New techniques for the pole placement of singular systems

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Abstract

New techniques for pole placement problem of single input singular systems are proposed in this paper. These techniques provide different ways to approach the generalized Ackermann's formula with better numerical properties and flexibility. Since the solution of the pole placement problem depend on the singularity of the matrix E. Two sets of recursive algorithms are presented separately corresponding to the matrix E is singular and nonsingular respectively. These algorithms are verified and implemented in MATLAB program.

1. Preliminaries

Consider the single input singular linear system

$$\overline{E}\dot{x}(t) = \overline{A}x(t) + \overline{b}u(t) \qquad (1)$$

where and $\overline{A} \in R^{n\times n}$, $\overline{b} \in R^{n\times 1}$, $\overline{E}$ is possibly singular matrix, and $u(t) \in R$ and $x(t) \in R^n$ are input and state vectors respectively. The problem of pole placement in singular systems is to find the state feedback control law $u(t) = -kx(t) + r(t)$, where $k \in R^{1\times n}$ and $r(t) \in R$ such that the closed-loop system has prescribed finite and infinite eigenvalues. To develop the generalized Ackermann's formula, the restricted equivalent transformation is performed as follows

$$E = (\mu\overline{E} - \overline{A})^{-1}\overline{E} \qquad (2a)$$

$$A = (\mu\overline{E} - \overline{A})^{-1}\overline{A} \qquad (2b)$$

$$b = (\mu\overline{E} - \overline{A})^{-1}\overline{b}\,. \qquad (2c)$$

Definition 1

The system (3) is called the standard singular system of system (2)

$$E\dot{x}(t) = Ax(t) + bu(t) \qquad (3)$$

if system (3) is obtained from the (2a), (2b) and (2c).

□

Lemma 1 [5]

The generalized system (1) is controllable if and only if

$$rank[s\overline{E} - \overline{A} \quad \overline{b}] = n \quad \text{for all s} \qquad (4)$$

□

For the standard singular system satisfying the controllable condition of Lemma 1 is referred as standard controllable singular system. In Lemma 2, we define a indeterminate parameter p and explore the determinant relationship of the open-loop system and closed-loop system in terms of p.

Lemma 2

For the case of $\mu E - A = I$ and $s \neq \mu$,

$$\det(sE - A) = (\frac{1}{p})^n \det(pI - E) \qquad (5)$$

$$\det[sE - (A - bK)] = (\frac{1}{p})^n \det[p(I + bk) - E] \qquad (6)$$

where $p = \dfrac{1}{\mu - s}$ or $s = \mu - \dfrac{1}{p}$

□

For a standard controllable generalized system, the following two theorems are derived.

Theorem 1 (Generalized Ackermann's formula)[1]

Let $E\dot{x}(t) = Ax(t) + bu(t)$ be a standard controllable generalized system, satisfying. Assume the state feedback control law is

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$u(t) = -kx(t) + r(t)$ and the desired closed-loop characteristics polynomial is

$$\Delta_d(p) = (p)^n \det[sE - (A - bk)] \,|_{s=\mu-\frac{1}{p}} = \sum_{i=0}^{n} d_{n-i} p^i$$

, where $d_n = \det(-E)$. Then

$$kE = e_n' C^{-1} \Delta_d(E) \tag{7}$$

where $e_n' = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$

$C = \begin{bmatrix} b & Eb & \cdots & E^{n-1}b \end{bmatrix}$

$\Delta_d(E) = \sum_{i=0}^{n} d_{n-i} E^i$

□

Theorem 2 [1]

Let $E\dot{x}(t) = Ax(t) + bu(t)$, with E singular, be a standard controllable generalized system, satisfying $\mu E - A = I$. Assume the desired closed-loop characteristic polynomial is $\Delta_d(p) = (p)^n \det(sE - (A - bk))]|_{s=\mu-\frac{1}{p}} = \sum_{i=1}^{n} d_{n-i} p^i$

. Then

$$k = e_n' C^{-1}[(d_{n-1} - a_{n-1})I + (d_{n-2} - a_{n-2})E + \cdots + (d_1 - a_1)E^{n-2} + (d_0 - 1)E^{n-1}] \tag{8}$$

where $e_n' = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$

$C = [b \quad Eb \quad \cdots \quad E^{n-1}b]$

$\det(pI - E) = \sum_{i=0}^{n-1} a_{n-i} p^i + p^n$

□

Remark : The Leverrier's algorithm [6] can be used to compute the coefficients of $\Delta_o(p)$. To compute the coefficients of $\Delta_d(p)$, the following procedure can be proceeded :

(1) Convert the desired eigenvalues s_{id} to p_{id} via formula $p_{id} = \dfrac{1}{\mu - s_{id}}$

(2) Compute $\Delta_d(p) = (p - p_{1d})(p - p_{2d}) \cdots (p - p_{nd})$

(3) Compute the scale factor $c = \det(-E)/[(-1)^n \prod_i^n p_{id}]$, and adjust the $\Delta_d(p) = c \bullet \Delta_d(p)$ to meet the condition $\det(-E) = d_n$.

