Eigenfunctions of the Canonical Transform and the Self-imaging Problems in Optical System

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ABSTRACT

Affine Fourier transform (AFT) also called as the canonical transform. It generalizes the fractional Fourier transform (FRFT), Fresnel transform, scaling operation, etc., and is a very useful tool for signal processing. In this paper, we will derive the eigenfunctions of AFT. The eigenfunctions seems hard to be derived, but since AFT can be represented by the time-frequency matrix (TF matrix), so we can just use the matrix operations to derive its eigenfunctions. Then, because many optical systems can be represented as a special case of AFT, so the eigenfunctions of the AFT are just the light distributions that will cause the self-imaging phenomena for some optical system. We will use the eigenfunctions we derive to discuss the self-imaging phenomena.

I. INTRODUCTION

Affine Fourier transform (AFT) [1][2] is defined as:

$$G_{(a,b,c,d)}(t) = O_F^{(a,b,c,d)}(g(t)) = \sqrt{\frac{1}{j2\pi b}} e^{\frac{j}{2}\frac{d}{b}u^2} \int_{-\infty}^{\infty} e^{-j\frac{u}{b}t} e^{\frac{j}{2}\frac{a}{b}t^2} g(t)\,dt \quad (1)$$

when $b \neq 0$; and when $b = 0$, it reduces to

$$G_{(a,b,c,d)}(t) = d^{1/2}\, e^{j\cdot cd\cdot u^2/2}\, g(d\,u) \quad (2)$$

And the constraint $ad-bc = 1$ must be satisfied. It is also called as the canonical transform. The AFT has the additivity property as:

$$O_F^{(a1,b1,c1,d1)}\left(O_F^{(a,b,c,d)}(g(t))\right) = O_F^{(n,o,p,q)}(g(t)) \quad (3)$$

where $\begin{bmatrix} n & o \\ p & q \end{bmatrix} = \begin{bmatrix} a1 & b1 \\ c1 & d1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. (4)

AFT is the generalization of many operations. When {a, b, c, d} = {cosα, sinα, −sinα, cosα}, the AFT will become the fractional Fourier transform (FRFT) [3] of order α:

$$O_{FRFT}^{\alpha}(g(t)) = \left(e^{j\alpha}\right)^{\frac{1}{2}} O_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}(g(t)) \quad (5)$$

When a=d=1, c=0, the AFT becomes the Fresnel transform. When a=d=1, b=0, the AFT becomes the chirp multiplication. And when b=c=0, the AFT will become the scaling operation. Since the special cases of the AFT described above are all the useful tools for signal processing, and the AFT is the combination of them, so AFT is very general and will be the very useful tool for signal processing. It has been used for the applications of optical system analysis, filter design, pattern recognition, etc.

AFT has very intimated relations with the optical system because many of the wave propagation operations in the optical system can be expressed as the special case of the AFT. We list two of them as below. They are all associated with the monochromic wave with the wavelength λ.

(a) Propagation in the free space with the distance z:

$$O_{Fres,x}^{z}(g(x)) = (j\lambda z)^{-\frac{1}{2}}\, e^{j\frac{\pi z}{\lambda}} \int_{-\infty}^{\infty} e^{j\frac{\pi}{z\lambda}(s-x)^2} g(x)\,dx \quad (6)$$

It is just the Fresnel transform, and corresponds to the AFT with the parameters as {a, b, c, d} = {1, zλ/2π, 0, 1}.

(b) Propagation through the lens with the focal length f.

$$O_{lens,x}^{f}(g(x)) = e^{j\frac{\pi}{\lambda}n\Delta}\, e^{-j\frac{\pi}{f\cdot\lambda}x^2}\, g(x) \quad (7)$$

where n is the refractive index and Δ is the thickness of the lens. It corresponds to the AFT with {a, b, c, d} = {1, 0, −2π/fλ, 1}.

