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# $\mathcal{H}_\infty$ Control for Nonlinear Differential-Algebraic Equations

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### Abstract

In this paper, we study the $\mathcal{H}_\infty$ control problem for nonlinear differential-algebraic equations (DAEs). Necessary and sufficient conditions are derived for the existence of a controller solving the problem. We first give various sufficient conditions for the solvability of $\mathcal{H}_\infty$ control problem for DAEs. Both state feedback and output feedback cases are considered. Then, necessary conditions for the output feedback control problem to be solvable are obtained in terms of two Hamilton - Jacobi inequalities plus a weak coupling condition. Moreover, a parameterization of a family of output feedback controllers solving the problem is also provided.

## 1 Introduction

For the purpose of control, nonlinear differential-algebraic equations(DAEs) are frequently described by a parameter dependent form

$$\dot{\mathrm{x}}_1 = f(\mathrm{x}_1, \mathrm{x}_2, u), \qquad (1)$$
$$0 = g(\mathrm{x}_1, \mathrm{x}_2, u), \qquad (2)$$

or in a more compact form

$$E\dot{\mathrm{x}} = F(\mathrm{x}, u),$$

where $E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathrm{x} = [\mathrm{x}_1^T \ \ \mathrm{x}_2^T]^T \triangleq (x_1, \ldots, x_n)$ are local coordinates for an $n$-dimensional state space manifold $\mathcal{X}$. In the state space $\mathcal{X}$, dynamic state variables $\mathrm{x}_1$ and instantaneous state variables $\mathrm{x}_2$ are distinguished. The dynamics of the states $\mathrm{x}_1$ is directly defined by (1), while the dynamics of $\mathrm{x}_2$ is such that the system satisfies the constraint (2). The parameter $u$, usually termed as "control input", defines a specific system configuration and the operating condition. In many cases, the algebraic constraint (2) of the full DAEs can be eliminated (usually due the consistence of initial conditions). As a consequence, the DAEs reduce to a well-known state-variable system. In fact, the state-variable descriptions have been the predominant tool in systems and control theory. Nevertheless, in some cases this kind of elimination is not possible (often due to inconsistent initial conditions), since it may result in loss of accuracy or loss of necessary information. As a result, the theory of DAEs has been well estabilished for many years with the practical outcome that the shortcomings of state-variable theory are often overcome [7]. A large class of physical systems can be modeled by this kind of DAEs. The paper of Newcomb *et al.* [7] gives many practical examples -including circuit and system design, robotics, neural network, etc.- and presents an excellent review on nonlinear DAEs. Many other applications of DAEs as well as numerical treatment can be found in [3].

In this paper, we investigate the contractive property of DAEs, namely the $\mathcal{H}_\infty$ control problem. Our paper is mainly divided into two parts. The first part concerns various sufficient conditions for the solvability of the $\mathcal{H}_\infty$ control problem. Both state feedback and output feedback cases are considered. We seek sufficient conditions under which a given DAE has an $\mathcal{L}_2$-gain no greater than a prescribed positive number $\gamma$ with internal stability, and in the mean time, eliminates possible singularity-induced bifurcation. This kind of bifurcation phenomena was first discovered by Venkatasubramanian *et al.* [9]. It is an unpleasant physical phenomenon, for we can not decide whether it is stable or not. It has a close relation to the notion concerning indices of DAEs, however. We will derive a family of output feedback controllers solving the $\mathcal{H}_\infty$ control problem. The underlying ideas are differential games[2] and dissipation inequailties[12]. These ideas were also used in Isidori *et al.* [4] and Yung *et al.* [15] in which they have given the central controller and a family of controllers, respectively, solving the $\mathcal{H}_\infty$ output feedback control problem for general nonlinear systems in usual state-variable form, i.e. the $E$-matrix is nonsingular.

The second part of this paper is devoted to a converse result, namely the derivation of necessary conditions for solutions of local disturbance attenuation to exist. We obtain necessary conditions given in terms of the existence of nonnegative solutions to two Hamilton-Jacobi inequalities, together with a weak coupling condition. Similar result has previously been published in [1] for

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($W$-)input affine nonlinear systems with nonsingular $E$-matrix. In a recent monograph [8], among many other important contributions, van der Schaft has addressed a number of issues related to necessary conditions for solutions of local disturbacne attenuation to exist. He treated more general nonlinear systems in which both the penalty output $z$ and the measured output $y$ were involved with the control input $u$ and exogenous input $w$. See also [4] for some related work. In the aforementioned literature, the system under consideration are the conventional state-variable systems, i.e. the $E$-matrix is nonsingular. Our results can be thought of as a parallel extension of the results of [8] to the DAEs case.