Step (3) can be skipped when theorem 2 is applied. Since the condition $\det(-E) = d_n$ is reached as matrix E is singular matrix.

2. Main Results

From previous section, it is clear that the solution of feedback gain k is depend on the singularity of matrix E. In subsection, algorithms are proposed separate respect to the singularity of matrix E.

There are two major problem been criticized about the Ackermann's formula (1) the inverse of controllability matrix C might cause numerical error due to the ill-conditioning of C (2) the numerical error computing feedback gain due to the multiple multiplication of matrix E . For the first problem, technique to solve this problem is recommended by [2] which is listed as follows :

Algorithm 1 : (Computation of $C_0 \equiv e_n' C^{-1}$)

$r_1 = b$

for $i = 1 : n$

$n_i = norm(r_i)$

$r_i = r_i / n_i$

$r_{i+1} = Er_i$

end

$e_n = e_n / \prod_{i=1}^{n} n_i$

$R = [r_1 \quad r_2 \quad \cdots \quad r_n]$

$C_0 = (R^{-T} e_n)^T$

In algorithm 1, the controllability matrix is normalized and the scaling adjustment is made as the computation of C_0. Algorithm 1 has been verified and claimed by [2] that have the advantage of numerical robustness.

2.1 Matrix E is nonsingular

Let $\Delta_d(p) = \sum_{i=0}^{n} d_{n-i} p^i$ be the closed-loop polynomial in terms of p with $d_n = \det(-E)$. There are two ways to approach the computation of $C_0 \Delta_d(E)$. First, we can factorize $\Delta_d(p)$ in terms of closed-loop eigenvalues $p_{d,i}$ as follows :

$$\begin{aligned}\Delta_d(p) &= d_0 p^n + d_1 p^{n-1} + \cdots + d_{n-1} p + d_n \\ &= d_0 (p - p_{d,1})(p - p_{d,2}) \cdots (p - p_{d,n}) \\ &= d_0 \prod_{i=1}^{n} (p - p_{d,i})\end{aligned}$$

Since $d_n = (-1)^n d_0 \prod_{i=1}^{n} p_{d,i} = \det(-E)$, we have $d_0 = \det(-E) / [(-1)^n \prod_{i=1}^{n} p_{d,i}]$. Hence,

$$C_0 \Delta_d(E) = d_0 C_0 \prod_{i=1}^{n} (E - p_{d,i} I)$$

where $d_0 = \det(-E) / [(-1)^n \prod_{i=1}^{n} p_{d,i}]$

Next, we will develop a recursive procedure to compute $C_0 \Delta_d(E)$ in terms of the coefficients of closed-loop systems d_i.

$$\begin{aligned}C_0 \Delta_d(E) &= d_0 C_0 E^n + d_1 C_0 E^{n-1} + \cdots + d_{n-1} C_0 E + d_n C_0 \\ &= d_0 l_n + d_1 l_{n-1} + \cdots + d_{n-1} l_1 + d_n l_0 \\ &= \sum_{i=0}^{n} d_{n-i} l_i\end{aligned}$$

where $l_i = l_{i-1} E$ and $l_0 = C_0$

From remark 2 and above derivation, k can be computed as

$$k = e_n' C^{-1} \Delta_d(E) E^{-1} = [\sum_{i=0}^{n} d_{n-i} l_i] E^{-1} = d_0 [C_0 \prod_{i=1}^{n} (E - p_{d,i} I)] E^{-1} \quad (9)$$

Algorithm 2 and 3 are provided to implement the approaching methods discussed above.

Algorithm 2 : (Use desired closed-loop eigenvalues)

Compute C_0 as Algorithm 1

$f_0 = \{\det(-E) / [(-1)^n \prod_{i=1}^{n} p_i]\} C_0$

for $\quad i = 1 : n-1$

$\qquad f_i = f_i (E - p_{d,i} I)$

end

$k = f_n E^{-1}$

Algorithm 3 : (Use the coefficients of the closed-loop system)

Compute C_0 as Algorithm 1

$l_0 = C_0$

$k = d_n l_0$

for $\quad i = 1 : n$

$\qquad l_i = l_{i-1} E$

$\qquad k = k + d_{n-i} l_i$

end

$k = k E^{-1}$

2.2 Matrix E is singular

From theorem 2, the solution of feedback gain k need not only the information of closed-loop system but also open-loop system. From last section, we see that the information of closed-loop system can be expressed in either eigenvalues of closed-loop systems or coefficients of closed-loop systems. Therefore, as matrix E is singular, there are four possible combination algorithms.

