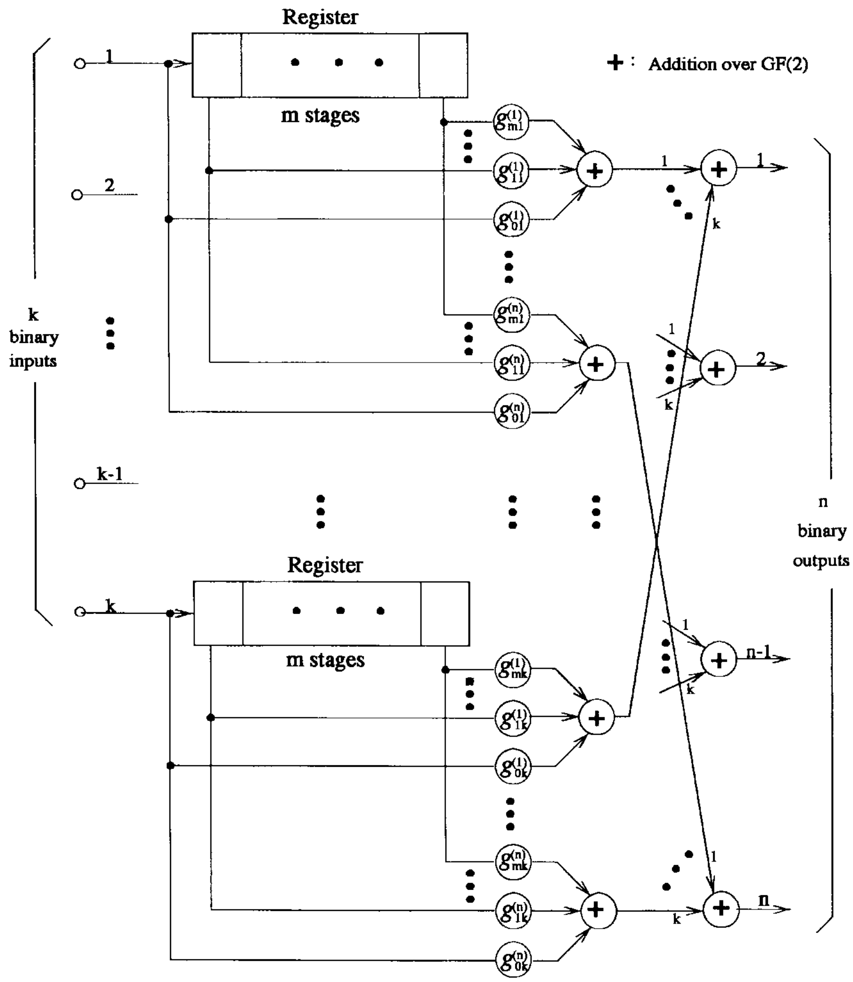

Fig. 2. The structure of the binary operation block.

listed in [6]–[10] are not the best codes because they only have maximum free distances but do not have the best information-weight spectra. Furthermore, we derive an upper bound on the free distances of binary-to-M-ary codes and q-ary-to-M-ary codes. This bound is derived based on the technique of deriving the Plotkin bound. Our bound is tighter than that derived by Trumpis [2] and fits well with the free distances found by the computer search.

II. Some Preliminaries

General forms of encoders for the two classes of codes are shown in Fig. 1. Each encoder is composed of an input block, an output block, and a binary operation block. The binary operation block shown in Fig. 2 is the common block for all the encoders in Fig. 1. At each time unit, k binary input bits are shifted into k parallel shift registers and through some logical operations n output bits are produced. For the operation block, the input sequence $\boldsymbol{x}$ and output sequence $\boldsymbol{y}$ can be expressed as

$$\begin{aligned}\boldsymbol{x} &= [\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots] \\ &= [x_0^{(1)} x_0^{(2)} \cdots x_0^{(k)}, x_1^{(1)} x_1^{(2)} \cdots x_1^{(k)}, \cdots]\end{aligned} \tag{3}$$

and

$$\begin{aligned}\boldsymbol{y} &= [\boldsymbol{y}_0, \boldsymbol{y}_1, \boldsymbol{y}_2, \cdots] \\ &= [y_0^{(1)} y_0^{(2)} \cdots y_0^{(n)}, y_1^{(1)} y_1^{(2)} \cdots y_1^{(n)}, \cdots]\end{aligned} \tag{4}$$

where $\boldsymbol{x}_i = [x_i^{(1)} x_i^{(2)} \cdots x_i^{(k)}]$ is the input vector which contains k bits and $\boldsymbol{y}_i = [y_i^{(1)} y_i^{(2)} \cdots y_i^{(n)}]$ is the output vector which contains n bits. The relationship between $\boldsymbol{x}$ and $\boldsymbol{y}$ is

$$\boldsymbol{y} = \boldsymbol{x}\boldsymbol{G} \tag{5}$$

where all operations on $\boldsymbol{x}\boldsymbol{G}$ are modulo-2 and the semi-infinite matrix

$$\boldsymbol{G} = \begin{pmatrix} \boldsymbol{G}_0 & \boldsymbol{G}_1 & \boldsymbol{G}_2 & \cdots & \boldsymbol{G}_m & & \\ 0 & \boldsymbol{G}_0 & \boldsymbol{G}_1 & \boldsymbol{G}_2 & \cdots & \boldsymbol{G}_m & \\ 0 & 0 & \boldsymbol{G}_0 & \boldsymbol{G}_1 & \boldsymbol{G}_2 & \cdots & \boldsymbol{G}_m \\ & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{6}$$

is called the generator matrix of the operation block. Note that each $\boldsymbol{G}_l$, $0 \le l \le m$, in (6) is a $k \times n$ submatrix of binary elements, which can be expressed as

$$\boldsymbol{G}_l = \begin{pmatrix} g_{1l}^{(1)} & g_{1l}^{(2)} & \cdots & g_{1l}^{(n)} \\ g_{2l}^{(1)} & g_{2l}^{(2)} & \cdots & g_{2l}^{(n)} \\ \vdots & \vdots & & \vdots \\ g_{k,l}^{(1)} & g_{k,l}^{(2)} & \cdots & g_{k,l}^{(n)} \end{pmatrix}. \tag{7}$$

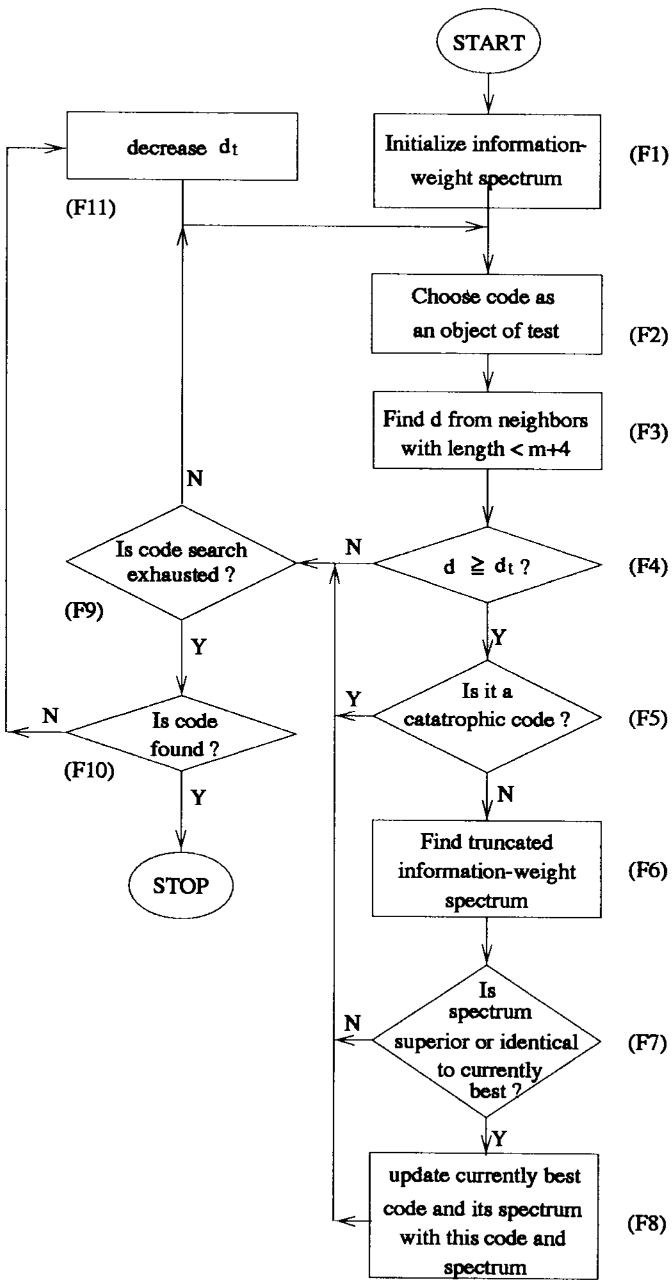

Fig. 3. The flowchart of the algorithm used to search the best convolutional code.

The relation between $\boldsymbol{x}$ and $\boldsymbol{y}$ can also be represented by the transfer function matrix

$$\boldsymbol{G}(D) = \begin{pmatrix} G_1^{(1)}(D) & G_1^{(2)}(D) & \cdots & G_1^{(n)}(D) \\ G_2^{(1)}(D) & G_2^{(2)}(D) & \cdots & G_2^{(n)}(D) \\ \vdots & \vdots & & \vdots \\ G_k^{(1)}(D) & G_k^{(2)}(D) & \cdots & G_k^{(n)}(D) \end{pmatrix} \tag{8}$$

where

$$G_i^{(j)}(D) = g_{i,m_i}^{(j)} D^{m_i} + \cdots + g_{i0}^{(j)} \tag{9}$$

is the generator polynomial representing the connections from the ith input to the jth output on the operation block for $1 \leq i \leq k$, $1 \leq j \leq n$. Let

$$m_i = \max_{1 \leq j \leq n} [\deg G_i^{(j)}(D)].$$

The memory order of the corresponding code is

$$m = \max_{1 \leq i \leq k} m_i \tag{10}$$

and the total number of memory elements actually used in the operation block is

$$\nu = \sum_{i=1}^{k} m_i. \tag{11}$$

In our study, we assume that $m - 1 \leq m_i \leq m$ for each i.

The input and output blocks of the encoders shown in Fig. 1 depend on the class of codes. For the binary convolutional encoder, its input block is a 1 to k multiplexer and its output block is an n to 1 demultiplexer. The distance properties of a convolutional code can be found by assuming the all-zero path to be a reference and all other paths to be the neighbors of the all-zero path. The all-zero path results from the all-zero information sequence $\boldsymbol{x} = [\mathbf{0}, \mathbf{0}, \cdots]$. Each neighbor is a path which departs from the all-zero path at time $t = 0$ and remerges to the all-zero path at time $L + m$ where $L \geq 1$. Then, this neighbor has a path length of $L + m$ branches or $(L + m) \times n$ bits. The distance of a neighbor to the all-zero path is the number of nonzero bits in the terminated coded sequence. The free distance of a binary convolutional code is the minimum distance for all its neighbors, which is denoted by d_∞. Let $N(d_\infty + i)$ denote the number of neighbors at a distance $d_\infty + i$ to the all-zero path. The list of $N(d_\infty + i)$ for $i = 0, 1, 2, \cdots$ is called the distance spectrum of a code. Let $N_{\mathrm{I}}(d_\infty + i)$ denote the total number of nonzero bits in the information sequences which correspond to all neighbors at a distance of $d_\infty + i$ to the all-zero path. A list of $N_{\mathrm{I}}(d_\infty + i)$ for $i = 0, 1, 2, \cdots$ is then called the information-weight spectrum of a code. Since both the distance and information-weight spectra are related to the distance properties of a code, we call these two spectra the distance properties of a code.