In the above we all reduce the 2-D optical system into the 1-D operation. We can recover the 2-D result by doing the 1-D operations two times. For example, for the Fresnel transform,

$$O_{Fres}^{z}(g(x,y)) = O_{Fres,x}^{z}\left(O_{Fres,y}^{z}(g(x,y))\right) \quad (8)$$

Since the additivity property of AFT can be represented by the matrix operation as Eq. (5), so it is convenient to use

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (9)$$

to represent the AFT. We call it as the time-frequency matrix (TF matrix) in this paper. There are many advantages for this representation. With TF matrix, we can use the operations of 2×2 matrix operations instead of the integration operations as Eq. (1) in many conditions. Many problems of the AFT, and hence the problems of optical system can be easily solved by TF matrix representation.

In this paper, we will first use the TF matrix to derive the eigenfunctions of AFT in section 2. In order to discuss the self-imaging phenomena, in section 3, we will discuss in what conditions the two AFTs will be equivalent in the optical system. Then, we will use the results in sections 2, 3 to discuss the self-imaging problems of the optical systems in section 4. In section 5, we make a conclusion.

II. DERIVING THE EIGENFUNCTIONS OF AFT BY TF MATRIX

The eigenfunctions of the fractional Fourier transform has been known. But there is less discussion about the eigenfunctions of the AFT. In [4], they have found the eigenfunctions of AFT in the case of $|a + d| < 2$. Here we will use a special method that uses the TF matrix to find the eigenfunctions of AFT for all the cases. We first state two important properties.

(Property A)

If e(t) is the eigenfunction of the operation O_P with the eigenvalue λ, i.e., $O_P(e(t)) = \lambda \cdot e(t)$, and suppose O_R is a reversible operation, then

$$O_R\left(O_P\left(O_R^{-1}\left(O_R\left(e(t)\right)\right)\right)\right) = \lambda \cdot O_R\left(e(t)\right) \tag{10}$$

So $O_R(e(t))$ will be the eigenfunction of $O_R O_P O_R^{-1}$, and the eigenvalue is also λ.

(Property B)

Suppose

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}^{-1} \tag{11}$$

Then

$$a + d = a_2 + d_2 \tag{12}$$

This can be proved by the fact that the eigenvalues of two similar matrices are the same, and the diagonal sum of a matrix is just its eigenvalues sum. Comparing with property A, we note the matrix {a, b, c, d} in Eq. (11) will have the same role as $O_R O_P O_R^{-1}$ in Eq. (10). The matrix $\{a_2, b_2, c_2, d_2\}$ has the same role as O_P. And the matrix $\{a_1, b_1, c_1, d_1\}$ will have the same role as O_R.

Thus, from these two properties, if we want to find the eigenfunctions of AFT with parameters {a, b, c, d}, we can

(1) First, separate the TF matrix of AFT into the form as Eq. (11). The eigenfunctions, eigenvalues of the AFT with the parameters $\{a_2, b_2, c_2, d_2\}$ must have been known.

(2) Then doing the AFT with parameters $\{a_1, b_1, c_1, d_1\}$ for the eigenfunctions of the AFT with parameters $\{a_2, b_2, c_2, d_2\}$, we will obtain the eigenfunctions and eigenvalues of AFT with parameters {a, b, c, d}.

{Case A: $|a + d| < 2$}

In the case of $|a + d| < 2$, we can choose $\{a_2, b_2, c_2, d_2\}$ = {cosα, sinα, −sinα, cosα} (i.e., the fractional Fourier transform (FRFT) with some difference in constant phase) in Eq. (11). Then we can separate the TF matrix of AFT as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ -\tau\sigma^{-1} & \sigma^{-1} \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \sigma^{-1} & 0 \\ \tau\sigma^{-1} & \sigma \end{bmatrix} \tag{13}$$

$$\alpha = \cos^{-1}\left((a+d)/2\right) = \sin^{-1}\left(\operatorname{sgn}(b)\,\zeta/2\right)$$
$$\sigma^2 = 2|b|/\zeta \qquad \tau = \operatorname{sgn}(b)\cdot(a-d)/\zeta$$
$$\zeta = \sqrt{4-(a+d)^2} \tag{14}$$