This paper is organized as follows. In section 2, we will review some notions of DAEs. Some terminology will be defined in this section as well. In section 3, some preliminary results for the theory of DAEs will be given, including stability theory and dissipativity. Our main results will be summarized in sections 4 and 5. We will concentrate on the output feedback case. However, for the sake of completeness, we will first investigate the state feedback control. We also give a parameterizaton of a family of output feedback controllers. In section 5, we will give necessary conditions for the $\mathcal{H}_\infty$ output feedback control problem to be solvable. Finally, conclusions will be given in section 6.

## 2 Differential-Algebraic Equations

Consider the following differential-algebraic equation(DAE)

$$E\dot{\mathbf{x}}(t) \;=\; F(\mathbf{x}, u), \qquad u \in \mathcal{U} \subset \mathbb{R}^m, \tag{3}$$

where $\mathbf{x} \triangleq (x_1, \ldots, x_n)$ are local coordinates for an $n$-dimensional state space manifold $\mathcal{X}$. $E$ is a constant matrix and $F(0, 0) = 0$. The constant matrix $E \in \mathbb{R}^{n\times n}$ is, in general, a singular matrix with rank$E = r < n$. The following definitions will be used (See [3]).

**Definition 1** *Let $\mathcal{T}$ be an open subinterval of $\mathbb{R}$, $\Omega$ a connected open subinterval of $\mathbb{R}^{2n}$, and $F(\bullet, u)$ a sufficiently smooth function from $\Omega$ to $\mathbb{R}^n$. Then the* DAE *(3) is solvable on $\mathcal{T}$ in $\Omega$ if there is an $r$-dimensional family of solutions $\phi(t, c)$ defined on a connected open set $\mathcal{T} \times \tilde{\Omega}$, $\tilde{\Omega} \subset \mathbb{R}^r$, such that*

1. *$\phi(t, c)$ is defined on all $\mathcal{T}$ for each $c \in \tilde{\Omega}$.*
2. *$(\phi(t, c), \dot{\phi}(t, c)) \in \Omega$ for $(t, c) \in \mathcal{T} \times \Omega$.*
3. *If $\varphi(t)$ is any other solution with $(\varphi(t), \dot{\varphi}(t)) \in \Omega$, then $\varphi(t) = \phi(t, c)$ for some $c \in \tilde{\Omega}$.*
4. *The graph of $\phi$ as a function of $(t, c)$ is an $r + 1$-dimensional manifold.* □

**Definition 2** [3] *The index of the DAE (3) is the the minimum number of times that all or part of (3) must be differentiated with respect to $t$ while setting $u \equiv 0$ in order to determine $(\dot{x}_{r+1}, \ldots, \dot{x}_n)$ as a continuous function of $(x_{r+1}, \ldots, x_n)$.* □

**Definition 3** [3] *The* DAE (3) *is said to be of (uniform) index one if the index of the constant coefficient system*

$$E\dot{\mathbf{w}}(t) \,-\, F_x(\hat{\mathbf{x}}(t), 0)\mathbf{w}(t) \,=\, g(t)$$

*is one for all $\hat{\mathbf{x}}$ in a neighborhood of the graph of the solution, where $F_{\mathbf{X}}$ denotes the Jacobian matrix $\frac{\partial F}{\partial \mathbf{X}}$.* □

The index of a DAE can be thought of as the generalization of the nilpotent index[3] of a linear time-invariant descriptor system.

## 3 Stability Results and Dissipation Inequality for DAEs

Consider a differential-algebraic system

$$E\dot{\mathbf{x}}(t) \;=\; F(\mathbf{x}), \tag{4}$$

where $\mathbf{x} \in \mathcal{X}$. In [13], some stability definitions and Lyapunov stability theorems for a nonlinear DAE of the type (4) have been given. For the sake of brevity, we do not reproduce those results here. Instead, we will derive an improved version of the Lyapunov stability theorem for DAE (4). It provides a sufficient condition for the existence of feasibility regions.

**Theorem 4** *Consider* DAE (4) *with $F(0) = 0$. Let $\mathbf{x}_0 \in \Im m\, E$ be given. Suppose that there exists a $C^3$ function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ vanishing at $E\mathbf{x} = 0$ and positive elsewhere which satisfies the following properties.*

- i) *$\frac{\partial}{\partial \mathbf{X}} V \;=\; \tilde{V}^T(\mathbf{x})E$ for some $C^2$ function $\tilde{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and*
- ii) *$\tilde{V}^T(\mathbf{x})F(\mathbf{x}) < 0$, and*
- iii) *$E^T\tilde{V}_{\mathbf{X}} \;=\; \tilde{V}_{\mathbf{X}}^T E \geq 0$, where $\tilde{V}_{\mathbf{X}}$ denote the Jacobian of $\tilde{V}$.*

*Then the equilibrium point $\mathbf{x} = 0$ is locally asymptotically stable and the* DAE *is of index one.* □

For the remainder of this section, we will investigate dissipative property[12] of a given DAE. Consider the following DAE

$$\begin{aligned} E\dot{\mathbf{x}} &= F(\mathbf{x}, u), \qquad u \in \mathcal{U} \subset \mathbb{R}^m, \\ y &= H(\mathbf{x}, u), \qquad y \in \mathcal{Y} \subset \mathbb{R}^p, \end{aligned} \tag{5}$$

where $x \in \mathcal{X}$, together with a function $s : \mathcal{U} \times \mathcal{Y} \longrightarrow \mathbb{R}$, called the supply rate. We have the following very important result.