For nonbinary codes, we only discuss the q-ary-to-M-ary codes because binary-to-M-ary codes constitute a special case of the former with $q = 2$. For a q-ary-to-M-ary convolutional code, the input block is a q-ary-to-binary converter which each time takes $\log_q 2^k$ q-ary symbols from the q-ary information sequence and converts these q-ary symbols into a k-tuple input vector as shown in (3). After the process of the operation block, an n-tuple output vector is produced. The n-tuple output vector is then converted into $\log_M 2^n$ M-ary symbols by an output block. Here, the distance of a neighbor to the all-zero path is the number of nonzero M-ary symbols in this neighbor. For a q-ary-to-M-ary code, its free distance and distance spectra are defined in ways similar to those for the binary convolutional code. Here, $N_{\mathrm{I}}(d_\infty + i)$ is the total number of nonzero q-ary symbols on the information sequences mapping to all neighbors at a distance of $d_\infty + i$ to the all-zero path and the information-weight spectrum of the codes is a list of $N_{\mathrm{I}}(d_\infty + i)$ for $0 \leq i \leq \infty$.

III. An Upper Bound on the Free Distances of M-ary Codes

We now derive an upper bound on the free distances of q-ary-to-M-ary codes, which is better than that derived in [2]. At the beginning, we consider the case of $M = 2^n$.

The free distance of a convolutional code is the smallest number of nonzero symbols in any possible terminated nonzero coded sequence

TABLE I
UPPER BOUNDS FOR 2^k-ARY-TO-2^n-ARY CODE OF RATE k/n ($M = 2^n$ AND $\nu = km$)

m \ k/n	1/2	1/3	2/3	1/4	2/4	3/4	1/5	2/5	3/5	4/5
1	2	2	2	2	2	2	2	2	2	2
2	3	3	3	3	3	3	3	3	3	3
3	4	4	4	4	4	4	4	4	4	4
4	5	5	5	5	5	5	5	5	5	5
5	6	6	6	6	6	6	6	6	6	6
6	7	7	7	7	7	7	7	7	7	7
7	8	8	8	8	8	8	8	8	8	8
8	8 +	9	9	9	9	9	9	9	9	9
9	9 +	10	10	10	10	10	10	10	10	10
10	10 +	11	10	11	11	11	11	11	11	11
11	11 +	12	11	12	12	12	12	12	12	12
12	12 +	13	12	13	13	13	13	13	13	13
13	13	14	13	14	14	14	14	14	14	14
14	13 +	15	14	15	15	15	15	15	15	15
15	14 +	16	15	16	16	16	16	16	16	16
16	15 +	17	16	17	17	17	17	17	17	17
17	16	18	17	18	18	17	18	18	18	18
18	17	19	18	19	19	18	19	19	19	19

TABLE II
UPPER BOUNDS FOR q-ARY-TO-M-ARY CODE OF RATE $k/n = pk'/pn'$ WHERE $M = 2^{n'}$

ν \ k/n	1/2 M=4	1/3 M=8	2/3 M=8	1/4 M=16	2/4 M=16	2/4 M=4	3/4 M=16	3/6 M=4	4/6 M=8
1	2	2	1	2	1	3	1	4	2
2	3	3	2	3	2	4	1	5	3
3	4	4	2	4	2	5	2	5	3
4	5	5	3	5	3	6	2	6	3
5	6	6	3	6	3	7	2	7	4
6	7	7	4	7	4	8	3	7	5
7	8	8	4	8	4	9	3	9	5
8	8	9	5	9	5	9	3	10	5
9	9	10	5	10	5	10	4	10	6
10	10	11	6	11	6	11	4	12	7
11	11	12	6	12	6	12	4	12	7
12	12	13	7	13	7	12	5	12	7
13	13	14	7	14	7	13	5	14	8
14	13	15	8	15	8	14	5	15	9
15	14	16	8	16	8	15	6	15	9
16	15	17	9	17	9	16	6	17	9
17	16	18	9	18	9	17	6	18	10
18	17	19	10	19	10	17	7	18	10

which can be expressed as

$$(\boldsymbol{y}_0, \boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_{L+m-1}) = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_{L-1})\boldsymbol{G}^{(L)}$$

where (see (12) at the bottom of this page) and $L \geq 1$. Note that the code symbols $\boldsymbol{y}_0, \boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_{L+m-1}$ are binary n-tuples which can be regarded as elements in GF $(M) =$ GF (2^n) and information symbols $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_{L-1}$ are binary k-tuples. The sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{L-1})$ can be considered as a binary sequence of length kL. Since GF (2) is a subfield of GF $(M) =$ GF (2^n), each element in this sequence of length kL is in GF (M) and is restricted in GF (2). The terminated coded sequences of $m + L$ symbols in the convolutional code can be considered as the codewords of a linear block code $C^{(L)}$ with generator matrix $\boldsymbol{G}^{(L)}$. As shown in (12), $\boldsymbol{G}^{(L)}$ is a $kL \times (L + m)$ matrix for which all its entries are elements in GF (M) and each submatrix $\boldsymbol{G}_l$ for $0 \leq l \leq m$ is a $k \times 1$ matrix. Let d_L be an upper bound on the minimum distance of the block code $C^{(L)}$. Then the free distance of the M-ary convolutional code is upper-bounded by

$$d_\infty \leq \min_{L \geq 1} d_L. \tag{13}$$

In [2], an upper bound on the free distance for a convolutional code is derived by employing the Plotkin upper bound on minimum distances for all the linear block codes $C^{(L)}$, $L \geq 1$. In [2], the generator matrix of a linear block code is taken to be in a general form, other than that given in (12). However, we observe that the generator matrix shown in (12) has a special structure. For example, there is only one nonzero submatrix G_0 on the first column. Therefore, in the following, we will also apply the concept of the Plotkin bound but carefully utilize the special structure inherent in (12) to obtain a bound tighter than that given in [2].

Recall that the total number of possible codewords in $C^{(L)}$ is 2^{kL}. Let $\boldsymbol{A}$ be a $2^{kL} \times (m + L)$ matrix which contains the 2^{kL} codewords of $C^{(L)}$ as its rows. Each entry in the matrix $\boldsymbol{A}$ is an element in GF (M). The matrix $\boldsymbol{A}$ is called the codeword matrix for the linear block code $C^{(L)}$. Let a_{ij} denote the entry of $\boldsymbol{A}$ at the ith row and the jth column. It follows from the concept of the Plotkin bound that the minimum distance d_L between any two codewords in matrix $\boldsymbol{A}$ can be upper-bounded by

$$2^{kL}(2^{kL} - 1)d_L \leq \sum_{j=0}^{m+L-1} W_L(j) \tag{14}$$

where

$$W_L(j) = \sum_{i \neq i'} d(a_{ij}, a_{i'j})$$

for $1 \leq i, i' \leq 2^{kL}$, where

$$d(a_{ij}, a_{i'j}) = \begin{cases} 1, & \text{for } a_{ij} \neq a_{i'j} \\ 0, & \text{for } a_{ij} = a_{i'j}. \end{cases} \tag{15}$$

$$\boldsymbol{G}^{(L)} = \overbrace{\begin{pmatrix} \boldsymbol{G}_0 & \boldsymbol{G}_1 & \boldsymbol{G}_2 & \cdots & \cdots & \boldsymbol{G}_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{G}_0 & \boldsymbol{G}_1 & \boldsymbol{G}_2 & \cdots & \cdots & \boldsymbol{G}_m & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \boldsymbol{G}_0 & \boldsymbol{G}_1 & \boldsymbol{G}_2 & \cdots & \boldsymbol{G}_m \end{pmatrix}}^{L+m \text{ columns}} \left.\vphantom{\begin{pmatrix} a \\ a \\ a \\ a \end{pmatrix}}\right\} L \text{ rows} \tag{12}$$

TABLE III
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY CONVOLUTIONAL CODES OF RATE $k/n = 1/2$

m	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$ $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	$(1\ \ 3)_8$	3	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$
2	$(5\ \ 7)_8^*$	5	$\frac{1}{1}$	$\frac{4}{2}$	$\frac{12}{4}$	$\frac{32}{8}$	$\frac{80}{16}$	$\frac{192}{32}$	$\frac{448}{64}$
3	$(15\ \ 17)_8^*$	6	$\frac{2}{1}$	$\frac{7}{3}$	$\frac{18}{5}$	$\frac{49}{11}$	$\frac{130}{25}$	$\frac{333}{55}$	$\frac{836}{121}$
4	$(23\ \ 35)_8^*$	7	$\frac{4}{2}$	$\frac{12}{3}$	$\frac{20}{4}$	$\frac{72}{16}$	$\frac{225}{37}$	$\frac{500}{68}$	$\frac{1324}{176}$
5	$(53\ \ 75)_8^*$	8	$\frac{2}{1}$	$\frac{36}{8}$	$\frac{32}{7}$	$\frac{62}{12}$	$\frac{332}{48}$	$\frac{701}{95}$	$\frac{2342}{281}$
6	$(133\ \ 171)_8^*$	10	$\frac{36}{11}$	$\frac{0}{0}$	$\frac{211}{38}$	$\frac{0}{0}$	$\frac{1404}{193}$	$\frac{0}{0}$	$\frac{11633}{1331}$
7	$(247\ \ 371)_8^*$	10	$\frac{2}{1}$	$\frac{22}{6}$	$\frac{60}{12}$	$\frac{148}{26}$	$\frac{340}{52}$	$\frac{1008}{132}$	$\frac{2642}{317}$
8	$(561\ \ 753)_8^*$	12	$\frac{33}{11}$	$\frac{0}{0}$	$\frac{281}{50}$	$\frac{0}{0}$	$\frac{2179}{286}$	$\frac{0}{0}$	$\frac{15035}{1630}$
9	$(1131\ \ 1537)_8$	12	$\frac{2}{1}$	$\frac{21}{7}$	$\frac{100}{19}$	$\frac{186}{28}$	$\frac{474}{69}$	$\frac{1419}{185}$	$\frac{3542}{411}$
10	$(2473\ \ 3217)_8$	14	$\frac{56}{14}$	$\frac{0}{0}$	$\frac{656}{92}$	$\frac{0}{0}$	$\frac{3708}{426}$	$\frac{0}{0}$	$\frac{27503}{2595}$
11	$(4325\ \ 6747)_8^{**}$	15	$\frac{66}{14}$	$\frac{98}{21}$	$\frac{220}{34}$	$\frac{788}{101}$	$\frac{2083}{249}$	$\frac{5424}{597}$	$\frac{13771}{1373}$
12	$(10627\ \ 16765)_8$	16	$\frac{60}{14}$	$\frac{188}{38}$	$\frac{288}{35}$	$\frac{952}{108}$	$\frac{2754}{342}$	$\frac{6628}{724}$	$\frac{16606}{1604}$
13	$(27251\ \ 37363)_8$	16	$\frac{2}{1}$	$\frac{99}{17}$	$\frac{234}{38}$	$\frac{513}{69}$	$\frac{1316}{158}$	$\frac{3890}{414}$	$\frac{9642}{944}$

* This code and its information-weight and distance spectra were respectively found by Odenwalder [6] and Conan [16].
** This code was found by Johannesson-Paaske [10].

