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BOUNDED REAL LEMMA AND H_∞ CONTROL FOR DESCRIPTOR SYSTEMS

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ABSTRACT

The purpose of this paper is to give a necessary and sufficient condition under which for a given plant of descriptor system model there exists a normal, internally stabilizing controller of order no greater than rank$E$ that satisfies a closed-loop H_∞ norm bound. The approach used in this paper is based on a generalized version of Bounded Real Lemma, thus the proofs are simple.

1. INTRODUCTION

H_∞ (sub)optimal control has become one of the most important notions in the field of automatic control theory. It has drawn considerable attention of many researchers from around the world. Although H_∞ control theory has been perfectly developed over the last decade, however, most of the results were developed based on state space equations[6][11][12]. State space models are very useful, but the state variables thus introduced often do not provide a physical meaning[13]. In addition, state space equations cannot represent algebraic restrictions between state variables. Besides, some physical phenomena, like impulse, hysterisis which are important in circuit theory, cannot be treated properly in the state space models[8].

Descriptor systems representation provides a suitable way to handle such problems and it has been proven in the literature that descriptor systems have higher capability in describing a physical system[7][10][15]. In fact, descriptor system models appear more convenient and natural than state space models in large scale systems, economics, networks, power, neural systems and elsewhere [8][9][10].

The control theory based on descriptor system models has been widely developed for many years: Cobb first gave a necessary and sufficient condition for the existence of an optimal solution to linear quadratic optimization problem[2] and also extensively studied the notions of controllability, observability and duality in descriptor systems[3]. Lewis[7], Bender et al.[1] and Takaba et al.[13] constructed different kinds of Riccati equations for solving linear quadratic regulator problems based on certain assumptions. Some excellent results on pole placement[17] and robust control[16], to name only a few, were also obtained.

Recently, Copeland and Safonov used the descriptor-system-like models to solve the singular H_2 and H_∞ control problems in which the plants have pure imaginary(including infinity) poles or zeros [4].

Most recently, Takaba et al.[13] gave solutions to H_∞ control problem for descriptor systems. They dealt with the problem using a J-spectral factorization, thus their proofs were involved. Moreover, only sufficient conditions for solutions to exist were given. Also, only controllers of descriptor system model were presented; this cause difficulties in implementation.

The purpose of this paper is to give a necessary and sufficient condition under which for a given plant of descriptor

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system model there exists a normal, internally stabilizing controller of order no greater than rank$E$ that satisfies a closed-loop H_∞ norm bound. The approach used in this paper is based on a generalized version of Bounded Real Lemma, thus the proofs are simple.

This paper is organized as follows: In section 2 we first review some notions of descriptor systems and give some preliminary results which are useful in our proofs. In section 3, we formulate the H_∞ output feedback control problem for a descriptor system. In section 4, the main results are given. Finally, some concluding remarks are given in section 5.

2. PRELIMINARIES

In this section, we will review some basic notions concerning descriptor systems. Consider a descriptor system described by the state equations

$$E\dot{x} = Ax + Bu \quad (1a)$$
$$y = Cx \quad (1b)$$

where $x \in I\!R^n$ is the state, and $u \in I\!R^m$ and $y \in I\!R^p$ are the input and output signals respectively. A, B, C and D are constant matrices with compatible dimensions and E is a square matrix of rank $r < n$. The pencil $(sE - A)$ is assumed to be regular. It is well known that a descriptor system contains three different modes: finite dynamic modes, impulsive modes and nondynamic modes. For a detailed definition, see [1]. Briefly, suppose that $\{E, A\}$ is regular with rank $E = r < n$ and $q \triangleq \deg \det(sE - A)$. Then $\{E, A\}$ have q finite dynamic modes, $r - q$ impulsive modes and $n - r$ nondynamic modes. Furthermore, if $r = q$, then there exist no impulsive modes and the system is said to be impulse-free.

The stability definition of descriptor systems is similar: $\{E, A\}$ is called stable if there exist no finite dynamic modes in $Re[s] \geq 0$. $\{E, A\}$ is admissible if $\{E, A\}$ is regular, impulse-free and stable.

Based on some basic assumptions, Takaba et al.[14] developed the following theorem related to linear quadratic regulator problems and generalized algebraic Riccati equations(GARE).