**Theorem 5** *Consider* DAE (5) *with $\mathbf{x}_0 \in \Im m\, E$ given. Suppose that the matrix $D^TD - \gamma^2 I$ is negative definite and $\{E, A, G\}$ is impulse observable*(The triple $\{E, A, G\}$ is called impulse observable if there exists a

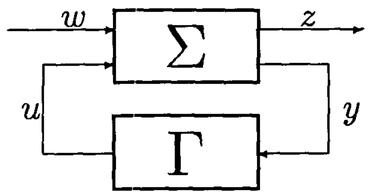


**Figure 1:** Standard Block Diagram

constant matrix $L$ such that $\{E, A + LG\}$ is impulse-free), *where*

$$\begin{aligned} D &= \left(\tfrac{\partial}{\partial u} H\right)_{(\mathrm{x},u)=(0,0)}, \\ A &= \left(\tfrac{\partial}{\partial \mathrm{x}} F\right)_{(\mathrm{x},u)=(0,0)}, \\ G &= \left(\tfrac{\partial}{\partial \mathrm{x}} H\right)_{(\mathrm{x},u)=(0,0)}. \end{aligned}$$

*Suppose that any bounded trajectory* $\mathrm{x}(t)$ *of the system* $E\dot{\mathrm{x}} = F(\mathrm{x}(t), 0)$ *satisfying* $H(\mathrm{x}(t), 0) = 0$ *for all* $t \geq 0$ *is such that* $\lim_{t\to\infty} \mathrm{x}(t) = 0$. *Suppose also that there exists a* $C^3$ *function* $V : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ *vanishing at* $E\mathrm{x} = 0$ *and positive elsewhere which satisfies the following properties.*

i) $\frac{\partial}{\partial \mathrm{x}} V = \tilde{V}^T(\mathrm{x})E$ *for some* $C^2$ *function* $\tilde{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

ii) $Y_0 \triangleq \tilde{V}^T(\mathrm{x})F(\mathrm{x}, u) + \|y\|^2 - \gamma^2\|u\|^2 \leq 0$.

iii) $E^T\tilde{V}_{\mathrm{x}} = \tilde{V}_{\mathrm{x}}^T E$.

*Then the* DAE *has an* $\mathcal{L}_2$ *gain less than or equal to* $\gamma$ *and the equilibrium point* $\mathrm{x} = 0$ *is locally asymptotically stable. Moreover, the* DAE *is of index one.* □

## 4 The $\mathcal{H}_\infty$ Control Problem

Consider the standard feedback configuration shown in Figure 1. Let $\Sigma$ be a nonlinear system described by the following DAE

$$\begin{aligned} E\dot{x} &= F(\mathrm{x}, w, u), \ w \in \mathcal{W} \subset \mathbb{R}^l, \ u \in \mathcal{U} \subset \mathbb{R}^m, \\ z &= Z(\mathrm{x}, w, u), \ z \in \mathcal{Z} \subset \mathbb{R}^s, \\ y &= Y(\mathrm{x}, w, u), \ y \in \mathcal{Y} \subset \mathbb{R}^p, \end{aligned} \tag{6}$$

where $\mathrm{x} \in \mathcal{X}$. Here $u$ stands for the vector of control inputs, $w$ is the exogenous input (disturbances to-be-rejected or signals to-be-tracked), $y$ is the measured output, and finally $z$ denotes the to-be-controlled outputs (tracking errors, cost variables). It is assumed throughout that $F(0, 0, 0) = 0$, $Z(0, 0, 0) = 0$ and $Y(0, 0, 0) = 0$. The standard $\mathcal{H}_\infty$ control problem consists of finding, if possible, a controller $\Gamma$ such that the resulting closed-loop system is locally asymptotically stable and has $\mathcal{L}_2$ gain (from $w$ to $z$) less than or equal to $\gamma$.

### 4.1 State Feedback $\mathcal{H}_\infty$ Control

In the state feedback $\mathcal{H}_\infty$ control problem we assume that $y = \mathrm{x}$ in (6), i.e., that the whole state is available for measurement. We suppose the following.

**(A1)** The matrix $D_{12}$ has rank $m$ and the matrix $D_{11}^T D_{11} - \gamma^2 I$ is negative definite, where $D_{12} = (\frac{\partial Z}{\partial u})_{(\mathrm{x},w,u)=(0,0,0)}$ and $D_{11} = (\frac{\partial Z}{\partial w})_{(\mathrm{x},w,u)=(0,0,0)}$.