TABLE IV
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY CONVOLUTIONAL CODES OF RATE $k/n = 1/3$

m	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$ $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	$(1\ \ 3\ \ 3)_8$	5	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$
2	$(5\ \ 7\ \ 7)_8^*$	8	$\frac{3}{2}$	$\frac{0}{0}$	$\frac{15}{5}$	$\frac{0}{0}$	$\frac{58}{13}$	$\frac{0}{0}$	$\frac{201}{34}$
3	$(13\ \ 15\ \ 17)_8^*$	10	$\frac{6}{3}$	$\frac{0}{0}$	$\frac{6}{2}$	$\frac{0}{0}$	$\frac{58}{15}$	$\frac{0}{0}$	$\frac{118}{24}$
4	$(25\ \ 33\ \ 37)_8^*$	12	$\frac{12}{5}$	$\frac{0}{0}$	$\frac{12}{3}$	$\frac{0}{0}$	$\frac{56}{13}$	$\frac{0}{0}$	$\frac{320}{62}$
5	$(47\ \ 53\ \ 75)_8^*$	13	$\frac{1}{1}$	$\frac{8}{3}$	$\frac{26}{6}$	$\frac{20}{4}$	$\frac{19}{5}$	$\frac{62}{12}$	$\frac{86}{14}$
6	$(117\ \ 127\ \ 155)_8^*$	15	$\frac{7}{3}$	$\frac{8}{3}$	$\frac{22}{6}$	$\frac{44}{9}$	$\frac{22}{4}$	$\frac{94}{18}$	$\frac{219}{35}$
7	$(225\ \ 331\ \ 367)_8^*$	16	$\frac{1}{1}$	$\frac{0}{0}$	$\frac{24}{8}$	$\frac{0}{0}$	$\frac{113}{24}$	$\frac{0}{0}$	$\frac{287}{51}$
8	$(575\ \ 623\ \ 727)_8$	18	$\frac{2}{1}$	$\frac{10}{4}$	$\frac{50}{10}$	$\frac{37}{9}$	$\frac{92}{17}$	$\frac{92}{14}$	$\frac{274}{39}$
9	$(1167\ \ 1375\ \ 1545)_8$	20	$\frac{6}{3}$	$\frac{16}{6}$	$\frac{72}{14}$	$\frac{68}{14}$	$\frac{170}{27}$	$\frac{162}{24}$	$\frac{340}{51}$
10	$(2325\ \ 2731\ \ 3747)_8$	22	$\frac{17}{7}$	$\frac{0}{0}$	$\frac{122}{27}$	$\frac{0}{0}$	$\frac{345}{55}$	$\frac{0}{0}$	$\frac{1102}{161}$
11	$(5745\ \ 6471\ \ 7553)_8$	24	$\frac{43}{13}$	$\frac{0}{0}$	$\frac{162}{32}$	$\frac{0}{0}$	$\frac{507}{78}$	$\frac{0}{0}$	$\frac{1420}{202}$
12	$(2371\ \ 13725\ \ 14733)_8^+$	24	$\frac{12}{5}$	$\frac{55}{15}$	$\frac{28}{6}$	$\frac{90}{14}$	$\frac{250}{44}$	$\frac{321}{51}$	$\frac{568}{83}$

* This code and its information-weight and distance spectra were respectively found by Odenwalder [6] and Conan [16].
+ This code is the currently best.

Equation (14) implies that an upper bound on d_L is

$$d_L = \left\lfloor \frac{\sum_{j=0}^{m+L-1} W_L(j)}{2^{kL}\cdot(2^{kL}-1)} \right\rfloor. \tag{16}$$

In order to find d_L, we must calculate all the values of $W_L(j)$ for $0 \le j \le m+L-1$. An upper bound on $W_L(j)$ can be found from the following theorem.

Theorem 1: For the M-ary code $C^{(L)}$ $(M = 2^n)$, suppose that the jth column $(0 \le j \le m+L-1)$ of the generator matrix in (12) contains $b\boldsymbol{G}_l$ nonzero submatrices and $L-b$ all-zero submatrices $\mathbf{0}$'s

TABLE V
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY CONVOLUTIONAL CODES OF RATE $k/n = 1/4$

	Generator		$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$						
m	sequences	d_∞	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	$(1\ 1\ 3\ 3)_8^*$	6	$\frac{1}{1}$	$\frac{0}{0}$	$\frac{2}{1}$	$\frac{0}{0}$	$\frac{3}{1}$	$\frac{0}{0}$	$\frac{4}{1}$
2	$(5\ 5\ 7\ 7)_8^*$	10	$\frac{1}{1}$	$\frac{0}{0}$	$\frac{4}{2}$	$\frac{0}{0}$	$\frac{12}{4}$	$\frac{0}{0}$	$\frac{32}{8}$
3	$(13\ 13\ 15\ 17)_8^{**}$	13	$\frac{4}{2}$	$\frac{2}{1}$	$\frac{0}{0}$	$\frac{10}{3}$	$\frac{3}{1}$	$\frac{16}{4}$	$\frac{34}{8}$
4	$(25\ 27\ 33\ 37)_8^{**}$	16	$\frac{8}{4}$	$\frac{0}{0}$	$\frac{7}{2}$	$\frac{0}{0}$	$\frac{17}{4}$	$\frac{0}{0}$	$\frac{60}{15}$
5	$(45\ 53\ 67\ 77)_8$	18	$\frac{5}{3}$	$\frac{0}{0}$	$\frac{19}{5}$	$\frac{0}{0}$	$\frac{14}{4}$	$\frac{0}{0}$	$\frac{70}{14}$
6	$(117\ 127\ 155\ 171)_8$	20	$\frac{3}{2}$	$\frac{0}{0}$	$\frac{17}{6}$	$\frac{0}{0}$	$\frac{32}{7}$	$\frac{0}{0}$	$\frac{66}{15}$
7	$(257\ 311\ 337\ 355)_8$	22	$\frac{2}{1}$	$\frac{4}{2}$	$\frac{4}{2}$	$\frac{24}{6}$	$\frac{22}{5}$	$\frac{33}{7}$	$\frac{44}{8}$
8	$(533\ 575\ 647\ 711)_8$	24	$\frac{1}{1}$	$\frac{0}{0}$	$\frac{15}{6}$	$\frac{0}{0}$	$\frac{56}{12}$	$\frac{0}{0}$	$\frac{69}{16}$
9	$(1173\ 1325\ 1467\ 1751)_8$	27	$\frac{7}{3}$	$\frac{10}{4}$	$\frac{0}{0}$	$\frac{28}{5}$	$\frac{54}{12}$	$\frac{58}{12}$	$\frac{54}{10}$

* This code was found by Bahl-Jelinek [18].

** This code and its information-weight and distance spectra were respectively found by Larsen [19] and Conan [16].

TABLE VI
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY CONVOLUTIONAL CODES OF RATE $k/n = 2/3$

		Generator		$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$						
m	ν	sequences	d_∞	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	2	$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 3 & 3 \end{pmatrix}_8$	3	$\frac{1}{1}$	$\frac{10}{4}$	$\frac{54}{14}$	$\frac{226}{40}$	$\frac{856}{116}$	$\frac{3072}{339}$	$\frac{10647}{991}$
2	3	$\begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 7 \end{pmatrix}_8$	4	$\frac{1}{1}$	$\frac{12}{5}$	$\frac{102}{24}$	$\frac{412}{71}$	$\frac{1821}{238}$	$\frac{7804}{862}$	$\frac{31819}{2991}$
	4	$\begin{pmatrix} 6 & 5 & 1 \\ 7 & 2 & 5 \end{pmatrix}_8$	5	$\frac{5}{3}$	$\frac{45}{11}$	$\frac{218}{39}$	$\frac{949}{135}$	$\frac{4518}{519}$	$\frac{19355}{1902}$	$\frac{81065}{6875}$
3	5	$\begin{pmatrix} 07 & 06 & 03 \\ 12 & 01 & 13 \end{pmatrix}_8$	6	$\frac{26}{7}$	$\frac{118}{25}$	$\frac{533}{79}$	$\frac{2455}{305}$	$\frac{10643}{1117}$	$\frac{46925}{4265}$	$\frac{198910}{15979}$
	6	$\begin{pmatrix} 06 & 13 & 13 \\ 13 & 06 & 17 \end{pmatrix}_8^*$	7	$\frac{86}{17}$	$\frac{360}{53}$	$\frac{1148}{133}$	$\frac{5767}{569}$	$\frac{27277}{2327}$	$\frac{114524}{8624}$	$\frac{481710}{32412}$
4	7	$\begin{pmatrix} 16 & 13 & 03 \\ 25 & 05 & 34 \end{pmatrix}_8$	8	$\frac{218}{43}$	$\frac{0}{0}$	$\frac{5102}{547}$	$\frac{0}{0}$	$\frac{101093}{8139}$	$\frac{0}{0}$	$\frac{1847624}{117815}$
	8	$\begin{pmatrix} 37 & 31 & 16 \\ 23 & 14 & 35 \end{pmatrix}_8$	8	$\frac{20}{6}$	$\frac{284}{42}$	$\frac{1312}{153}$	$\frac{5164}{510}$	$\frac{22192}{1853}$	$\frac{99382}{7338}$	$\frac{428364}{28378}$
5	9	$\begin{pmatrix} 27 & 23 & 16 \\ 46 & 17 & 41 \end{pmatrix}_8$	9	$\frac{89}{17}$	$\frac{577}{81}$	$\frac{2078}{228}$	$\frac{10311}{933}$	$\frac{43647}{3469}$	$\frac{185957}{13203}$	$\frac{808372}{51286}$
	10	$\begin{pmatrix} 63 & 51 & 34 \\ 52 & 37 & 55 \end{pmatrix}_8$	10	$\frac{435}{69}$	$\frac{0}{0}$	$\frac{9445}{925}$	$\frac{0}{0}$	$\frac{177734}{13189}$	$\frac{0}{0}$	$\frac{3296251}{197340}$

* This code was found by Paaske [9].

where each submatrix contains k symbols in GF(M). Let t be the number of zero symbols in the $b\boldsymbol{G}_l$ submatrices. Then, the value of $W_L(j)$ for the corresponding codeword matrix $\boldsymbol{A}$ is upper-bounded by

$$W_L(j) \le \begin{cases} 2^{2kL} - 2^{2kL-kb+t}, & \text{for } bk - t \le n \\ 2^{2kL} - 2^{2kL-n}, & \text{for } bk - t > n. \end{cases} \tag{17}$$

Proof: In the jth column of codeword matrix $\boldsymbol{A}$, let z_{ij} be the number of symbols which equal i for $i \in GF(M)$. Since $\sum_i z_{ij} = 2^{kL}$, we have

$$W_L(j) = \sum_i (2^{kL} - z_{ij})\, z_{ij} = 2^{2kL} - \sum_i z_{ij}^2. \tag{18}$$

The maximum in (18) occurs when all z_{ij}'s are equal. When $bk - t \le n$, the number of distinct symbols in the jth column is at most 2^{bk-t}. By taking

$$z_{ij} = 2^{kL}/2^{bk-t} = 2^{kL-kb+t}$$

TABLE VII
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY CONVOLUTIONAL CODES OF RATE $k/n = 2/4$

m	ν	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$ $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	2	$\begin{pmatrix}0&3&1&2\\3&1&2&1\end{pmatrix}_8$	5	$\frac{2}{2}$	$\frac{8}{4}$	$\frac{24}{8}$	$\frac{64}{16}$	$\frac{160}{32}$	$\frac{390}{67}$	$\frac{938}{142}$
2	3	$\begin{pmatrix}1&3&1&3\\7&1&5&4\end{pmatrix}_8$	6	$\frac{1}{1}$	$\frac{20}{8}$	$\frac{34}{9}$	$\frac{76}{16}$	$\frac{295}{57}$	$\frac{679}{106}$	$\frac{1598}{219}$
	4	$\begin{pmatrix}3&7&1&6\\4&7&6&3\end{pmatrix}_8$	8	$\frac{30}{12}$	$\frac{0}{0}$	$\frac{258}{52}$	$\frac{0}{0}$	$\frac{1714}{260}$	$\frac{0}{0}$	$\frac{12278}{1483}$
3	5	$\begin{pmatrix}06&07&03&03\\13&04&13&16\end{pmatrix}_8$	8	$\frac{2}{1}$	$\frac{24}{9}$	$\frac{57}{15}$	$\frac{173}{33}$	$\frac{494}{80}$	$\frac{1107}{156}$	$\frac{3064}{396}$
	6	$\begin{pmatrix}17&13&05&02\\10&03&17&15\end{pmatrix}_8$	10	$\frac{50}{16}$	$\frac{0}{0}$	$\frac{446}{77}$	$\frac{0}{0}$	$\frac{2926}{396}$	$\frac{0}{0}$	$\frac{20703}{2311}$
4	7	$\begin{pmatrix}27&07&02&33\\17&14&13&05\end{pmatrix}_8^+$	11	$\frac{44}{13}$	$\frac{167}{32}$	$\frac{326}{52}$	$\frac{741}{108}$	$\frac{2093}{255}$	$\frac{5686}{639}$	$\frac{15347}{1582}$
	8	$\begin{pmatrix}07&32&15&34\\35&15&21&31\end{pmatrix}_8^+$	12	$\frac{42}{11}$	$\frac{181}{38}$	$\frac{345}{51}$	$\frac{785}{109}$	$\frac{2288}{286}$	$\frac{6253}{697}$	$\frac{16363}{1690}$

$^+$ This code is the currently best.