Since the eigenfunctions of FRFT have been known to be the Hermite functions multiplied by $\exp(-t^2/2)$:

$$\phi_m(t) = e^{-t^2/2}\cdot H_m(t) \qquad m \in [0, 1, 2, 3, \ldots\ldots] \tag{15}$$

and its eigenvalue is $\mu_m = \exp(-jm\pi/2)$. Then from the relation of Eq. (5), The eigenfunctions of AFT with the parameters {cosα, sinα, −sinα, cosα} are as the Eq. (15), and the eigenvalues are:

$$\kappa_m = (e^{-j\alpha})^{1/2} e^{-j\cdot m\cdot\alpha} \tag{16}$$

Thus, the eigenfunctions of the AFT in the case that $|a + d| < 2$ is:

$$\phi_m^{(\sigma,\tau)}(t) = A\cdot O_F^{(\sigma,0,-\tau\cdot\sigma^{-1},\sigma^{-1})}\left(\exp(-t^2/2)\cdot H_m(t)\right) \tag{17}$$

where A is some constant, so

$$\phi_m^{(\sigma,\tau)}(t) = \exp\left(-(1+i\tau)t^2/2\sigma^2\right)\cdot H_m(t/\sigma) \tag{18}$$

And the eigenvalues are

$$\kappa_m = (e^{-j\alpha})^{1/2}\cdot e^{-jm\cdot\alpha} \tag{19}$$

{Case B: $a + d = 2,\ b \neq 0$}

In this case, we can choose $\{a_2, b_2, c_2, d_2\}$ = {1, η, 0, 1} (i.e., the Fresnel transform with some difference in constant phase) in Eq. (11). Then we can separate the TF matrix of AFT as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\tau & 1 \end{bmatrix} \tag{20}$$

$$\text{where} \qquad \tau = (d-a)/2b \tag{21}$$

The eigenfunctions of the Fresnel transform is the periodic functions. This is the well-known Talbot effect [5]. If f(t) = f(t+q), then it will be the eigenfunctions of the Fresnel transform with distance $z = 2Nq^2/\lambda$. Thus, from Eq. (6), we can conclude if

$$f(t) = f(t + \sqrt{|b|\,\pi/S}) \qquad \text{S is some integer} \tag{22}$$

then it will be the eigenfunctions of the AFT with parameters {1, η, 0, 1}, and the eigenvalue is 1. So in the case that a + d = 2, the functions as below will be the eigenfunctions of AFT:

$$\phi^{(\tau,b)}(t) = e^{j\cdot\tau^2/2}\cdot f(t) \tag{23}$$

where the f(t) is the periodic function satisfies the Eq. (22), and the eigenvalue is 1.

{Case C: $a + d = -2,\ b \neq 0$}

In this case, we can separate the TF matrix as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\tau & 1 \end{bmatrix} \tag{24}$$

$$\text{where} \qquad \tau = (d-a)/2b \tag{25}$$

The AFT with the parameters {−1, η, 0, −1} corresponds to the Fresnel transform with the reverse operation. Thus, if

$$f(t) = f(t + \sqrt{|b|\,\pi/S}) \qquad \text{S is some integer}$$
$$f(t) = \pm f(-t) \tag{26}$$

then the functions as below will be the eigenfunctions of the AFT when a+d = −2, and the eigenvalues will be $\pm(-1)^{1/2}$:

$$\phi^{(\tau,b)}(t) = e^{j\cdot\tau\cdot t^2/2}\cdot f(t) \tag{27}$$

{Case D: a + d = 2 and b = 0, Case E: a + d = −2 and b = 0}

In these cases, {a, b, c, d} must be the values as {±1, 0, c, ±1}, and the AFT is just the chirp multiplication. So the eigenfunctions will just be the impulse train. In the case that {a, b, c, d} = {1, 0, c, 1}, the eigenfunctions are:

$$\phi^{c,k}(t) = \sum_{n=0}^{\infty} A_n\cdot\delta\left(t - \sqrt{4n\pi|c|^{-1}+k}\right) + \sum_{m=0}^{\infty} B_m\cdot\delta\left(t + \sqrt{4m\pi|c|^{-1}+k}\right) \tag{28}$$

In the case that {a, b, c, d} = {−1, 0, c, −1}, the eigenfunctions are also as above, but $B_m = A_m$ for all m, or $B_m = -A_m$ for all m.