**(A2)** Any bounded trajectory $\mathrm{x}(t)$ of the system $E\dot{\mathrm{x}}(t) = F(\mathrm{x}(t), 0, u(t))$ satisfying $Z(\mathrm{x}(t), 0, u(t)) = 0$ for all $t \geq 0$ is such that $\lim_{t\to\infty} \mathrm{x}(t) = 0$.

**(A3)** The matrix pencil

$$\begin{bmatrix} A - \mathrm{j}\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix}$$

has full column rank for all $\omega \in \mathbb{R} \cup \{\infty\}$, where $A = (\frac{\partial F}{\partial \mathrm{x}})_{(\mathrm{x},w,u)=(0,0,0)}$, $B_2 = (\frac{\partial F}{\partial u})_{(\mathrm{x},w,u)=(0,0,0)}$, and $C_1 = (\frac{\partial Z}{\partial \mathrm{x}})_{(\mathrm{x},w,u)=(0,0,0)}$.

**Lemma 6** *Consider the* DAE (6). *Assume that assumptions* A1-A3 *are satisfied. Suppose the following hypothesis also holds.*

**H1:** There exists a smooth function $V(\mathrm{x})$ vanishing at $E\mathrm{x} = 0$ and positive elsewhere, locally defined on a neighborhood of the equilibrium point 0 in $\mathcal{X}$, such that the function

$$Y_1 = H(\mathrm{x}, \tilde{V}(\mathrm{x}), \alpha_1(\mathrm{x}), \alpha_2(\mathrm{x}))$$

is negative semidefinite near $\mathrm{x} = 0$, where the function $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is defined on a neighborhood of $(\mathrm{x}, p, w, u) = (0, 0, 0, 0)$ as

$$H(\mathrm{x}, p, w, u) = p^T F(\mathrm{x}, w, u) + \|Z(\mathrm{x}, w, u)\|^2 - \gamma^2\|w\|^2, \tag{7}$$

$\frac{\partial V}{\partial \mathrm{x}} = \tilde{V}^T(\mathrm{x})E$ defined as above (Theorem 5), $\alpha_1(\mathrm{x}) = w^*(\mathrm{x}, \tilde{V}(\mathrm{x}))$ and $\alpha_2(\mathrm{x}) = u^*(\mathrm{x}, \tilde{V}(\mathrm{x}))$, and $w^*(\mathrm{x}, p)$ and $u^*(\mathrm{x}, p)$ are defined on a neighborhood of $(\mathrm{x}, p) = (0, 0)$ satisfying

$$\begin{aligned} \frac{\partial H}{\partial w}(\mathrm{x}, p, w^*(\mathrm{x}, p), u^*(\mathrm{x}, p)) &= 0, \\ \frac{\partial H}{\partial u}(\mathrm{x}, p, w^*(\mathrm{x}, p), u^*(\mathrm{x}, p)) &= 0 \end{aligned}$$

with $w^*(0, 0) = 0$ and $u^*(0, 0) = 0$.

*Then the feedback law* $u = \alpha_2(\mathrm{x})$ *solves the* $\mathcal{H}_\infty$ *state feedback control problem.* □

### 4.2 Output Feedback $\mathcal{H}_\infty$ Control

In this subsection, we consider the case in which the state $\mathrm{x}$ of the DAE (6) is not available for direct measurement. Motivated by the work of Yung *et al.*[15], we consider a dynamic controller of the form

$$\begin{aligned} \hat{E}\dot{\xi} &= F(\xi, \alpha_1(\xi), \alpha_2(\xi)) + G(\xi)(y - Y(\xi, \alpha_1(\xi), \alpha_2(\xi)), \\ u &= \alpha_2(\xi) \end{aligned} \tag{8}$$

where $\xi = [\xi_1, \ldots, \xi_n]$ are local coordinates for the state space manifold $\mathcal{X}_c$ of the controller $\Gamma$. The matrix $G(\xi)$, called the output injection gain, is to be determined.

Substitute the controller (8) in (6) to obtain the corresponding closed-loop system as

$$E^e\dot{x}^e = F^e(x^e, w), \qquad z = Z^e(x^e, w) = Z(x, w, \alpha(\xi)), \tag{9}$$

where $E^e = \begin{bmatrix} E & 0 \\ 0 & \hat{E} \end{bmatrix}$

$$x^e = \begin{bmatrix} x \\ \xi \end{bmatrix}, \text{ and}$$

$$F^e(x^e, w) = \begin{bmatrix} F(x, w, \alpha_2(\xi)) \\ F(\xi, \alpha_1(\xi), \alpha_2(\xi)) + G(\xi)(y - Y(\xi, \alpha_1(\xi), \alpha_2(\xi)) \end{bmatrix}.$$

Again, we try to render the closed-loop system locally dissipative with respect to the supply rate $\gamma^2\|w\|^2 - \|z\|^2$. Clearly, it suffices to show that there exists a smooth nonnegative function $U(x^e)$ with $\frac{\partial U}{\partial x^e} = \tilde{U}^T E^e$ and $E^{eT}\tilde{U}_x = \tilde{U}_x^T E^e$ such that

$$\tilde{U}^T F^e(x^e, w) + \|Z(x^e, w)\|^2 - \gamma^2\|w\|^2 \leq 0, \qquad \text{for all } w, \tag{10}$$

subject to the closed-loop system is locally asymptotically stable. As a matter of fact, we have the following result.