TABLE VIII
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY CONVOLUTIONAL CODES OF RATE $k/n = 3/4$

m	ν	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$ $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	2	$\begin{pmatrix}1&1&1&0\\3&0&0&1\\3&2&0&2\end{pmatrix}_8$	3	$\frac{12}{6}$	$\frac{84}{23}$	$\frac{428}{80}$	$\frac{1984}{284}$	$\frac{8807}{1027}$	$\frac{37818}{3724}$	$\frac{158316}{13480}$
	3	$\begin{pmatrix}3&2&1&0\\3&1&2&1\\2&2&2&3\end{pmatrix}_8$	4	$\frac{31}{10}$	$\frac{237}{46}$	$\frac{1565}{202}$	$\frac{9389}{949}$	$\frac{53751}{4444}$	$\frac{297976}{20840}$	$\frac{1610542}{97528}$
2	4	$\begin{pmatrix}0&1&2&3\\3&0&1&2\\2&4&1&5\end{pmatrix}_8$	4	$\frac{2}{2}$	$\frac{59}{18}$	$\frac{486}{82}$	$\frac{3349}{402}$	$\frac{21074}{2017}$	$\frac{127322}{10093}$	$\frac{748206}{50536}$
	5	$\begin{pmatrix}3&3&2&2\\5&2&7&0\\4&7&0&1\end{pmatrix}_8$	5	$\frac{23}{7}$	$\frac{280}{47}$	$\frac{1680}{208}$	$\frac{11375}{1096}$	$\frac{70134}{5625}$	$\frac{423243}{28867}$	$\frac{2489834}{148226}$
	6	$\begin{pmatrix}5&4&3&2\\4&6&5&5\\6&1&4&3\end{pmatrix}_8$	6	$\frac{159}{27}$	$\frac{910}{118}$	$\frac{5536}{529}$	$\frac{37465}{2978}$	$\frac{223804}{15201}$	$\frac{1348149}{79518}$	$\frac{7943828}{414729}$
3	7	$\begin{pmatrix}02&03&04&07\\03&07&03&05\\15&02&02&17\end{pmatrix}_8$	6	$\frac{16}{5}$	$\frac{517}{65}$	$\frac{3088}{292}$	$\frac{18742}{1442}$	$\frac{116312}{7618}$	$\frac{700970}{39734}$	$\frac{4199390}{209693}$
	8	$\begin{pmatrix}04&06&07&07\\01&12&05&14\\00&07&14&11\end{pmatrix}_8$	7	$\frac{167}{27}$	$\frac{1384}{157}$	$\frac{9156}{770}$	$\frac{56524}{3925}$	$\frac{339830}{20440}$	$\frac{2035418}{107679}$	$\frac{12090596}{568758}$
	9	$\begin{pmatrix}03&06&10&15\\00&16&03&13\\16&05&02&17\end{pmatrix}_8$	8	$\frac{1097}{136}$	$\frac{0}{0}$	$\frac{47271}{3583}$	$\frac{0}{0}$	$\frac{1789855}{99698}$	$\frac{0}{0}$	$\frac{62662961}{2767158}$

we have

$$W_L(j) \le 2^{2kL} - 2^{2kL-kb+t}$$

for $bk - t < n$. When $bk - t > n$, the number of distinct symbols in the jth column is at most 2^n. By taking

$$z_{ij} = 2^{kL}/2^n = 2^{kL-n}$$

we have

$$W_L(j) \le 2^{2kL} - 2^{2kL-n}$$

for $bk - t > n$. □

We are now able to derive an upper bound on the free distances for q-ary-to-M-ary codes. By observing the generator matrix in (12), we find that the total number of $\boldsymbol{G}_l$ submatrices in the jth column is

$$b = \begin{cases} \min(L,\ j+1), & \text{for } 0 \le j \le m-1 \\ \min(m+L-j,\ m+1), & \text{for } m \le j \le m+L-1. \end{cases} \tag{19}$$

Since we assume that $m-1 \le m_i \le m$ for each i, it can be shown that there are $k \cdot m - \nu$ zero row vectors appearing in the submatrix $\boldsymbol{G}_m$ when m and ν are specified, where ν is defined in

TABLE IX
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY CONVOLUTIONAL CODES OF RATE $k/n = 4/6$

m	ν	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$, $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	2	$\begin{pmatrix} 3&0&2&3&2&2\\ 1&2&2&0&3&0\\ 1&0&0&1&1&0\\ 0&1&1&1&0&1 \end{pmatrix}_8^+$	3	$\frac{1}{1}$	$\frac{17}{6}$	$\frac{115}{28}$	$\frac{437}{76}$	$\frac{1586}{218}$	$\frac{6049}{708}$	$\frac{22178}{2234}$
	3	$\begin{pmatrix} 3&0&3&2&2&2\\ 0&2&3&0&3&0\\ 3&1&1&3&3&0\\ 0&1&1&1&0&1 \end{pmatrix}_8^+$	4	$\frac{2}{1}$	$\frac{44}{14}$	$\frac{260}{49}$	$\frac{1112}{154}$	$\frac{4654}{530}$	$\frac{19272}{1858}$	$\frac{78816}{6572}$
	4	$\begin{pmatrix} 1&2&0&3&0&3\\ 3&1&0&2&2&1\\ 3&2&2&0&1&2\\ 1&3&3&2&0&0 \end{pmatrix}_8^+$	6	$\frac{186}{46}$	$\frac{0}{0}$	$\frac{4128}{552}$	$\frac{0}{0}$	$\frac{79646}{7546}$	$\frac{0}{0}$	$\frac{1400194}{102563}$
2	5	$\begin{pmatrix} 2&7&0&4&7&7\\ 1&3&2&0&2&2\\ 3&0&2&1&2&1\\ 0&2&3&2&2&1 \end{pmatrix}_8^+$	6	$\frac{24}{9}$	$\frac{328}{52}$	$\frac{929}{130}$	$\frac{4973}{543}$	$\frac{22782}{2077}$	$\frac{91555}{7394}$	$\frac{409027}{29001}$
	6	$\begin{pmatrix} 4&1&2&1&6&1\\ 1&6&2&1&0&7\\ 2&2&3&1&3&1\\ 3&0&2&3&1&2 \end{pmatrix}_8^+$	7	$\frac{141}{28}$	$\frac{822}{104}$	$\frac{3224}{318}$	$\frac{12984}{1130}$	$\frac{58483}{4434}$	$\frac{262540}{17656}$	$\frac{1123883}{67650}$
	7	$\begin{pmatrix} 5&2&4&3&2&4\\ 0&3&2&4&1&7\\ 3&4&4&4&1&5\\ 2&2&3&2&3&2 \end{pmatrix}_8^+$	8	$\frac{442}{80}$	$\frac{0}{0}$	$\frac{9098}{1001}$	$\frac{0}{0}$	$\frac{182315}{14748}$	$\frac{0}{0}$	$\frac{3369304}{216437}$
	8	$\begin{pmatrix} 5&6&5&6&7&4\\ 7&0&7&3&6&2\\ 4&5&2&6&5&0\\ 6&1&5&7&2&5 \end{pmatrix}_8^+$	8	$\frac{76}{15}$	$\frac{782}{97}$	$\frac{2571}{269}$	$\frac{11049}{973}$	$\frac{49770}{3841}$	$\frac{211356}{14646}$	$\frac{917121}{57026}$

$^+$ This code is the currently best.

(11). Therefore, the value of t for the jth column can be found to be

$$t \geq \begin{cases} 0, & \text{for } 0 \leq j \leq m-1 \\ k\cdot m - \nu, & \text{for } j \geq m \end{cases} \tag{20}$$

It follows from (17), (19), and and (20) that the value of $W_L(j)$ can be bounded by (21) (see the bottom of this page) where

$$\begin{aligned} b &= \min(L,\, j+1), && \text{for } 0 \leq j \leq m-1 \\ b &= \min(m+L-j,\, m+1), && \text{for } m \leq j \leq m+L-1 \end{aligned}$$

and

$$p_1 = \lfloor (km+n-\nu)/k \rfloor.$$

With (13), (16), and (21), an upper bound on the free distance of the M-ary code $(M = 2^n)$ can be obtained.

Now we consider the M-ary codes $(M = 2^{n'})$ with $k/n = pk'/pn'$. The formula of the upper bound must be modified. In (12), each $\boldsymbol{G}_l$ in $\boldsymbol{G}^{(L)}$ must be replaced by

$$\boldsymbol{G}_l = \begin{pmatrix} G_l^{(1,1)} & G_l^{(1,2)} & \cdots & G_l^{(1,p)} \\ G_l^{(2,1)} & G_l^{(2,2)} & \cdots & G_l^{(2,p)} \\ \vdots & \vdots & & \vdots \\ G_l^{(p,1)} & G_l^{(p,2)} & \cdots & G_l^{(p,p)} \end{pmatrix} \tag{22}$$

where each $\boldsymbol{G}_l^{(e,f)}$ is a $k' \times 1$ matrix. Thus $G^{(L)}$ is a $pk'L \times p(L+m)$ matrix with entries in GF(M) where $M = 2^{n'}$. Then (14) and (16) must be replaced by

$$2^{pk'L}(2^{pk'L}-1)d_L \leq \sum_{j=0}^{p(m+L)-1} W_L(j) \tag{23}$$

$$W_L(j) \leq \begin{cases} \left.\begin{array}{ll} 2^{2kL} - 2^{k(2L-b)}, & \text{if } b \leq \frac{n}{k} \\ 2^{2kL} - 2^{2kL-n}, & \text{if } b > \frac{n}{k} \end{array}\right\}, & \text{for } 0 \leq j \leq m-1 \\ \left.\begin{array}{ll} 2^{2kL} - 2^{k(2L+m-b)-\nu}, & \text{if } b \leq p_1 \\ 2^{2kL} - 2^{2kL-n}, & \text{if } b > p_1 \end{array}\right\}, & \text{for } m \leq j \leq m+L-1. \end{cases} \tag{21}$$

TABLE X
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY-TO-4-ARY CONVOLUTIONAL CODES OF RATE $k/n = 1/2$

m	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$, $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
2	$(5\ \ 7)_8^*$	3	$\frac{1}{1}$	$\frac{4}{2}$	$\frac{12}{4}$	$\frac{32}{8}$	$\frac{80}{16}$	$\frac{192}{32}$	$\frac{448}{64}$
3	$(13\ \ 16)_8^*$	4	$\frac{1}{1}$	$\frac{14}{5}$	$\frac{21}{5}$	$\frac{94}{19}$	$\frac{261}{45}$	$\frac{818}{118}$	$\frac{2173}{270}$
4	$(26\ \ 37)_8^{**}$	5	$\frac{3}{2}$	$\frac{7}{3}$	$\frac{39}{11}$	$\frac{131}{28}$	$\frac{428}{76}$	$\frac{1413}{214}$	$\frac{4559}{598}$
5	$(56\ \ 75)_8^{**}$	6	$\frac{3}{2}$	$\frac{26}{8}$	$\frac{43}{10}$	$\frac{227}{44}$	$\frac{783}{129}$	$\frac{2442}{341}$	$\frac{7970}{997}$
6	$(133\ \ 176)_8^*$	7	$\frac{7}{4}$	$\frac{39}{10}$	$\frac{104}{19}$	$\frac{352}{62}$	$\frac{1348}{195}$	$\frac{4540}{574}$	$\frac{14862}{1665}$
7	$(265\ \ 373)_8$	8	$\frac{17}{6}$	$\frac{54}{13}$	$\frac{198}{34}$	$\frac{644}{97}$	$\frac{2148}{281}$	$\frac{7425}{874}$	$\frac{24028}{2534}$
8	$(437\ \ 651)_8$	8	$\frac{1}{1}$	$\frac{25}{7}$	$\frac{101}{21}$	$\frac{365}{59}$	$\frac{1020}{143}$	$\frac{3463}{427}$	$\frac{12569}{1385}$
9	$(732\ \ 1451)_8^{***}$	9	$\frac{1}{1}$	$\frac{56}{14}$	$\frac{107}{23}$	$\frac{475}{74}$	$\frac{1473}{197}$	$\frac{4992}{616}$	$\frac{17018}{1844}$
10	$(2655\ \ 3477)_8$	10	$\frac{5}{3}$	$\frac{85}{20}$	$\frac{303}{50}$	$\frac{911}{126}$	$\frac{2712}{337}$	$\frac{9845}{1113}$	$\frac{31314}{3150}$
11	$(5512\ \ 7671)_8$	11	$\frac{23}{8}$	$\frac{104}{20}$	$\frac{357}{57}$	$\frac{1185}{166}$	$\frac{3868}{466}$	$\frac{12803}{1376}$	$\frac{42910}{4224}$
12	$(13716\ \ 17243)_8$	12	$\frac{48}{12}$	$\frac{217}{37}$	$\frac{561}{80}$	$\frac{1878}{239}$	$\frac{6413}{707}$	$\frac{20302}{2046}$	$\frac{71040}{6485}$
13	$(24717\ \ 33226)_8$	12	$\frac{4}{1}$	$\frac{128}{25}$	$\frac{250}{40}$	$\frac{842}{119}$	$\frac{3200}{371}$	$\frac{10077}{1080}$	$\frac{33133}{3192}$

* This code and its information-weight and distance spectra were found by Trumpis [2].