{Case F: $a + d > 2$}

In this case, we can choose $\{a_2, b_2, c_2, d_2\} = \{\sigma^{-1}, 0, 0, \sigma\}$ in Eq. (11), and separate the TF matrix as below:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \eta \\ \tau & \tau\eta+1 \end{bmatrix} \begin{bmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} \tau\eta+1 & -\eta \\ -\tau & 1 \end{bmatrix} \quad (29)$$

$$\sigma = \left(a + d + \text{sgn}(d-a)\sqrt{(a+d)^2 - 4} \right)/2 \quad (30)$$

$$\tau = \frac{2\text{sgn}(a-d)\cdot c}{|d-a| + \sqrt{(a+d)^2 - 4}}, \quad \eta = \frac{\text{sgn}(d-a)\cdot b}{\sqrt{(a+d)^2 - 4}} \quad (31)$$

where $\text{sgn}(t) = 1$ for $t \geq 0$ and $\text{sgn}(t) = 0$ for $t < 0$. The AFT with the parameters $\{\sigma^{-1}, 0, 0, \sigma\}$ will be the scaling operation:

$$O_F^{(\sigma^{-1},0,0,\sigma)}(g(t)) = \sigma^{1/2} g(t/\sigma) \quad (32)$$

The eigenfunctions of the scaling operation are just the self-similar functions, they are also called as the fractals. The simplest examples of several self-similar functions are constant, t^n, $\delta(t)$, etc. From Eq. (29), we find if

$$\sigma^{1/2} g(t/\sigma) = \kappa \cdot g(t) \quad (33)$$

then

$$\phi^{(\sigma,\eta,\tau)}(t) = A \cdot O_F^{(1,\eta,\tau,1+\tau\eta)}(g(r)) \quad \text{A is some constant}$$

$$= e^{j\pi t^2/2} \int_{-\infty}^{\infty} e^{j(t-r)^2/2\eta} g(r)\, dr \quad (34)$$

will be the eigenfunctions of AFT in the case that $a + d > 2$, and the eigenvalue is also κ.

(Case G: $a + d < -2$)

In this case, we can separate the TF matrix as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \eta \\ \tau & \tau\eta+1 \end{bmatrix} \begin{bmatrix} -\sigma^{-1} & 0 \\ 0 & -\sigma \end{bmatrix} \begin{bmatrix} \tau\eta+1 & -\eta \\ -\tau & 1 \end{bmatrix} \quad (35)$$

$$\sigma = \left(-a - d + \text{sgn}(d-a)\sqrt{(a+d)^2 - 4} \right)/2 \quad (36)$$

τ, η are the same as Eq. (31).

So if

$$(-\sigma)^{1/2} g(-t/\sigma) = \kappa \cdot g(t) \quad (37)$$

then

$$\phi^{(\sigma,\tau,\eta)} = e^{j\pi t^2/2} \int_{-\infty}^{\infty} e^{j(t-r)^2/2\eta} g(r)\, dr \quad (38)$$

will be the eigenfunctions of AFT in the case of $a + d < -2$ and the eigenvalue is κ.

We summary the forms of the AFT's eigenfunctions as below:

Condition	Form of the eigenfunctions
$\lvert a + d \rvert < 2$	Hermite functions with the multiplication of chirp and $\exp(-t^2/2)$ and with the scaling
$\lvert a+d \rvert = 2$, $b \neq 0$	(symmetry) periodic functions with the multiplication of chirp
$\lvert a+d \rvert = 2$, $b=0$	(symmetry) impulse trains
$\lvert a + d \rvert > 2$	(symmetry) self-similar functions with the chirp convolution and the chirp multiplication

The phrase of symmetry must be appended when $a + d \leq -2$. We must emphasize that the eigenfunctions described in this section will not be the only possible eigenfunctions of AFT. There will be some other eigenfunctions of AFT for all the cases.