Proposition 1 *Consider the descriptor system* (1) *and suppose that the system is regular. Suppose that* $\{E, A, B\}$ *is finite dynamics stabilizable and impulse controllable and* $\{E, A, C\}$ *is finite dynamics detectable and impulse observable. Furthermore, assume that the Hamiltonian system*

$$\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & BB^T \\ -C^TC & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

is regular, impulse-free and has no finite dynamic modes lying on the imaginary axis. Then there exists an admissible solution X *to the GARE*

$$\begin{cases} X^TA + A^TX + C^TC + X^TBB^TX = 0, \\ E^TX = X^TE \end{cases}$$

□

Recall that a solution X to the GARE is called an admissible solution if $\{E, A + BB^TX\}$ is admissible. It is noted that X might not be unique, but $X^TE = E^TX$ is unique.

The following proposition is an extension of Lyapunov stability theorem for descriptor systems and is taken from [14] but with some further modifications.

Proposition 2 *Consider the descriptor system* (1). *Suppose that* $\{E, A\}$ *is regular. Then we have the following* (i) *Suppose that* $\{E, A, C\}$ *is finite dynamics detectable and impulse observable. Then* $\{E, A\}$ *is stable and impulse-free if and only if there exists a matrix* X *satisfying the generalized Lyapunov inequality:*

$$A^TX + X^TA + C^TC \leq 0, \qquad X^TE = E^TX \geq 0$$

(ii) *Suppose there exists a nonsingular matrix* P *satisfying the generalized Lyapunov inequality:*

$$A^TP + P^TA < 0, \qquad P^TE = E^TP \geq 0.$$

Then $\{E, A\}$ *is stable and impulse-free.* □

Proposition 3 *Given two constant matrices* B_2, C_2 *and a positive real number* γ. *Then we have* $\gamma^2 I - B_2^T C_2^T C_2 B_2 > 0$ *if and only if there exists a antistabilizing solution* X^+ *satisfying the ARE*

$$X + X + C_2^TC_2 + \frac{1}{\gamma^2} X B_2 B_2^T X = 0$$

provided that B_2 *and* C_2 *are multiplicable.* □

Proof. Consider a linear dynamical system of the form,

$$\begin{aligned} \dot{x} &= x + B_2 u \\ y &= C_2 x \end{aligned}$$

It is easy to verify that $\gamma^2 I - B_2^T C_2^T C_2 B_2 > 0$ if and only if the system is strictly Bounded Real with an upper bound γ. This completes the proof.

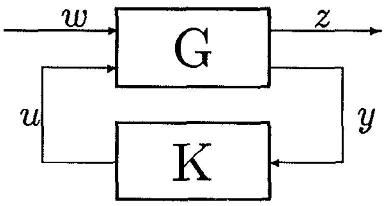

Figure 1: Standard Block Diagram

3. PROBLEM FORMULATION

The system considered in this paper is described by the standard block diagram shown in Figure 1, where G is the plant and K is the controller.

Assume that G has a realization of the form

$$E\dot{x} = Ax + B_1w + B_2u \quad (2a)$$
$$z = C_1x + D_{12}u \quad (2b)$$
$$y = C_2x + D_{21}w \quad (2c)$$

where $x \in I\!R^n$ is the state, and $w \in I\!R^m$ represents a set of exogenous inputs which includes disturbances to be rejected and/or reference commands to be tracked. $z \in I\!R^p$ is the output to be controlled and $y \in I\!R^q$ is the measured output. $u \in I\!R^l$ is the control input. $A, B_1, B_2, C_1, C_2, D_{12}$, and D_{21} are constant matrices with compatible dimensions. $E \in I\!R^{n\times n}$ and $\text{rank}E = r < n$.

The objective of this paper is to find a controller K of the normal form

$$\dot{\xi}_1 = \hat{A}_1\xi_1 + \hat{B}_1 y$$
$$u = \hat{C}_1\xi_1 \quad (2c)$$

where $\hat{A}_1 \in I\!R^{r\times r}$, $\hat{B}_1 \in I\!R^{r\times q}$ and $\hat{C}_1 \in I\!R^{l\times r}$, such that the resulting closed-loop system is internally stable and T_{zw}, the closed-loop system from w to z, has H_∞ norm strictly less than a prescribed positive number γ. Here closed-loop internal stability means that the closed-loop system is regular and impulse-free, and that the states of G and K go to zero from all initial values when $w = 0$.

The following assumptions are made throughout the paper(see also[13]).

Assumptions:

(A1) $\{E, A\}$ is regular.

(A2) $\{E, A, B_2\}$ is finite dynamics stabilizable and impulse controllable.