** This code and its information-weight and distance spectra were found by Ryan [5].

*** This code was found by Rajpal *et al.* [15].

TABLE XI
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY-TO-4-ARY CONVOLUTIONAL CODES OF RATE $k/n = 2/4$

m	ν	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$, $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	2	$\begin{pmatrix} 0 & 3 & 1 & 2 \\ 3 & 1 & 2 & 1 \end{pmatrix}_8$	4	$\frac{4}{3}$	$\frac{16}{6}$	$\frac{56}{15}$	$\frac{176}{36}$	$\frac{524}{87}$	$\frac{1504}{210}$	$\frac{4208}{507}$
2	3	$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 7 & 1 & 5 & 0 \end{pmatrix}_8$	4	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{37}{11}$	$\frac{103}{24}$	$\frac{337}{64}$	$\frac{1125}{176}$	$\frac{3671}{501}$
	4	$\begin{pmatrix} 0 & 7 & 2 & 7 \\ 7 & 0 & 5 & 2 \end{pmatrix}_8$	6	$\frac{12}{6}$	$\frac{40}{12}$	$\frac{144}{33}$	$\frac{488}{90}$	$\frac{1688}{270}$	$\frac{5568}{768}$	$\frac{18064}{2196}$
3	5	$\begin{pmatrix} 03 & 06 & 03 & 04 \\ 15 & 05 & 10 & 07 \end{pmatrix}_8$	6	$\frac{1}{1}$	$\frac{10}{5}$	$\frac{155}{33}$	$\frac{270}{51}$	$\frac{945}{136}$	$\frac{3390}{447}$	$\frac{12383}{1439}$
	6	$\begin{pmatrix} 01 & 17 & 05 & 16 \\ 16 & 01 & 13 & 05 \end{pmatrix}_8$	8	$\frac{52}{18}$	$\frac{56}{12}$	$\frac{504}{90}$	$\frac{1472}{216}$	$\frac{4428}{576}$	$\frac{16448}{1962}$	$\frac{51168}{5292}$
4	7	$\begin{pmatrix} 05 & 16 & 04 & 13 \\ 31 & 03 & 23 & 07 \end{pmatrix}_8$	8	$\frac{2}{2}$	$\frac{85}{20}$	$\frac{251}{52}$	$\frac{634}{99}$	$\frac{2516}{341}$	$\frac{8832}{1001}$	$\frac{29448}{3157}$
	8	$\begin{pmatrix} 07 & 32 & 15 & 34 \\ 35 & 15 & 21 & 31 \end{pmatrix}_8$	9	$\frac{16}{6}$	$\frac{52}{15}$	$\frac{404}{75}$	$\frac{1044}{153}$	$\frac{3552}{459}$	$\frac{11904}{1374}$	$\frac{38252}{4047}$

and

$$d_L = \left\lfloor \frac{\sum_{j=0}^{p(m+L)-1} W_L(j)}{2^{pk'L} \cdot (2^{pk'L} - 1)} \right\rfloor \tag{24}$$

respectively. We have Theorem 2.

Theorem 2: For the M-ary code $C^{(L)}$ $(M = 2^{n'})$, suppose that the jth column $(0 \le j \le p(m+L)-1)$ of the generator matrix in (12) contains $b\boldsymbol{G}_l^{(e,f)}$ nonzero submatrices and $L-b$ all-zero submatrices $\mathbf{0}$'s where each submatrix contains k' symbols in GF(M). Let t be the number of zero symbols in the $b\boldsymbol{G}_l^{(e,f)}$ submatrices. Then, the value of $W_L(j)$ for the corresponding codeword matrix $\boldsymbol{A}$ is upper-bounded by

$$W_L(j) \le \begin{cases} 2^{2pk'L} - 2^{2pk'L - pk'b + t}, & \text{for } bpk' - t \le n' \\ 2^{2pk'L} - 2^{2pk'L - n'}, & \text{for } bpk' - t > n'. \end{cases} \tag{25}$$

□

TABLE XII
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY-TO-4-ARY CONVOLUTIONAL CODES OF RATE $k/n = 3/6$

m	ν	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$ $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	2	$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 3 & 0 & 3 \\ 3 & 1 & 3 & 0 & 2 & 0 \end{pmatrix}_8$	3	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{43}{15}$	$\frac{156}{33}$	$\frac{542}{88}$	$\frac{1928}{261}$	$\frac{6475}{737}$
1	3	$\begin{pmatrix} 2 & 0 & 2 & 3 & 0 & 3 \\ 1 & 2 & 0 & 2 & 1 & 2 \\ 3 & 1 & 3 & 1 & 2 & 0 \end{pmatrix}_8$	5	$\frac{10}{6}$	$\frac{40}{13}$	$\frac{132}{32}$	$\frac{478}{93}$	$\frac{1490}{244}$	$\frac{4852}{683}$	$\frac{15214}{1868}$
2	4	$\begin{pmatrix} 3 & 2 & 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 & 1 & 3 \\ 5 & 7 & 5 & 5 & 2 & 2 \end{pmatrix}_8$	6	$\frac{21}{10}$	$\frac{73}{19}$	$\frac{205}{45}$	$\frac{821}{145}$	$\frac{2725}{414}$	$\frac{8939}{1186}$	$\frac{28890}{3382}$
2	5	$\begin{pmatrix} 3 & 0 & 3 & 2 & 1 & 3 \\ 1 & 5 & 3 & 7 & 3 & 6 \\ 4 & 2 & 6 & 3 & 7 & 0 \end{pmatrix}_8^+$	6	$\frac{1}{1}$	$\frac{50}{14}$	$\frac{166}{34}$	$\frac{455}{76}$	$\frac{1573}{230}$	$\frac{5462}{694}$	$\frac{17794}{2027}$
2	6	$\begin{pmatrix} 5 & 1 & 6 & 7 & 7 & 6 \\ 0 & 6 & 1 & 6 & 2 & 5 \\ 7 & 2 & 1 & 0 & 4 & 2 \end{pmatrix}_8^+$	7	$\frac{4}{2}$	$\frac{95}{25}$	$\frac{269}{47}$	$\frac{781}{121}$	$\frac{2673}{365}$	$\frac{9400}{1121}$	$\frac{31492}{3348}$
3	7	$\begin{pmatrix} 04 & 03 & 05 & 06 & 07 & 07 \\ 03 & 04 & 03 & 07 & 02 & 06 \\ 15 & 01 & 05 & 03 & 15 & 06 \end{pmatrix}_8^+$	8	$\frac{3}{2}$	$\frac{128}{34}$	$\frac{350}{65}$	$\frac{1125}{159}$	$\frac{3484}{463}$	$\frac{12306}{1477}$	$\frac{41872}{4481}$
3	8	$\begin{pmatrix} 07 & 02 & 07 & 06 & 00 & 07 \\ 04 & 17 & 07 & 13 & 04 & 17 \\ 13 & 01 & 17 & 07 & 13 & 02 \end{pmatrix}_8^+$	9	$\frac{28}{11}$	$\frac{181}{36}$	$\frac{592}{94}$	$\frac{1958}{267}$	$\frac{5936}{730}$	$\frac{20740}{2284}$	$\frac{67344}{6679}$

$^+$ This code is the currently best.

The total number of $G_l^{(e,f)}$ matrices in the jth column is shown in (26) at the bottom of this page. The value of t for the jth column is

$$t \geq \begin{cases} 0, & \text{for } 0 \leq j \leq mp-1 \\ k \cdot m - \nu, & \text{for } j \geq mp. \end{cases} \tag{27}$$

The value of $W_L(j)$ is bounded by (28) (see the second expression at the bottom of this page), where $b = p \cdot \min(L, \lfloor j/p \rfloor + 1)$ for $0 \leq j \leq mp-1$, $b = p \cdot \min(m+L-\lfloor j/P \rfloor, m+1)$ for $mp \leq j \leq (m+L)p-1$, and $p_1 = \lfloor (km+n'-\nu)/k \rfloor$. With (13), (24), and (28), an upper bound on the free distance of the M-ary $(M = 2^{n'})$ code can be obtained.

A list of upper bounds for the M-ary $(M = 2^n)$ codes with k/n ratios of $1/2$, $1/3$, $2/3, \cdots, 4/5$ and memory order m from 1 up to 18 is shown in Table I. Note that codes with rate k/n in this table can be q-ary-to-M-ary codes with $q = 2$ and $M = 2^n$ (i.e., binary-to-M-ary code), or q-ary-to-M-ary codes with $q = 2^k$ and $M = 2^n$. In addition, we only list those codes with $\nu = km$ in the table. Upper bounds in the shaded area of this table are achieved by codes obtained by the computer search of Section IV. We may compare the bound derived in this section to the one derived in [2]. In Table I, the "+" mark is used to indicate that our prediction is tighter than that in [2] for the associated code.

A list of upper bounds for the M-ary codes $(M = 2^n$ or $M = 2^{n'})$ with k/n ratios $1/2, 1/3, 2/3, \cdots, 3/4, 3/6$ and numbers of memory elements ν varying from 1 to 18 is shown in Table II.