III. EQUIVALENT SYSTEM IN OPTICS

For optical system, only the intensity of the light will be observed. Thus, if two light distributions have the same amplitude and are only different in phase, then they will be observed equivalently in optical system. From Eq. (2), we know the AFT with parameters $\{a, b, c, d\} = \{1, 0, \eta, 1\}$ is just the chirp multiplication. It will only change the phase of the input function, and the not affect the amplitude. So we can conclude the AFT with parameters $\{a_1, b_1, c_1, d_1\}$ is equivalent to the AFT with the parameters $\{a_2, b_2, c_2, d_2\}$ in optical system if

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \eta & 1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ \eta a_2 + c_2 & \eta b_2 + d_2 \end{bmatrix} \quad (39)$$

Sometimes for the optical system analysis, we just care if we have obtained the desired images, and the difference of scaling is ignored. The scaling operation corresponds to the AFT with the parameters $\{\sigma^{-1}, 0, 0, \sigma\}$. So in the case that the scaling is ignored, then the AFT with the parameters $\{a_1, b_1, c_1, d_1\}$ will be equivalent to the AFT with the parameters $\{a_2, b_2, c_2, d_2\}$ if

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} \sigma^{-1} a_2 & \sigma^{-1} b_2 \\ \sigma c_2 & \sigma d_2 \end{bmatrix} \quad (40)$$

Since the values of η, σ in Eqs. (39), (40) can be any real numbers, so from Eqs. (39), (40), we can conclude for the two AFTs with the parameters $\{a_1, b_1, c_1, d_1\}$ and $\{a_2, b_2, c_2, d_2\}$,

(a) If only the intensity is considered, then the two AFTs are equivalent if

$$a_1 = a_2 \qquad b_1 = b_2 \quad (41)$$

(b) If the scaling is ignored, then the two AFTs are equivalent if

$$a_1 : b_1 = a_2 : b_2, \qquad c_1 : d_1 = c_2 : d_2 \quad (42)$$

(c) If only the intensity is considered, and the scaling is ignored, then the two AFTs are equivalent if

$$a_1 : b_1 = a_2 : b_2 \quad (43)$$

Although the AFT has 4 parameters $\{a, b, c, d\}$, but sometimes only the values of a, b or only the ratio of a : b will be important.

IV. THE SELF-IMAGING PROBLEMS

By the relations between AFT and the optical system, we can use the eigenfunctions of AFT to study the self-imaging phenomena of the optical system, i.e., searching the input images that will have the output the same as the input image. If an optical system is the combination of free spaces and lenses, then from the discussion in section 1, we can represent it as a special case of AFT. So if we want to discuss the self-imaging phenomena of an optical system, we can follow the process as:

(1) Find the parameters of AFT to represent this optical system.

(2) Use the results in section 2 to search the eigenfunctions of AFT with the parameters found in step 1. Then the light distrib-

uted as these eigenfunctions will cause the self-imaging phenomena for this optical system.

But for the optical system, only the intensity is considered. And for the discussion of the self-imaging phenomena, the difference of the scaling is usually tolerable. Thus, suppose the optical system can be represented by the AFT with parameters $\{a_1, b_1, c_1, d_1\}$. Then from section 3, not only the eigenfunctions of the AFT with parameters $\{a_1, b_1, c_1, d_1\}$, and all the eigenfunctions of the AFT with the parameters $\{a_2, b_2, c_2, d_2\}$ in which $a_1 : b_1 = a_2 : b_2$ will also cause the self-imaging phenomena for this optical system. And the scaling ratio (the ratio σ such that $|f_o(t)| = \kappa \cdot |f_i(\sigma t)|$) is

$$\sigma = a_1/a_2 = b_1/b_2 \tag{44}$$

When $a_1 = a_2$, $b_1 = b_2$, there will be no scaling. And when $a_1/a_2<0$, $b_1/b_2 < 0$, the image will be reversed.

For example, for the optical system as below:

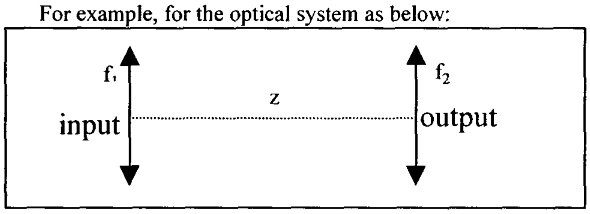

Fig. 1 The optical system with 2 lenses and 1 free space

Then it can be represented by the AFT with the parameters as:

$$\begin{bmatrix} 1 & 0 \\ -2\pi/f_2\lambda & 1 \end{bmatrix}\begin{bmatrix} 1 & z\lambda/2\pi \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2\pi/f_1\lambda & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - z/f_1 & z\lambda/2\pi \\ -\dfrac{2\pi}{f_1\lambda} - \dfrac{2\pi}{f_2\lambda} - \dfrac{z}{f_2} & 1 - \dfrac{z}{f_2} \end{bmatrix} \tag{45}$$