(A3) $\{E, A, C_2\}$ is finite dynamics detectable and impulse observable.

(A4) $\begin{bmatrix} A - j\omega E & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank $\forall\omega \in I\!R$

(A5) $\begin{bmatrix} A - j\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank $\forall\omega \in I\!R$

(A6) $R_1 \triangleq D_{12}^TD_{12} > 0$, $R_2 \triangleq D_{21}D_{21}^T > 0$.

4. MAIN RESULTS

In this section, we first give a generalized version of Bounded Real Lemma for descriptor systems. Based on the generalized Bounded Real Lemma, a necessary and sufficient condition is provided for the existence of a controller of the form (2c) that achieves closed-loop internal stability and H_∞ norm bound.

All proofs are omitted due to space limitation.

Lemma 4 (Generalized Bounded Real Lemma) *Consider the system* (1) *given by a Weierstrass form:*

$$E = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, A = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \quad (2d)$$

where N is a nilpotent matrix.The following statements are equivalent.

(1) *$\{E, A\}$ is stable, impulse-free (i.e. $N = 0$) and $||G(s)||_\infty < \gamma$, where*

$$G(s) \triangleq C(sE - A)^{-1}B.$$

(2) *$\{E, A\}$ is stable, impulse - free and $\gamma^2 I - B_2^T C_2^T C_2 B_2 > 0$. Furthermore, the Hamiltonian system*

$$\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & \frac{1}{\gamma^2}BB^T \\ -C^TC & -A^T \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix}$$

is regular, impulse-free and has no finite dynamic modes on the imaginary axis.

(3) *There exists an admissible solution to the GARE:*

$$\begin{cases} X^TA + A^TX + C^TC + \frac{1}{\gamma^2}X^TBB^TX = 0 \\ E^TX = X^TE \geq 0 \end{cases}$$

(4) *There exists a nonsingular matrix P, satisfying the generalized algebraic Riccati inequality*

$$\begin{cases} P^TA + A^TP + C^TC + \frac{1}{\gamma^2}P^TBB^TP < 0 \\ E^TP = P^TE \geq 0 \end{cases}$$

□

Remark. The nonsingular matrix P can be further selected to be a block diagonal matrix.

Next, we give a prelimilary result which is essentially taken from [13] and is useful in the subsequent development.

Lemma 5 *Consider* (2). *Suppose that the assumptions* **(A1)** *to* **(A6)** *hold. Suppose also that the following conditions are satisfied.*

(i): *There exists an admissible solution X_∞ to the GARE*

$$\begin{aligned} R_1(X) &= (A - B_2R_1^{-1}D_{12}^TC_1)^TX + X^T(A - B_2R_1^{-1}D_{12}^TC_1) \\ &\quad + C_1^T(I - D_{12}R_1^{-1}D_{12}^T)C_1 \\ &\quad + X^T(\tfrac{1}{\gamma^2}B_1B_1^T - B_2R_1^{-1}B_2^T)X = 0, \\ &\quad E^TX = X^TE \end{aligned} \tag{2e}$$

with $E^TX_\infty = X_\infty^TE \geq 0$.
(ii): *There exists an admissible solution Y_∞ to the GARE*

$$\begin{aligned} R_2(Y) &= (A - B_1D_{21}^TR_2^{-1}C_2)Y + Y^T(A - B_1D_{21}^TR_2^{-1}C_2)^T \\ &\quad + Y^T(\tfrac{1}{\gamma^2}C_1^TC_1 - C_2^TR_2^{-1}C_2)Y \\ &\quad + B_1(I - D_{21}^TR_2^{-1}D_{21})B_1^T = 0, \\ &\quad EY = Y^TE^T \end{aligned} \tag{2f}$$

with $EY_\infty = Y_\infty^TE^T \geq 0$.
(iii): $\rho(X_\infty Y_\infty) < \gamma^2$.

Then the controller

$$\begin{aligned} E\dot{\xi} &= \hat{A}\xi + \hat{B}y \\ u &= \hat{C}\xi \end{aligned} \tag{2g}$$

internally stabilizes G and render $\|T_{zw}\|_\infty < \gamma$, where

$$\begin{aligned} \hat{A} &= A + B_2\hat{C} - \hat{B}C_2 + \tfrac{1}{\gamma^2}(B_1 - \hat{B}D_{21})B_1^TX_\infty \\ \hat{B} &= (I - \tfrac{1}{\gamma^2}Y_\infty^TX_\infty^T)^{-1}(Y_\infty^TC_2^T + B_1D_{21}^T)R_2^{-1} \\ \hat{C} &= -R_1^{-1}(B_2^TX_\infty + D_{12}^TC_1) \triangleq F_\infty \end{aligned}$$

□

We are now in a position to give our main result, which is summarized in the following statements.