IV. COMPUTER-AIDED SEARCH FOR GOOD CODES

An exhausted search which is used to find the best codes is depicted by the flowchart shown in Fig. 3. The program starts when all the

$$b = \begin{cases} p \cdot \min\left(L, \left\lfloor \dfrac{j}{p} \right\rfloor + 1\right), & \text{for } 0 \leq j \leq mp-1 \\ p \cdot \min\left(m + L - \left\lfloor \dfrac{j}{p} \right\rfloor, m+1\right), & \text{for } mp \leq j \leq (m+L)p-1. \end{cases} \tag{26}$$

$$W_L(j) \leq \begin{cases} \left.\begin{array}{ll} 2^{2kL} - 2^{k(2L-b)}, & \text{if } b \leq \dfrac{n}{k} = \dfrac{n'}{k'} \\ 2^{2kL} - 2^{2kL-n'}, & \text{if } b > \dfrac{n}{k} = \dfrac{n'}{k'} \end{array}\right\}, & \text{for } 0 \leq j \leq mp-1 \\ \left.\begin{array}{ll} 2^{2kL} - 2^{k(2L+m-b)-\nu}, & \text{if } b \leq p_1 \\ 2^{2kL} - 2^{2kL-n'}, & \text{if } b > p_1 \end{array}\right\}, & \text{for } mp \leq j \leq (m+L)p-1 \end{cases} \tag{28}$$

TABLE XIII
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY-TO-4-ARY CONVOLUTIONAL CODES OF RATE $k/n = 4/6$

m	ν	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$, $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	2	$\begin{pmatrix} 2&1&2&2&3&3 \\ 0&2&1&2&1&3 \\ 1&0&1&0&0&0 \\ 1&0&0&0&0&1 \end{pmatrix}_8^+$	3	$\frac{4}{3}$	$\frac{84}{27}$	$\frac{532}{99}$	$\frac{2832}{381}$	$\frac{14664}{1602}$	$\frac{72356}{6543}$	$\frac{346156}{26721}$
	3	$\begin{pmatrix} 2&0&3&1&3&0 \\ 1&3&1&2&0&3 \\ 2&0&2&3&0&3 \\ 1&0&0&1&0&0 \end{pmatrix}_8^+$	3	$\frac{1}{1}$	$\frac{23}{9}$	$\frac{284}{63}$	$\frac{1541}{235}$	$\frac{9330}{1142}$	$\frac{52251}{5245}$	$\frac{286576}{24497}$
	4	$\begin{pmatrix} 0&3&1&3&0&2 \\ 3&1&3&0&2&0 \\ 3&1&2&2&1&2 \\ 2&1&0&2&1&1 \end{pmatrix}_8^+$	4	$\frac{7}{4}$	$\frac{101}{29}$	$\frac{721}{126}$	$\frac{4572}{629}$	$\frac{26959}{2969}$	$\frac{157085}{14592}$	$\frac{882664}{70570}$
2	5	$\begin{pmatrix} 1&7&3&7&1&6 \\ 3&1&2&0&2&1 \\ 1&2&2&3&3&0 \\ 2&2&3&0&1&2 \end{pmatrix}_8^+$	5	$\frac{50}{16}$	$\frac{369}{66}$	$\frac{2472}{334}$	$\frac{14549}{1603}$	$\frac{87895}{8038}$	$\frac{513131}{40358}$	$\frac{2900073}{199609}$
	6	$\begin{pmatrix} 0&6&2&5&3&5 \\ 7&0&4&3&6&2 \\ 1&2&3&3&3&0 \\ 2&3&3&1&1&3 \end{pmatrix}_8^+$	5	$\frac{15}{4}$	$\frac{177}{39}$	$\frac{1397}{192}$	$\frac{8430}{908}$	$\frac{47074}{4360}$	$\frac{281348}{22417}$	$\frac{1629775}{113481}$
	7	$\begin{pmatrix} 2&7&0&4&1&6 \\ 6&2&7&0&7&0 \\ 1&7&6&5&5&2 \\ 2&3&2&1&1&3 \end{pmatrix}_8^+$	6	$\frac{56}{14}$	$\frac{689}{103}$	$\frac{4166}{477}$	$\frac{22336}{2122}$	$\frac{137556}{11191}$	$\frac{795149}{56604}$	$\frac{4581937}{288866}$
	8	$\begin{pmatrix} 3&6&3&5&1&6 \\ 4&1&5&3&7&3 \\ 1&6&4&7&4&1 \\ 5&5&4&3&1&1 \end{pmatrix}_8^+$	6	$\frac{20}{5}$	$\frac{431}{64}$	$\frac{2280}{258}$	$\frac{13208}{1254}$	$\frac{80268}{6457}$	$\frac{471993}{33203}$	$\frac{2686348}{167904}$

$^+$ This code is the currently best.

TABLE XIV
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR 4-ARY CONVOLUTIONAL CODES OF RATE $k/n = 2/4$

m	ν	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$, $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	2	$\begin{pmatrix} 0&3&1&2 \\ 3&1&2&1 \end{pmatrix}_8^*$	4	$\frac{3}{3}$	$\frac{12}{6}$	$\frac{42}{15}$	$\frac{132}{36}$	$\frac{393}{87}$	$\frac{1128}{210}$	$\frac{3156}{507}$
2	3	$\begin{pmatrix} 1&2&1&3 \\ 7&1&5&0 \end{pmatrix}_8$	4	$\frac{1}{1}$	$\frac{2}{2}$	$\frac{30}{11}$	$\frac{80}{24}$	$\frac{261}{64}$	$\frac{870}{176}$	$\frac{2836}{501}$
	4	$\begin{pmatrix} 0&7&2&7 \\ 7&0&5&2 \end{pmatrix}_8^*$	6	$\frac{9}{6}$	$\frac{30}{12}$	$\frac{108}{33}$	$\frac{366}{90}$	$\frac{1266}{270}$	$\frac{4176}{768}$	$\frac{13548}{2196}$
3	5	$\begin{pmatrix} 03&06&03&04 \\ 15&05&10&07 \end{pmatrix}_8$	6	$\frac{1}{1}$	$\frac{8}{5}$	$\frac{114}{33}$	$\frac{214}{51}$	$\frac{705}{136}$	$\frac{2583}{447}$	$\frac{9361}{1439}$
	6	$\begin{pmatrix} 01&17&05&16 \\ 16&01&13&05 \end{pmatrix}_8^*$	8	$\frac{39}{18}$	$\frac{42}{12}$	$\frac{378}{90}$	$\frac{1104}{216}$	$\frac{3321}{576}$	$\frac{12336}{1962}$	$\frac{38376}{5292}$
4	7	$\begin{pmatrix} 05&16&04&13 \\ 31&03&23&07 \end{pmatrix}_8^+$	8	$\frac{2}{2}$	$\frac{64}{20}$	$\frac{201}{52}$	$\frac{498}{99}$	$\frac{1934}{341}$	$\frac{6401}{1001}$	$\frac{22495}{3157}$
	8	$\begin{pmatrix} 07&32&15&34 \\ 35&15&21&31 \end{pmatrix}_8^*$	9	$\frac{12}{6}$	$\frac{39}{15}$	$\frac{303}{75}$	$\frac{783}{153}$	$\frac{2664}{459}$	$\frac{8928}{1374}$	$\frac{28689}{4047}$

* This code was found by Ryan-Wilson [9].

$^+$ This code is the currently best.

values of n, k, m, and ν as well as the class of codes are given. The actions used in Fig. 3 are described as follows.

F1) (Initialize the information-weight spectrum.) The initialized information-weight spectrum contains one spectral component at the target free distance d_t that we estimate, which is also declared as the spectrum of the currently best code. The value of $N_I(d_t)$ is assumed to be a large value such as 10^4 or even larger. The target free distance d_t is determined by

TABLE XV
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY-TO-8-ARY CONVOLUTIONAL CODES OF RATE $k/n = 1/3$

m	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$ $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
2	$(3\ 5\ 7)_8^*$	3	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{5}{2}$	$\frac{10}{3}$	$\frac{20}{5}$	$\frac{38}{8}$	$\frac{71}{13}$
3	$(6\ 15\ 17)_8^*$	4	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{7}{3}$	$\frac{16}{5}$	$\frac{41}{11}$	$\frac{94}{21}$	$\frac{219}{43}$
4	$(16\ 31\ 35)_8^{**}$	5	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{11}{4}$	$\frac{20}{6}$	$\frac{52}{13}$	$\frac{156}{34}$	$\frac{344}{64}$
5	$(56\ 67\ 75)_8^{**}$	6	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{13}{5}$	$\frac{37}{9}$	$\frac{58}{14}$	$\frac{219}{44}$	$\frac{500}{87}$
6	$(136\ 155\ 171)_8^{**}$	7	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{20}{7}$	$\frac{34}{8}$	$\frac{96}{22}$	$\frac{212}{41}$	$\frac{669}{118}$
7	$(165\ 322\ 371)_8$	8	$\frac{1}{1}$	$\frac{4}{2}$	$\frac{13}{4}$	$\frac{56}{15}$	$\frac{103}{22}$	$\frac{310}{59}$	$\frac{737}{123}$
8	$(371\ 423\ 645)_8$	9	$\frac{1}{1}$	$\frac{4}{2}$	$\frac{21}{7}$	$\frac{72}{15}$	$\frac{116}{24}$	$\frac{391}{67}$	$\frac{1039}{165}$
9	$(144\ 1351\ 1733)_8^+$	10	$\frac{1}{1}$	$\frac{9}{4}$	$\frac{41}{11}$	$\frac{87}{19}$	$\frac{179}{35}$	$\frac{503}{86}$	$\frac{1391}{219}$
10	$(133\ 2575\ 3351)_8^+$	11	$\frac{1}{1}$	$\frac{19}{7}$	$\frac{52}{12}$	$\frac{78}{16}$	$\frac{291}{54}$	$\frac{651}{107}$	$\frac{1705}{254}$
11	$(1747\ 2426\ 7534)_8^+$	12	$\frac{1}{1}$	$\frac{10}{4}$	$\frac{49}{12}$	$\frac{125}{23}$	$\frac{303}{53}$	$\frac{653}{99}$	$\frac{1710}{244}$

* This code and its information-weight and distance spectra were found by Trumpis [2].

** This code and its information-weight and distance spectra were found by Ryan [5].

+ This code is the currently best.

TABLE XVI
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY-TO-8-ARY CONVOLUTIONAL CODES OF RATE $k/n = 2/3$

m	ν	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$ $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	2	$\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}_8^*$	2	$\frac{6}{4}$	$\frac{46}{15}$	$\frac{260}{56}$	$\frac{1300}{209}$	$\frac{6082}{780}$	$\frac{27290}{2911}$	$\frac{118984}{10864}$
2	3	$\begin{pmatrix} 2 & 3 & 3 \\ 3 & 5 & 7 \end{pmatrix}_8$	2	$\frac{1}{1}$	$\frac{23}{10}$	$\frac{203}{49}$	$\frac{1516}{263}$	$\frac{10365}{1393}$	$\frac{67318}{7393}$	$\frac{422559}{39225}$
	4	$\begin{pmatrix} 4 & 7 & 7 \\ 1 & 5 & 6 \end{pmatrix}_8$	3	$\frac{8}{5}$	$\frac{144}{32}$	$\frac{1092}{172}$	$\frac{9536}{1124}$	$\frac{70139}{6620}$	$\frac{516814}{40780}$	$\frac{3645744}{246692}$
3	5	$\begin{pmatrix} 01 & 02 & 07 \\ 17 & 01 & 11 \end{pmatrix}_8^*$	3	$\frac{1}{1}$	$\frac{55}{17}$	$\frac{569}{100}$	$\frac{4650}{614}$	$\frac{39329}{4135}$	$\frac{307534}{26771}$	$\frac{2349771}{174475}$
	6	$\begin{pmatrix} 01 & 07 & 16 \\ 05 & 14 & 13 \end{pmatrix}_8^*$	4	$\frac{13}{7}$	$\frac{322}{63}$	$\frac{2223}{322}$	$\frac{20478}{2338}$	$\frac{166645}{15656}$	$\frac{1332172}{105778}$	$\frac{10264380}{707950}$
4	7	$\begin{pmatrix} 13 & 16 & 16 \\ 35 & 07 & 31 \end{pmatrix}_8$	4	$\frac{1}{1}$	$\frac{132}{34}$	$\frac{1123}{168}$	$\frac{9618}{1161}$	$\frac{83254}{8106}$	$\frac{679359}{55785}$	$\frac{5402473}{382089}$
	8	$\begin{pmatrix} 36 & 31 & 26 \\ 23 & 17 & 21 \end{pmatrix}_8$	5	$\frac{26}{11}$	$\frac{584}{99}$	$\frac{4128}{535}$	$\frac{37906}{3844}$	$\frac{316458}{26828}$	$\frac{2588068}{186945}$	$\frac{20427388}{1286927}$
5	9	$\begin{pmatrix} 30 & 25 & 17 \\ 67 & 20 & 64 \end{pmatrix}_8^+$	5	$\frac{2}{2}$	$\frac{308}{60}$	$\frac{2374}{307}$	$\frac{19798}{2083}$	$\frac{168654}{14670}$	$\frac{1383862}{102727}$	$\frac{11073738}{715148}$
	10	$\begin{pmatrix} 75 & 66 & 65 \\ 52 & 37 & 60 \end{pmatrix}_8^+$	6	$\frac{72}{22}$	$\frac{1174}{171}$	$\frac{8700}{1006}$	$\frac{73794}{6929}$	$\frac{618126}{49061}$	$\frac{5040837}{344693}$	$\frac{39969021}{2402122}$

* The generator sequences of this code are the same as those of 4-ary-to-8-ary codes of rate $k/n = 2/3$ convolutional codes with same m and ν.