So all the eigenfunctions of AFT with the parameters $\{a, b, c, d\}$ where

$$a : b = 1 - z/f_1 : z\lambda/2\pi \tag{46}$$

will cause the self-imaging phenomena for this optical system, and the scaling ratio will be

$$\sigma = (1 - z/f_1)/a = z\lambda/(2\pi b) \tag{47}$$

If $a = 1 - z/f_1$, $b = z\lambda/2\pi$, then the eigenfunction of AFT will not only cause the self-imaging phenomena, and there will be no scaling between the input and output.

We first discuss the functions that will case the self-imaging phenomena without scaling. The value of a + d determines the form of the eigenfunctions of AFT. Since d can be any value, and

$$a + d = 1 - z/f_1 + d$$

So the value of a + d can also be any values. Varying the values of d from $-\infty$ to ∞, we can find all the possible inputs that will cause the self-imaging phenomena for this optical system.

(1) Setting d to be in the range of $d < -3 + z/f_1$, we will find the functions defined as Eq. (38) (i.e., the chirp multiplication, chirp convolution of the symmetry self-similar functions) will cause the self-imaging phenomena without scaling. The value of σ, τ, η can be calculated from Eqs. (36), (31), from the substitution that $\{a, b, c, d\} = \{1 - z/f_1, z\lambda/2\pi, c, d\}$.

(2) Setting $d = -3 + z/f_1$, we find the functions as below will cause the self-imaging phenomena without scaling:

$$\phi(t) = e^{j\tau \cdot t^2/2} \cdot g(t) \tag{48}$$

where $g(t) = g(t + \sqrt{z\lambda/2S})$ S is any integer (49)

$g(t) = \pm g(-t)$ $\tau = 2\pi(-2 + z/f_1)/z\lambda$ (50)

(3) Setting d to satisfy $-3 + z/f_1 < d < 1 + z/f_1$, we find the functions as below cause the self-imaging phenomena without scaling:

$$\phi_m^{(\sigma,\tau)}(t) = \exp\left(-(1 + i\tau)t^2/2\sigma^2\right) \cdot H_m(t/\sigma) \tag{51}$$

where $\sigma^2 = z\lambda/\pi\zeta$, $\tau = (1 - d - z/f_1)/\zeta$

$$\zeta = \sqrt{4 - (1 + d - z/f_1)^2} \tag{52}$$

(4) Setting $d = 1 + z/f_1$, we find the functions as below will cause the self-imaging phenomena without scaling:

$$\phi(t) = e^{j\pi \cdot t^2/f_1\lambda} \cdot g(t) \tag{53}$$

where $g(t) = g(t + \sqrt{z\lambda/2S})$ S is any integer (54)

(5) Setting $d > 1 + z/f_2$, we will find the functions defined as Eq. (34) (i.e., the chirp multiplication, chirp convolution of the self-similar functions) will cause the self-imaging phenomena without scaling, where the value of σ, τ, η can be calculated from Eqs. (30), 31).

Thus, there are many types of input images that cause the self-imaging phenomena without scaling for the optical system as Fig. 1. We note the last lens (with focal length f_2) will have no effects on the self-imaging phenomena.

We can use the similar method as above and both vary the value of d and b in the interval of $(-\infty, \infty)$ to find all the possible inputs that will cause the self-imaging phenomena when the difference of scaling is tolerated.

V. CONCLUSION

In this paper we have used the time-frequency matrix (TF matrix) to derive the eigenfunctions of AFT, discuss the conditions for two AFTs to be equivalent in optical system, and discussing the self-imaging phenomena of the optical system. The TF matrix is a very useful tool to solve the problems associated with the AFT. Besides the self-imaging phenomena, many other problems about the optics (such as the optical implementation and the generalized Talbot effect) and filter design can be easily solved by the TF matrix. With the aid of TF matrix representation, the AFT will become a very useful tool to solve many signal-processing problems in the future.

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