Theorem 6 *Consider the system* (2). *Suppose that assumptions* **(A1)**-**(A6)** *hold. Suppose also that the following hypothesis holds.*

Hypothesis (H1): *There exists a constant matrix K such that if we set $u = Ky + v$ then the system (2) would be regular and impulse-free with the new control input v. Rename the input v as u, and now the descriptor system is regular and impulse-free. Hence we can find two nonsingular matrices which transform the system into the following form.*

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} w + \underbrace{\begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}}_{B_2} u \\ z &= \begin{bmatrix} C_{11} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_{12}u \\ y &= \underbrace{\begin{bmatrix} C_{21} & C_{22} \end{bmatrix}}_{C_2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_{21}w \end{aligned} \tag{2h}$$

where A_1 is an $r \times r$ matrix, $r = rankE$.

Then there exists a controller of the form (2c) *achieves closed-loop internal stability and H_∞ norm bound if and only if there exist two matrices $X^- \geq 0$ and $Y^- \geq 0$ which are the unique positive semidefinite stabilizing solutions to the AREs*

$$\begin{aligned} &(A_1 - B_{21}R_1^{-1}D_{12}^TC_{11})^TX + X(A_1 - B_{21}R_1^{-1}D_{12}^TC_{11}) \\ &\quad + C_{11}^T(I - D_{12}R_1^{-1}D_{12}^T)C_{11} \\ &\quad + X(\tfrac{1}{\gamma^2}B_{11}B_{11}^T - B_{21}R_1^{-1}B_{21}^T)X = 0, \end{aligned} \tag{2i}$$

and respectively

$$\begin{aligned} &(A_1 - B_{11}D_{21}^TR_2^{-1}C_{21})Y + Y(A_1 - B_{11}D_{21}^TR_2^{-1}C_{21})^T \\ &\quad + Y(\tfrac{1}{\gamma^2}C_{11}^TC_{11} - C_{21}^TR_2^{-1}C_{21})Y \\ &\quad + B_{11}(I - D_{21}^TR_2^{-1}D_{21})B_{11}^T = 0, \end{aligned} \tag{2j}$$

and satisfy $\rho(X^-Y^-) < \gamma^2$. In this case, one solution is given in the form (2c) *with*

$$\begin{aligned} \hat{A}_1 &= A_1 + B_{21}\hat{C}_1 - \hat{B}_1C_{21} + \tfrac{1}{\gamma^2}(B_{11} - \hat{B}_1D_{21})B_{11}^TX^- \\ \hat{B}_1 &= (I - \tfrac{1}{\gamma^2}Y^-X^-)^{-1}(Y^-C_{21}^T + B_{11}D_{21}^T)R_2^{-1} \\ \hat{C}_1 &= -R_1^{-1}(B_{21}^TX^- + D_{12}^TC_{11}) \end{aligned}$$

□

5. CONCLUDING REMARKS

We have given a necessary and sufficient condition for the solvability of the H_∞ output feedback control problem for a descriptor system. It is noted that a preliminary transformation is needed. We make some comments on this:

1. In hypothesis **(H1)**, a static output feedback matrix K can always be found since the descriptor system is assumed to be impulse controllable and observable(see [5])
2. Form (2h) is just a standard Weierstrass decomposition [5] with $B_{12} = 0$ and $C_{12} = 0$.
3. $B_{12} = 0$ implies that the disturbance has no influence on the nondynamic modes. If $B_{12} \neq 0$, we can form

an equivalent problem with $B_{12} = 0$ by considering the output signal of the form

$$z = \begin{bmatrix} C_{11} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_{11}w + D_{12}u.$$

The general H_∞ control problem in which $D_{11} \neq 0$ for a descriptor system is not done in this paper and is left for future work.

4. Similarly, the case in which $C_{12} \neq 0$ does not pose any problem since it is easy to form an equivalent problem with $C_{12} = 0$ by suitably adjusting D_{12}.

Finally, it should be pointed out that the controller thus found is in the normal state space model. It is much easier to be implemented than a controller in the descriptor system model.

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