**Lemma 7** *Consider* (9). *Suppose that assumptions* A1-A3 *hold. Suppose the DAE*

$$\hat{E}\dot{\xi} = F(\xi, \alpha_1(\xi), 0) - G(\xi)Y(\xi, \alpha_1(\xi), 0) \tag{11}$$

*has index one and a locally asymptotically stable equilibrium at* $\xi = 0$. *Furthermore, suppose that there exists a smooth function* $U(x^e)$, *vanishing at* $E^e x^e = 0$ *and positive elsewhere, which satisfies the inequality* (10) *for all* $w$. *Then the closed-loop system* (9) *has an* $\mathcal{L}_2$-*gain no greater than* $\gamma$ *and a locally asymptotically stable equilibrium at* $x^e = 0$. □

The inequality (10), in fact, contains undetermined function $G(\bullet)$, and involves $2n$ independent variables. The next theorem shows how the conditions in Lemma 7 can be met, while reducing the number of independent variables of (10). A further assumption is needed.

**(A4)** The matrix $D_{21} = (\frac{\partial Y}{\partial w})_{(x,w,u)=(0,0,0)}$ has rank $p$.

Define

$$\begin{aligned} r_{11}(x) &= \tfrac{1}{2}\left(\frac{\partial^2 H(x, \tilde{V}^T(x), w, u)}{\partial w^2}\right)_{w=\alpha_1(x), u=\alpha_2(x)} \\ r_{12}(x) &= \tfrac{1}{2}\left(\frac{\partial^2 H(x, \tilde{V}^T(x), w, u)}{\partial u \partial w}\right)_{w=\alpha_1(x), u=\alpha_2(x)} \\ r_{21}(x) &= \tfrac{1}{2}\left(\frac{\partial^2 H(x, \tilde{V}^T(x), w, u)}{\partial w \partial u}\right)_{w=\alpha_1(x), u=\alpha_2(x)} \\ r_{22}(x) &= \tfrac{1}{2}\left(\frac{\partial^2 H(x, \tilde{V}^T(x), w, u)}{\partial u^2}\right)_{w=\alpha_1(x), u=\alpha_2(x)} \end{aligned}$$

and set

$$R(x) = \begin{bmatrix} (1-\epsilon_1)r_{11}(x) & r_{12}(x) \\ r_{21}(x) & (1+\epsilon_2)r_{22}(x) \end{bmatrix}$$

where $\epsilon_1$ and $\epsilon_2$ are any real numbers satisfying $0 < \epsilon_1 < 1$ and $\epsilon_2 > 0$, respectively. The following theorem is readily obtained.

**Theorem 8** *Consider* (9). *Suppose assumptions* A1-A4 *are satisfied. Suppose hypothesis* H1 *of* Lamma 6 *holds. Suppose the following hypothesis also holds.*

**H2:** There exists a smooth function $Q(x)$ vanishing at $Ex = 0$ and positive elsewhere, locally defined on a neighborhood of $x = 0$, such that the function

$$Y_2(x) = K(x, \tilde{Q}(x), \hat{w}(x, \tilde{Q}(x), \hat{y}(x, \tilde{Q}(x))), \hat{y}(x, \tilde{Q}(x)))$$

is negative definite near $x = 0$, and its Hessian matrix is nonsingular at $x = 0$. Here $\tilde{Q}$ is defined by $\frac{\partial Q}{\partial x} = \tilde{Q}^T E$ with $E^T\tilde{Q}_x = \tilde{Q}_x E$, the function $K : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^p \to \mathbb{R}$ is defined on a neighboorhood of the origin as

$$K(x, p, w, y) = p^T F(x, w + \alpha_1(x), 0) - y^T Y(x, w + \alpha_1(x), 0) + \begin{bmatrix} w \\ -\alpha_2(x) \end{bmatrix}^T R(x) \begin{bmatrix} w \\ -\alpha_2(x) \end{bmatrix}.$$

and the function $\hat{w}(x, p, y)$, respectively $\hat{y}(x, p)$, defined on a neighborhood of $(0, 0, 0)$, respectively $(0, 0)$, satisfies

$$\left(\frac{\partial K(x, p, w, y)}{\partial w}\right)_{w=\hat{w}(x,p,y)} = 0, \quad \hat{w}(0, 0, 0) = 0,$$

respectively,

$$\left(\frac{\partial K(x, p, \hat{w}(x, p, y), y)}{\partial y}\right)_{y=\hat{y}(x,p)} = 0, \quad \hat{y}(0, 0) = 0.$$