+ This code is the currently best.

n, k, m, ν, and the class of codes. For binary codes, the values of d_t can be found from [13, Table 11.1] or by using [11, eqs. (4)–(7)]. For nonbinary codes, the values of d_t can be determined from the upper bound derived in Section III.

F2) (Choose a code to be tested.) In choosing codes to be tested, we have to avoid testing equivalent codes. Two codes are equivalent if they satisfy one of the following conditions.

a) This condition is valid for binary codes and M-ary codes with $M = 2^n$. The transfer function matrix of one code is obtained from interchanging two columns in the transfer function matrix of the other code.

b) For $1 \le i \le k$ and $1 \le j \le n$, each $G_i^{(j)}(D)$ of one code is the reciprocal polynomial of the $G_i^{(j)}(D)$ of the other code.

TABLE XVII
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY-TO-8-ARY CONVOLUTIONAL CODES OF RATE $k/n = 4/6$

m	ν	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$ $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	2	$\begin{pmatrix} 2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}_8^+$	2	$\frac{4}{3}$	$\frac{84}{24}$	$\frac{484}{104}$	$\frac{3484}{536}$	$\frac{21292}{2696}$	$\frac{128128}{13440}$	$\frac{748064}{67664}$
1	3	$\begin{pmatrix} 1 & 0 & 3 & 2 & 3 & 2 \\ 1 & 1 & 0 & 2 & 2 & 3 \\ 0 & 2 & 2 & 3 & 1 & 3 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}_8^+$	2	$\frac{1}{1}$	$\frac{16}{8}$	$\frac{324}{80}$	$\frac{2128}{368}$	$\frac{18296}{2452}$	$\frac{129848}{14224}$	$\frac{936584}{86336}$
1	4	$\begin{pmatrix} 2 & 0 & 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 3 \\ 1 & 2 & 3 & 3 & 0 & 2 \end{pmatrix}_8^+$	3	$\frac{8}{4}$	$\frac{134}{37}$	$\frac{1739}{262}$	$\frac{13767}{1563}$	$\frac{118451}{10795}$	$\frac{939463}{70689}$	$\frac{7282554}{469331}$
2	5	$\begin{pmatrix} 1 & 7 & 7 & 0 & 0 & 4 \\ 1 & 2 & 3 & 2 & 3 & 3 \\ 0 & 3 & 1 & 3 & 0 & 2 \\ 3 & 1 & 1 & 3 & 1 & 0 \end{pmatrix}_8^+$	4	$\frac{54}{20}$	$\frac{683}{126}$	$\frac{5652}{761}$	$\frac{49593}{5328}$	$\frac{401643}{35530}$	$\frac{3168509}{238362}$	$\frac{24483529}{1602975}$
2	6	$\begin{pmatrix} 6 & 4 & 1 & 3 & 6 & 1 \\ 1 & 2 & 2 & 5 & 2 & 3 \\ 0 & 1 & 3 & 0 & 1 & 2 \\ 1 & 2 & 3 & 2 & 1 & 3 \end{pmatrix}_8^+$	4	$\frac{9}{5}$	$\frac{410}{79}$	$\frac{2963}{412}$	$\frac{25229}{2806}$	$\frac{214928}{19710}$	$\frac{1722166}{134248}$	$\frac{13502861}{915859}$
2	7	$\begin{pmatrix} 2 & 5 & 2 & 0 & 6 & 0 \\ 5 & 4 & 5 & 3 & 5 & 3 \\ 5 & 1 & 1 & 1 & 2 & 5 \\ 1 & 3 & 3 & 2 & 0 & 3 \end{pmatrix}_8^+$	4	$\frac{1}{1}$	$\frac{122}{31}$	$\frac{1577}{235}$	$\frac{12496}{1429}$	$\frac{106815}{9965}$	$\frac{884252}{69575}$	$\frac{7080520}{480952}$
2	8	$\begin{pmatrix} 7 & 4 & 7 & 4 & 3 & 4 \\ 1 & 4 & 1 & 1 & 4 & 6 \\ 6 & 3 & 4 & 0 & 5 & 0 \\ 2 & 5 & 5 & 7 & 2 & 5 \end{pmatrix}_8^+$	5	$\frac{28}{10}$	$\frac{972}{140}$	$\frac{6492}{754}$	$\frac{56481}{5223}$	$\frac{468958}{36944}$	$\frac{3772668}{256762}$	$\frac{29843053}{1788220}$

$^+$ This code is the currently best.

TABLE XVIII
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY-TO-16-ARY CONVOLUTIONAL CODES OF RATE $k/n = 1/4$

m	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$ $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
2	$(1\ 2\ 3\ 4)_8$	3	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{5}{2}$	$\frac{10}{3}$	$\frac{20}{5}$	$\frac{38}{8}$	$\frac{71}{13}$
3	$(1\ 2\ 4\ 13)_8$	4	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{5}{2}$	$\frac{12}{4}$	$\frac{26}{7}$	$\frac{56}{13}$	$\frac{118}{24}$
4	$(12\ 14\ 35\ 37)_8$	5	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{5}{2}$	$\frac{14}{5}$	$\frac{32}{9}$	$\frac{74}{18}$	$\frac{171}{37}$
5	$(16\ 27\ 54\ 75)_8$	6	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{5}{2}$	$\frac{17}{6}$	$\frac{35}{9}$	$\frac{87}{21}$	$\frac{219}{46}$
6	$(25\ 53\ 104\ 162)_8$	7	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{5}{2}$	$\frac{20}{7}$	$\frac{36}{9}$	$\frac{95}{23}$	$\frac{247}{53}$
7	$(1\ 57\ 146\ 373)_8^+$	8	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{17}{6}$	$\frac{32}{9}$	$\frac{41}{9}$	$\frac{156}{35}$	$\frac{337}{66}$
8	$(1\ 107\ 255\ 577)_8^+$	9	$\frac{1}{1}$	$\frac{4}{2}$	$\frac{20}{6}$	$\frac{30}{8}$	$\frac{63}{16}$	$\frac{175}{36}$	$\frac{451}{85}$
9	$(1\ 64\ 1311\ 1647)_8^+$	10	$\frac{1}{1}$	$\frac{12}{5}$	$\frac{28}{7}$	$\frac{30}{8}$	$\frac{106}{23}$	$\frac{291}{57}$	$\frac{554}{101}$

$^+$ This code is the currently best.

c) This condition of equivalent codes is valid for M-ary convolutional codes only when $M > 2$. i) Suppose $M = 2^n$. Two codes are equivalent if the transfer function matrix of one is obtained through a certain column operation on the other code. The equivalence results from the fact that any nonzero output of M-ary symbol in vector form of $(a_1, a_2, \cdots, a_n)$ with $a_i \in \{0, 1\}$ is never turned into an all-zero vector by any addition operation on $a_1, a_2, \cdots, a_n$. It should be noted that this equivalent relationship is invalid for binary codes

TABLE XIX

BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR BINARY-TO-16-ARY CONVOLUTIONAL CODES OF RATE $k/n = 3/4$

m	ν	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$ $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
1	3	$\begin{pmatrix} 0 & 1 & 1 & 3 \\ 2 & 2 & 2 & 3 \\ 1 & 2 & 3 & 3 \end{pmatrix}_8^*$	2	$\frac{23}{11}$	$\frac{492}{116}$	$\frac{7814}{1222}$	$\frac{110032}{12873}$	$\frac{1451093}{135609}$	$\frac{18362128}{1428556}$
2	4	$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 5 & 1 & 5 & 5 \end{pmatrix}_8$	2	$\frac{5}{4}$	$\frac{263}{64}$	$\frac{5197}{784}$	$\frac{91015}{9927}$	$\frac{1465436}{125180}$	$\frac{22497500}{1579343}$
	5	$\begin{pmatrix} 0 & 2 & 3 & 3 \\ 3 & 7 & 2 & 5 \\ 4 & 3 & 4 & 7 \end{pmatrix}_8$	2	$\frac{1}{1}$	$\frac{111}{34}$	$\frac{2546}{435}$	$\frac{49250}{5987}$	$\frac{871793}{81969}$	$\frac{14639322}{1122457}$
	6	$\begin{pmatrix} 3 & 4 & 4 & 5 \\ 7 & 0 & 0 & 6 \\ 1 & 1 & 2 & 7 \end{pmatrix}_8$	3	$\frac{38}{16}$	$\frac{1496}{239}$	$\frac{29318}{3251}$	$\frac{560025}{46956}$	$\frac{9958653}{669725}$	$\frac{170034175}{9559053}$
3	7	$\begin{pmatrix} 06 & 07 & 07 & 07 \\ 06 & 03 & 04 & 05 \\ 00 & 01 & 01 & 14 \end{pmatrix}_8^+$	3	$\frac{7}{5}$	$\frac{765}{134}$	$\frac{14287}{1651}$	$\frac{289024}{24936}$	$\frac{5276598}{362907}$	$\frac{92677421}{5296454}$
	8	$\begin{pmatrix} 06 & 07 & 07 & 07 \\ 15 & 00 & 11 & 13 \\ 10 & 15 & 02 & 10 \end{pmatrix}_8^+$	3	$\frac{3}{2}$	$\frac{252}{54}$	$\frac{8011}{925}$	$\frac{143828}{12585}$	$\frac{2716386}{188364}$	$\frac{48496433}{2787245}$
	9	$\begin{pmatrix} 07 & 10 & 11 & 14 \\ 15 & 04 & 06 & 10 \\ 00 & 11 & 02 & 16 \end{pmatrix}_8^+$	4	$\frac{91}{28}$	$\frac{3371}{446}$	$\frac{64050}{6166}$	$\frac{1243923}{93178}$	$\frac{22648788}{1387346}$	$\frac{398226588}{20639130}$

* The generator sequences for this code are the same as those for 8-ary-to-16-ary codes of rate $k/n = 3/4$.

$^+$ This code is the currently best.