From eq. (8), we can separate the solution k into two parts as follows :

$$\begin{aligned}k &= C_0[(d_{n-1} - a_{n-1}) I + (d_{n-2} - a_{n-2}) E + \cdots + (d_0 - 1) E^{n-1}] \\ &= C_0 (d_{n-1} I + d_{n-2} E + \cdots + d_0 E^{n-1}) - C_0 (a_{n-1} I + a_{n-2} E + \cdots + E^{n-1}) \\ &= C_0 \overline{\Delta}_c(E) - C_0 \overline{\Delta}_o(E)\end{aligned}$$

$C_0 \overline{\Delta}_c(E)$ represents the information of closed-loop systems while $C_0 \overline{\Delta}_o(E)$ represents the information of open-loop systems.

Use the same technique as (9) in previous section, we have

$$C_0 \overline{\Delta}_c(E) = \sum_{i=0}^{n-1} d_{n-i-1} l_i = C_0 \prod_{i=1}^{n-1} (E - p_{d,i} I)$$

$$C_0 \overline{\Delta}_o(E) = \sum_{i=0}^{n-1} a_{n-i-1} l_i = C_0 \prod_{i=1}^{n-1} (E - p_i I)$$

where $l_i = l_{i-1} E$

$p_{d,i}$: desired eigenvalues

p_i : eigenvalues of open-loop system

a_i, d_i : coefficients of open-loop systems and closed-loop systems respectively.

Lemma 3 [4]

The closed-loop system has at most $rank\, E$ finite poles for any feedback control. □

From lemma 3, there are only rank E eigenvalues can be assigned. Hence, there are $n - rank\, E$ closed-loop poles ($p_{i,d}$, $i = rankE + 1, \cdots, n$) being assigned as 0.

Four algorithms corresponding to different combination of the information of open-loop system and closed-loop system are given as following.

Algorithm 4 : (Use the coefficients of the closed-loop system and open-loop system)

Compute C_0 as Algorithm 1

$l_0 = C_0$

$k = (d_{n-1} - a_{n-1})l_0$

for $i = 1 : n-1$

$\quad l_i = l_{i-1}E$

$\quad k = k + (d_{n-i-1} - a_{n-i-1})l_i$

end

Algorithm 5 : (Use eigenvalues of closed-loop system and open-loop system)

Compute C_0 as Algorithm 1

$l_0 = f_0 = C_0$

for $i = 1 : n-1$

$\quad l_i = l_{i-1}(E - p_{d,i}I)$

$\quad f_i = f_{i-1}(E - p_i I)$

end

$k = f_{n-1} - l_{n-1}$

Algorithm 6 : (Use the coefficients of closed-loop system and eigenvalues of open-loop system)

Compute C_0 as Algorithm 1

$l_0 = f_0 = C_0$

$k = d_{n-1}l_0$

for $i = 1 : n-1$

$\quad l_i = l_{i-1}E$

$\quad f_i = f_{i-1}(E - p_i I)$

$\quad k = k + d_{n-i-1}l_i$

end

$k = k - f_{n-1}$

Algorithm 7 : (Use the eigenvalues of closed-loop system and coefficients of open-loop system)

Compute C_0 as Algorithm 1

$l_0 = f_0 = C_0$

$k = a_{n-1}l_0$

for $i = 1 : n-1$

$\quad l_i = l_{i-1}E$

$\quad f_i = f_{i-1}(E - p_{i,d}I)$

$\quad k = k + a_{n-i-1}l_i$

end

$k = f_{n-1} - k$

3. Conclusion

Algorithms to solve the pole placement problem of the singular systems utilizing the generalized Ackermann's formula are proposed. The weakness of numerical properties of generalized Ackermann's formula are improved by these algorithms. Since the open-loop characteristic polynomial can be obtained by the Leverrier's algorithm and the desired closed-loop poles are more straightforward needed, algorithm 7 is better than the others (algorithm 4 to 6). The algorithm 4 has the least computation flops in the four algorithms of section 2.2. All of the proposed algorithms in this paper are implemented and verified by MATLAB software package.

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