*Then, if the equation*

$$\tilde{Q}^T(x)G(x) = \hat{y}(x, \tilde{Q}(x)) \tag{12}$$

*has a smooth solution* $G(x)$ *near* $x = 0$, *the nonlinear* $\mathcal{H}^\infty$ *output feedback control problem is solved by the output feedback controller* (8) *with* $\hat{E} = E$. □

### 4.3 Parameterization of Output Feedback Controllers

For affine nonlinear systems, state space formulae of a family of output feedback controllers are given in Lu and Doyle [6] and Yung *et al.* [14]. Recently, Yung *et al.*[15] have derived a set of parameterized solutions for general nonlinear systems. They have considered both output feedback and state feedback cases. Indeed, we can extend the technique developed in [15] to give a family of nonlinear $\mathcal{H}_\infty$ controllers for differential-algebraic systems.

Motivated by the work of [14][15], we consider the family of controllers described by the following DAEs

$$\begin{aligned} \hat{E}\dot{\xi} &= F(\xi, \alpha_1(\xi), \alpha_2(\xi) + c(\eta)) \\ &+ G(\xi)(y - Y(\xi, \alpha_1(\xi), \alpha_2(\xi) + c(\eta))) \\ &+ \hat{g}_1(\xi)c(\eta) + \hat{g}_2(\xi)d(\eta), \\ E_Q\dot{\eta} &= a(\eta, y - Y(\xi, \alpha_1(\xi), \alpha_2(\xi) + c(\eta))), \\ u &= \alpha_2(\xi) + c(\eta), \end{aligned} \tag{13}$$

where $\xi$ and $\eta$ are defined on some neighborhoods of the origins in $\mathcal{X}_c$ and $\mathbb{R}^q$, respectively. $G(\bullet)$ satisfies (12). $a(\bullet, \bullet)$ and $c(\bullet)$ are smooth functions with $a(0, 0) = 0$ and $c(0) = 0$. $\hat{g}_1(\bullet)$, $\hat{g}_2(\bullet)$ and $d(\bullet)$ are $C^k$ functions ($k \geq 1$). $E_Q$ is a constant matrix, and, in general, is singular. The functions $a(\bullet, \bullet)$, $c(\bullet)$, $\hat{g}_1(\bullet)$, $\hat{g}_2(\bullet)$, $d(\bullet)$, and the matrix $E_Q$ are to-be-determined variables such that the closed-loop system (6) -(13) is dissipative with respect to the supply rate $\gamma^2||w||^2 - ||z||^2$, and is locally asymptotically stable with index one.

Observe first that the DAEs describing the closed-loop system (6)-(13) can be put in the form

$$\begin{aligned} E_a\dot{\mathrm{x}}_a &= F_a(\mathrm{x}_a, w), \\ z &= Z(\mathrm{x}, w, \alpha_2(\xi) + c(\eta)), \end{aligned}$$

where $\mathrm{x}_a = col[\mathrm{x}, \xi, \eta]$, $E_a \triangleq \begin{bmatrix} E & 0 & 0 \\ 0 & \hat{E} & 0 \\ 0 & 0 & E_Q \end{bmatrix}$, $\mathrm{x}_a \triangleq col[\mathrm{x}, \xi, \eta]$ and

$$F_a \triangleq \begin{bmatrix} F(\mathrm{x}, w, \alpha_2(\xi) + c(\eta)) \\ \tilde{F}(\xi, \eta) + G(\xi)Y(\mathrm{x}, w, \alpha_2(\xi) + c(\eta)) + \hat{g}_1(\xi)c(\eta) + \hat{g}_2(\xi)d(\eta) \\ a(\eta, Y(\mathrm{x}, w, \alpha_2(\xi) + c(\eta)) - Y(\xi, \alpha_1(\xi), \alpha_2(\xi) + c(\eta))) \end{bmatrix}.$$

In the above equation,

$$\tilde{F}(\xi, \eta) \triangleq F(\xi, \alpha_1(\mathrm{x}), \alpha_2(\xi) + c(\eta)) - G(\xi)Y(\xi, \alpha_1(\xi), \alpha_2(\xi) + c(\eta)).$$

Consider a Hamiltonian function $J : \mathbb{R}^{2n+q} \times \mathbb{R}^{2n+q} \times \mathbb{R}^r \to \mathbb{R}$ defined as follows.