TABLE XX

BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR 4-ARY-TO-7-ARY CONVOLUTIONAL CODES OF RATE $k/n = 2/3$

m	ν	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$ $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
1	2	$\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}_8^*$	2	$\frac{5}{4}$	$\frac{38}{15}$	$\frac{214}{56}$	$\frac{1068}{209}$	$\frac{4991}{780}$	$\frac{22378}{2911}$	$\frac{97516}{10864}$
2	3	$\begin{pmatrix} 2 & 3 & 3 \\ 7 & 1 & 6 \end{pmatrix}_8$	2	$\frac{1}{1}$	$\frac{20}{10}$	$\frac{159}{49}$	$\frac{1162}{263}$	$\frac{7812}{1393}$	$\frac{50231}{7393}$	$\frac{313070}{39225}$
	4	$\begin{pmatrix} 2 & 5 & 7 \\ 7 & 1 & 4 \end{pmatrix}_8$	3	$\frac{7}{5}$	$\frac{96}{31}$	$\frac{740}{172}$	$\frac{6376}{1114}$	$\frac{46202}{6564}$	$\frac{342050}{40671}$	$\frac{2394142}{245572}$
3	5	$\begin{pmatrix} 01 & 02 & 07 \\ 17 & 01 & 11 \end{pmatrix}_8^*$	3	$\frac{1}{1}$	$\frac{41}{17}$	$\frac{429}{100}$	$\frac{3546}{614}$	$\frac{29885}{4135}$	$\frac{233958}{26771}$	$\frac{1788552}{174475}$
	6	$\begin{pmatrix} 01 & 07 & 16 \\ 05 & 14 & 13 \end{pmatrix}_8^*$	4	$\frac{11}{7}$	$\frac{247}{63}$	$\frac{1676}{322}$	$\frac{15628}{2338}$	$\frac{126035}{15656}$	$\frac{1007828}{105778}$	$\frac{7752954}{707950}$
4	7	$\begin{pmatrix} 16 & 13 & 13 \\ 31 & 35 & 03 \end{pmatrix}_8^+$	4	$\frac{1}{1}$	$\frac{103}{34}$	$\frac{869}{168}$	$\frac{7387}{1161}$	$\frac{63798}{8106}$	$\frac{519940}{55785}$	$\frac{4126946}{382089}$
	8	$\begin{pmatrix} 36 & 31 & 26 \\ 23 & 17 & 21 \end{pmatrix}_8$	5	$\frac{20}{11}$	$\frac{435}{99}$	$\frac{3084}{535}$	$\frac{28420}{3844}$	$\frac{236828}{26828}$	$\frac{1938094}{186945}$	$\frac{15290127}{1286927}$
5	9	$\begin{pmatrix} 30 & 25 & 17 \\ 67 & 20 & 64 \end{pmatrix}_8^+$	5	$\frac{2}{2}$	$\frac{233}{60}$	$\frac{1796}{307}$	$\frac{15002}{2083}$	$\frac{127480}{14670}$	$\frac{1044846}{102727}$	$\frac{8354956}{715148}$
	10	$\begin{pmatrix} 64 & 60 & 57 \\ 53 & 75 & 25 \end{pmatrix}_8^+$	6	$\frac{60}{22}$	$\frac{922}{171}$	$\frac{6786}{1006}$	$\frac{57161}{6929}$	$\frac{476581}{49061}$	$\frac{3869671}{344693}$	$\frac{30604590}{2402122}$

* The generator sequences of this code are the same as those of the codes of binary-to-8-ary convolutional codes of rate $k/n = 2/3$ with same m and ν.

$^+$ This code is the currently best.

because the distances for these codes are counted by bits, not by M-ary symbols. ii) Suppose that $M = 2^{n'}$ with $k/n = pk'/pn'$. Since p M-ary symbols are produced for each time unit, the column operation is restricted within those columns that are used to generate the same M-ary symbol.

TABLE XXI
BEST CODES AND THEIR RELATED INFORMATION—WEIGHT AND DISTANCE SPECTRA FOR 8-ARY-TO-16-ARY CONVOLUTIONAL CODES OF RATE $k/n = 3/4$

m	ν	Generator sequences	d_∞	$\frac{N_I(d_\infty+i)}{N(d_\infty+i)}$, $i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
1	3	$\begin{pmatrix} 0 & 1 & 1 & 3 \\ 2 & 2 & 2 & 3 \\ 1 & 2 & 3 & 3 \end{pmatrix}_8^{*,**}$	2	$\frac{16}{11}$	$\frac{340}{116}$	$\frac{5387}{1222}$	$\frac{75766}{12873}$	$\frac{998483}{135609}$	$\frac{12628804}{1428556}$
2	4	$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 3 & 3 & 2 \\ 0 & 2 & 7 & 5 \end{pmatrix}_8$	2	$\frac{4}{4}$	$\frac{171}{64}$	$\frac{3288}{784}$	$\frac{56758}{9927}$	$\frac{906611}{125180}$	$\frac{13846154}{1579343}$
	5	$\begin{pmatrix} 0 & 2 & 3 & 3 \\ 0 & 7 & 1 & 7 \\ 6 & 7 & 7 & 6 \end{pmatrix}_8$	2	$\frac{1}{1}$	$\frac{77}{34}$	$\frac{1829}{442}$	$\frac{34007}{6004}$	$\frac{603219}{82604}$	$\frac{10064473}{1128979}$
	6	$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & 6 & 6 & 5 \\ 7 & 5 & 6 & 7 \end{pmatrix}_8^{+}$	3	$\frac{28}{16}$	$\frac{931}{239}$	$\frac{17833}{3251}$	$\frac{336691}{46956}$	$\frac{5933724}{669725}$	$\frac{100763933}{9559053}$
3	7	$\begin{pmatrix} 06 & 07 & 07 & 07 \\ 06 & 02 & 04 & 05 \\ 00 & 11 & 02 & 16 \end{pmatrix}_8^{+}$	3	$\frac{6}{5}$	$\frac{479}{134}$	$\frac{8769}{1651}$	$\frac{175470}{24936}$	$\frac{3184796}{362907}$	$\frac{55691359}{5296454}$
	8	$\begin{pmatrix} 06 & 07 & 07 & 07 \\ 15 & 04 & 10 & 16 \\ 12 & 05 & 02 & 14 \end{pmatrix}_8$	3	$\frac{3}{2}$	$\frac{170}{56}$	$\frac{4334}{861}$	$\frac{80828}{12159}$	$\frac{1516820}{181979}$	$\frac{26917444}{2687740}$
	9	$\begin{pmatrix} 07 & 10 & 12 & 14 \\ 15 & 04 & 07 & 10 \\ 00 & 11 & 05 & 16 \end{pmatrix}_8^{+}$	4	$\frac{65}{28}$	$\frac{2124}{446}$	$\frac{39540}{6166}$	$\frac{759572}{93178}$	$\frac{13732390}{1387346}$	$\frac{240236986}{20639130}$

* The generator sequences for this code are the same as those for binary-to-16-ary codes of rate $k/n = 3/4$.

** This code was found by Rajpal *et al.* [15].

+ This code is the currently best.

F3) (Find d for all neighbors of path length $\leq m+3$ branches.) Terminating the semi-infinite generator matrix shown in (6) to result in a generator matrix in the form of (12). Here, we use $L = 3$. Then we can find the minimum distance d of the associated block code.

F4) (Is $d \geq d_t$?) If the minimum distance d is less than d_t, then we give up the code under test and go to **F9**. Otherwise, go to **F5**. Actions **F3** and **F4** are used to reject any convolutional code whose free distance d_∞ is apparently less than d_t.

F5) (Is it a catastrophic code?) If the code is catastrophic, then we give up the code under test and go to **F9**. Otherwise, go to **F6**. Catastrophic code can be found by the catastrophic code test criterion shown in [13].

F6) (Find the truncated information-weight spectrum.) We use the modified FAST to analyze the spectrum of the code under test. The distance spectrum in truncated form is simultaneously found in this action. The accuracy of the modified FAST algorithm has been verified by comparing the outputs produced by the modified FAST program with the known results given in [10] and [16], [17].

F7) (Is the code superior to the currently best?) If the information-weight spectrum is superior the currently best, then go to **F8**. Otherwise, give up this code and go to **F9**. The criterion to justify the superiority of one code over the other is as follows. If d_∞ of one code is larger than that of the other, then the former code is the superior. If two codes have the same d_∞ and $N_{\mathrm{I}}(d_\infty + i)$ for $i = 0, \cdots, l-1$, then the code with smaller $N_{\mathrm{I}}(d_\infty + l)$ is the superior.

F8) (Update results.) Declare the information-weight spectrum of the code under test to be the currently best.

F9) (Is the search completed?) If the code search is exhausted, then go to **F10**. Otherwise, go to **F2** to choose another code as an object to be tested.

F10) (Is the best code found?) If the best code is found, then terminate the search program, otherwise go to **F11**. The "otherwise" case means that there is no code with free distance larger than or equal to d_t.

F11) (Decrease d_t.) Decrease the value of d_t by one, reset the information weight of d_t to be a large value again, and then go to **F2** to restart the code search.

By the exhausted search, the best codes or the currently best codes (due to an incomplete search) for the two classes of convolutional codes are found and are listed in Tables III–XXI. Each of codes in these tables are specified by the transfer function matrices containing $k \times n$ generator sequences expressed in octal form. Only the first few components are tabulated for each of the codes. These spectral components are expressed in a sequence of fractions. For a fraction, the value at the numerator position is the associated information weight and that at the denominator position is the number of the associated neighbors. For some cases, the time required for the exhausted search of some types of codes is too long to identify the best codes, we are only able to list codes which are the currently best from our incompleted search as results. Most of the codes previously found are exactly the best codes. For completeness, these previously found codes are also included in our tables.

Lists of the best (or currently best) binary convolutional codes for rates $1/2, 1/3, 1/4, 2/3, 2/4, 3/4,$ and $4/6$ are given in Tables III–IX. The results of binary-to-M-ary convolutional codes are listed in Tables X–XIII and Tables XV–IXX. The results for q-ary-to-M-ary convolutional codes including the 4-ary-to-8-ary codes of rate

$k/n = 2/3$ and 8-ary-to-16-ary codes of rate $k/n = 3/4$ are listed in Tables XX and XI, respectively. Note that the binary-to-16-ary codes of rate $k/n = 1/4$ found by computer search in Table XVIII are better than the convolutional codes obtained from Reed–Solomon codes by Ryan and Wilson [5].

Binary codes with rate $k/n = 2/4$ and $4/6$ have not been searched as far as we know. We use computer search to find the information-weight spectra of the $2/4$ and $4/6$ binary convolutional codes with $\nu = 2, 3, \cdots, 8$. Results in Table VII show that in some cases, e.g., $\nu = 4$ and 7, the $2/4$ binary convolutional code has a better free distance profile than the comparable $1/2$ binary convolutional code. Results in Table IX show that for $\nu = 4$, the $4/6$ binary convolutional code has a better free distance profile than the comparable $2/3$ binary convolutional code.

The q-ary convolutional code have been treated by Ryan and Wilson [5]. In [5], not only the input and output symbols are from GF(q) but the size of the memory unit is also the size of a q-ary symbol. A rate $1/2$ 4-ary code with memory order m is equivalent to a 4-ary convolutional code of rate $k/n = 2/4$ and $\nu = 2m$. Ryan and Wilson [5] have found $1/2$ 4-ary codes with $m = 2, 3, 4$, which are equivalent to 4-ary codes with $k/n = 2/4$ and $\nu = 4, 6, 8$. We then search for 4-ary codes of rate $k/n = 2/4$ with $\nu = 3, 5, 7$. A complete table is shown in Table XIV.

If we treat the input 4-ary symbol as two binary bits, the rate $1/2$ 4-ary code (i.e., 4-ary-to-4-ary of rate $k/n = 2/4$) is then converted to a binary-to-4-ary code of rate $k/n = 2/4$. A list of the best (or currently best) binary-to-4-ary codes of rate $k/n = 2/4$ is given in Table II. We find that the best binary-to-4-ary code of rate $k/n = 2/4$ is better than the best binary-to-4-ary code of rate $k/n = 1/2$ for $\nu = 2, 3, \cdots, 8$. We have also searched the binary-to-4-ary codes of rate $k/n = 3/6$. We only find a best binary-to-4-ary code of rate $k/n = 3/6$ is better than the best binary-to-4-ary code of rate $k/n = 2/4$ for the special case of $\nu = 3$. The results are shown in Table XII.

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Sequential Prediction and Ranking in Universal Context Modeling and Data Compression

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***Abstract*—Most state-of-the-art lossless image compression schemes use prediction followed by some form of context modeling. This might seem redundant at first, as the contextual information used for prediction is also available for building the compression model, and a universal coder will eventually learn the “predictive” patterns of the data. In this correspondence, we provide a formal justification to the combination of these two modeling tools, by showing that a combined scheme may result in faster convergence rate to the source entropy. This is achieved via a reduction in the model cost of universal coding. In deriving the main result, we develop the concept of sequential ranking, which can be seen as a generalization of sequential prediction, and we study its combinatorial and probabilistic properties.**

***Index Terms*—Context algorithm, image compression, prediction, ranking, universal coding.**

I. INTRODUCTION

Prediction is one of the oldest and most successful tools in the data compression practitioner’s toolbox. It is particularly useful in situations where the data (e.g., a digital image) originates from a natural physical process (e.g., sensed light), and the data samples (e.g., real numbers) represent a continuously varying physical magnitude (e.g., brightness). In these cases, the value of the next sample can often be accurately predicted using a simple function (e.g., a linear combination) of previously observed neighboring samples

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