$$J(\mathrm{x}_a, p_a, w) = p_a^T F_a(\mathrm{x}_a, w) + \begin{bmatrix} w - \alpha_2(\mathrm{x}) \\ \alpha_2(\xi) + c(\eta) - \alpha_2(\mathrm{x}) \end{bmatrix}^T R(\mathrm{x}) \begin{bmatrix} w - \alpha_2(\mathrm{x}) \\ \alpha_2(\xi) + c(\eta) - \alpha_2(\mathrm{x}) \end{bmatrix}. \tag{14}$$

It is easy to check that

$$\left(\frac{\partial^2 J(\mathrm{x}_a, p_a, w)}{\partial w^2}\right)_{(\mathrm{x}_a, p_a, w) = (0,0,0)} = 2(1 - \epsilon_1)(D_{11}^T D_{11} - \gamma^2 I),$$

which is negative definite by A1. Then, by the Implicit Function Theorem, there exists a unique smooth function $\tilde{w}(\mathrm{x}_a, p_a)$, defined on a neighborhood of the origin, satisfying

$$\left(\frac{\partial J(\mathrm{x}_a, p_a, w)}{\partial w}\right)_{w = \tilde{w}(\mathrm{x}_a, p_a)} = 0, \quad \tilde{w}(0, 0) = 0.$$

**Lemma 9** *Consider* (6) *and* (13). *Suppose assumptions* A1-A4 *are satisfied. Suppose hypotheses* H1 *of* Lemma 6 *and* H2 *of* Theorem 8 *hold. Furthermore, suppose that the following hypothesis also holds.*

**H3:** there exists a smooth function $M(\mathrm{x}_a)$, locally defined on a neighborhood of the origin in $\mathbb{R}^{2n+q}$, vanishing at $\mathrm{x}_a = col[E\mathrm{x}, E\mathrm{x}, 0]$ and positive elsewhere which satisfies $\frac{\partial M(\mathrm{x}_a)}{\partial \mathrm{x}_a} = \tilde{M}^T(\mathrm{x}_a)E_a$ with $\tilde{M}_{\mathrm{X}_a}^T(\mathrm{x}_a)E_a = E_a^T \tilde{M}_{\mathrm{X}_a}(\mathrm{x}_a)$, and is such that the function $Y_3(\mathrm{x}_a) = J(\mathrm{x}_a, \tilde{M}(\mathrm{x}_a), \tilde{w}(\mathrm{x}_a, \tilde{M}(\mathrm{x}_a)))$ vanishes at $\mathrm{x}_a = col[\mathrm{x}, \mathrm{x}, 0]$ and is negative elsewhere

*Then the family of controllers* (13) *with* $\hat{E} = E$ *solves the* $\mathcal{H}_\infty$ *output feedback control problem.* □

The previous lemma gives a general form of the output feedback controllers. However, it does not explicitly specify how we can choose the free system parameters $E_Q$, $a(\bullet, \bullet)$ and $c(\bullet)$ to meet the hypothesis in Lemma 9. In the sequel, we give a way to meet the condition in Lemma 9, and in the mean time, to reduce the number of independent variables.

Consider the following DAE.

$$E_Q\dot{\eta} = a(\eta, \bullet). \tag{15}$$

If there exists a smooth function $L(\eta)$, locally defined on a neighborhood of $\eta = 0$, which vanishes at $E_Q\eta = 0$, is positive elsewhere, and satisfies that $\frac{\partial L(\eta)}{\partial \eta} = \tilde{L}(\eta)^T E_Q$ with $E_Q^T \tilde{L}_\eta(\eta) = \tilde{L}_\eta(\eta)^T E_Q$, such that

$$\tilde{L}^T a(\eta, \bullet) < 0,$$

then we can conclude from Theorem 4 that DAE (15) is locally asymptotically stable and has index one. Henceforth, if some further hypotheses are imposed in the above inequality, the condition in Lemma 9 can be met. This is summarized in the following theorem.

**Theorem 10** *Consider* (6) *and* (13). *Suppose assumptions* A1-A4 *are satisfied. Suppose hypotheses* H1 *of* Lemma 6 *and* H2 *of* Theorem 8 *hold. Suppose also that the following hypothesis holds.*

**H4:** There exists a smooth function $L(\eta)$ defined as above such that the function

$$Y_4(\eta, w) = \tilde{L}^T(\eta)a(\eta, Y(0, w, 0)) + \begin{bmatrix} w \\ c(\eta) \end{bmatrix}^T R(0) \begin{bmatrix} w \\ c(\eta) \end{bmatrix}$$

at $w = w^+(\eta)$, viewed as a function of $\eta$, is negative definite near $\eta = 0$, and its Hessian matrix is nonsingular at $\eta = 0$. The function $w^+(\eta)$ is defined on a neighborhood of $\eta = 0$, which satisfies $(\frac{\partial Y_4(\eta, w)}{\partial w})_{w = w^+(\eta)} = 0$ with $w^+(0) = 0$ (This function exists, for $R(0)$ is nonsingular).

*Then, if* $\hat{g}_1(\bullet)$ *and* $\hat{g}_2(\bullet)$ *satisfy*

$$\begin{aligned} \tilde{Q}(\mathrm{x})\hat{g}_1(\mathrm{x}) &= \tilde{Q}(\mathrm{x})(\tfrac{\partial F}{\partial u}(\mathrm{x}, 0, 0) - G(\mathrm{x})\tfrac{\partial Y}{\partial u}(\mathrm{x}, 0, 0)) \\ &\quad + 2\beta^T(\mathrm{x}, 0, 0)r_{12}(\mathrm{x}) - 2(1 + \epsilon_2)\alpha_2^T(\mathrm{x})r_{22}(\mathrm{x}) \end{aligned}$$

*and*

$$\tilde{Q}(\mathrm{x})\hat{g}_2(\mathrm{x}) = -Y^T(\mathrm{x}, \alpha_1(\mathrm{x}) + \beta(\mathrm{x}, 0, 0), 0)$$

*respectively, where*

$$\beta(\mathrm{x},\xi,\eta) = \tilde{w}(\mathrm{x}_a, [\ \tilde{Q}(\mathrm{x}-\xi) \quad -\tilde{Q}(\mathrm{x}-\xi) \quad \tilde{L}(\eta)\ ])$$

*, the family of controllers* (13) *with* $d(\eta) \triangleq \tilde{L}(\eta)$ *solves the* $\mathcal{H}_\infty$ *output feedback control problem.* □

## 5 Converse Result - A Necessary Condition

In this section, we give necessary conditions for a storage function to exist in terms of two Hamilton-Jacobi inequalities.

Suppose that the $\mathcal{H}_\infty$ control problem is solved by the output feedback controller $\Gamma$ which has the following representation

$$\begin{array}{rcl} \hat{E}\dot{\xi} & = & \Phi(\xi, y), \\ u & = & \Theta(\xi) \end{array} \tag{16}$$

and let $U$ be a smooth function satisfying

$$\begin{array}{rl} W(\mathrm{x},\xi,w) & = \begin{bmatrix} \tilde{U}_{\mathrm{x}} & \tilde{U}_{\xi} \end{bmatrix} \begin{bmatrix} F(\mathrm{x},\, w,\, \Theta(\xi)) \\ \Phi(\xi,\, Y(\mathrm{x},\, w,\, \Theta(\xi)) \end{bmatrix} \\ & + \|Z(\mathrm{x},\, w,\, \Theta(\xi))\|^2 - \gamma^2\|w\|^2 \leq 0 \end{array} \tag{17}$$

for all $(\mathrm{x},\, \xi,\, w)$ in a neighborhood of $(0,\, 0,\, 0)$. We make the following assumption.

**(A5)** $\tilde{U}_\xi(\mathrm{x},\xi) = 0$ if and only if $\xi = \ell(\mathrm{x})$ for some smooth function $\ell$ with $\ell(0) = 0$. Furthermore, $\tilde{U}_{\xi\xi}(\mathrm{x},\xi)|_{\xi=\ell(\mathrm{x})}$ is nonsingular.

Define $V(\mathrm{x}) \triangleq S(\mathrm{x},\, \ell(\mathrm{x}))$. We have the following result.

**Theorem 11** *Consider system* (6) *and suppose assumptions* A1-A5 *hold. Suppose that the* $\mathcal{H}_\infty$ *control problem is solved by the output feedback controller* (16). *Suppose that there exists a smooth function* $U(\mathrm{x},\xi)$, *which vanishes at* $E^e\mathrm{x}^e = 0$ *and positive elsewhere with* $E^{eT}\tilde{U}_{\mathrm{X}^e} = \tilde{U}^T_{\mathrm{X}^e}E^e$, *satisfying* (10) *for all* $(\mathrm{x},\, \xi,\, w)$ *in a neighborhood of* $(0,\, 0,\, 0)$. *Then the Hamilton - Jacobi inequalities* $Y_1(\mathrm{x}) \leq 0$ *and*

$$K_\gamma(\mathrm{x}, \tilde{P}(\mathrm{x}), w^*(x, \tilde{P}(\mathrm{x})), y^*(\mathrm{x}, \tilde{P}(\mathrm{x}))) \leq 0$$

*have solutions* $V(\mathrm{x})$ *and, respectively,* $P(\mathrm{x})$ *(with* $\frac{\partial P}{\partial \mathrm{x}} = \tilde{P}^T E$*) given by* $V(\mathrm{x}) = U(\mathrm{x},\, \ell(\mathrm{x})) \geq 0$ *and, respectively,* $P(\mathrm{x}) = U(\mathrm{x},\, 0) \geq 0$. *Furthermore,* $Q(\mathrm{x}) \triangleq P(\mathrm{x}) - V(\mathrm{x}) \geq 0$. □

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