Robust H_∞ control for uncertain linear time-invariant descriptor systems

J.-C.Huang, H.-S.Wang and F.-R.Chang

Abstract: The H_∞ control problem for uncertain descriptor systems with time-invariant norm-bound uncertainty in the state matrix is considered. Necessary and sufficient conditions for robust H_∞ control of descriptor systems by state feedback and dynamic output feedback are derived. The design results are expressed in terms of generalised algebraic Riccati inequalities, which may be considered an extension of results given in recent literature. Explicit formulae for controllers which solve the corresponding problems are provided. The generalised algebraic Riccati inequalities approach used is based on a version of the bounded real lemma for descriptor systems, thus making the given proofs simpler.

1 Introduction

The H_∞ control of descriptor systems has received increasing interest in recent years [1, 2]. Although H_∞ control theory for linear systems is well established, its counterpart in descriptor systems has only recently been investigated. The descriptor system models, as mentioned in [3, 4], can more aptly describe a physical system than the linear system models. However, descriptor systems contain three different modes, namely finite dynamic modes, impulsive modes and nondynamic modes (see [5] for a detailed definition). This accounts for why the H_∞ control problem for linear descriptor systems is more intricate than the corresponding one for linear state-space systems.

While most control designs are based on nominal models, modeling errors and system uncertainties are inevitable. For preciseness, a design technique must accommodate these errors and uncertainties to be practically feasible. Recent, interest has focused on the robust H_∞ control of linear systems with parameter uncertainties [6–11]. This investigation continues this line of research to consider the robust H_∞ control problem for descriptor systems.

This investigation first proposes a descriptor state feedback H_∞ control design, which robustly stabilises a given descriptor system with norm-bounded parameter uncertainty in the state matrix. The robust H_∞ control problem is then solved via a dynamic output feedback controller. It is shown that the robust H_∞ control problem can be transformed to a standard H_∞ control problem for an auxiliary descriptor system. This investigation largely focuses on deriving necessary and sufficient conditions for the robust stabilisation of uncertain descriptor systems via two types of feedback configuration: state feedback and dynamic output feedback. The desired feedback control law is then constructed by solving certain constant-coefficient generalised algebraic Riccati inequalities (GARI). The feedback design technique developed herein can be viewed as an extension of the existing H_∞ control results for linear descriptor systems to the case of uncertain descriptor systems.

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2 Problem statement and definitions

Consider a class of uncertain descriptor systems described by the following set of differential algebraic equations.

$$\mathbf{E}\dot{\mathbf{x}} = (\mathbf{A} + \Delta\mathbf{A}(\sigma))\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u},$$
$$\mathbf{z} = \mathbf{C}_1\mathbf{x} + \mathbf{D}_{12}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w} \tag{1}$$

where $\mathbf{x} \in \Re^n$ is the descriptor state variable, $\mathbf{u} \in \Re^m$ is the control input, $\mathbf{w} \in \Re^q$ is the exogenous input, $\mathbf{y} \in \Re^p$ is the measured output, and $\mathbf{z} \in \Re^s$ is the controlled output. The matrix $\mathbf{E} \in \Re^{n \times n}$ has rank $\mathbf{r}(\leq \mathbf{n})$ and the other matrices have appropriate sizes. The parameter uncertainty of the system is denoted by $\Delta\mathbf{A}(\sigma)$, where σ is an uncertain parameter vector. With regard to the work of [9, 10], the uncertainty considered herein is time-invariant and has the following form:

$$\Delta\mathbf{A}(\sigma) = \mathbf{G}\mathbf{L}(\sigma)\mathbf{H} \tag{2}$$

where $\mathbf{L}(\sigma) \in \Re^{i \times j}$ is a norm-bounded uncertain matrix, and $\mathbf{G}, \mathbf{H}$ are known matrices of appropriate dimensions. Assume that the uncertain matrix $\mathbf{L}(\sigma)$ is such that $\mathbf{L}(\sigma)^T\mathbf{L}(\sigma) \leq \rho^2\mathbf{I}$ with $\rho > 0$ and $\sigma \in \Sigma$, where Σ is a compact set. Moreover, we assume that given any matrix $\mathbf{L}{:}\mathbf{L}^T\mathbf{L} \leq \rho^2\mathbf{I}$, there exists $\sigma \in \Sigma$ such that $\mathbf{L} = \mathbf{L}(\sigma)$. We will recall some notions and some preliminary results concerning descriptor systems to motivate the technique discussed in this investigation.

Definition 2.1:

(i) A pencil **sE-A** (or a pair {**E**, **A**}) is *regular* if det(**sE-A**) is not identically zero.

(ii) For a regular pencil **sE-A**, the finite eigenvalues of **sE-A** are called *finite modes* of $\{\mathbf{E}, \mathbf{A}\}$. Assume that $\mathbf{E}\mathbf{v}_1 = 0$. Then, the infinite eigenvalues associated with the generalised principal vectors $\mathbf{v}_k$ satisfying $\mathbf{E}\mathbf{v}_k = \mathbf{A}\mathbf{v}_{k-1}$, $k = 2,3, \ldots$ are *impulsive modes* of $\{\mathbf{E}, \mathbf{A}\}$.
(iii) A pair $\{\mathbf{E}, \mathbf{A}\}$ is *admissible* if it is regular and has neither impulsive modes nor unstable finite modes.

Definition 2.2: Consider the following unforced and unperturbed system

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w}, \quad \mathbf{z} = \mathbf{C}_1\mathbf{x} \tag{3}$$

Given a scalar $\gamma > 0$, the system in eqn. 3 is stated to be *admissible with disturbance attenuation* γ if it satisfies the following conditions:

(i) the pair $\{\mathbf{E}, \mathbf{A}\}$ is admissible; and
(ii) the transfer function from exogenous input $\mathbf{w}$ to controlled output $\mathbf{z}$, represented by $\mathbf{T}_w$, satisfies

$$\|\mathbf{T}_w(\mathbf{s})\|_\infty = \|\mathbf{C}_1(\mathbf{sE} - \mathbf{A})^{-1}\mathbf{B}_1\|_\infty < \gamma.$$

The following lemma gives a necessary and sufficient condition for the system to be admissible with disturbance attenuation γ. The condition is characterised by GARI.

Lemma 2.3(1): The unforced and unperturbed system in eqn. 3 is *admissible with disturbance attenuation* γ if and only if

(i) there exists a solution $\mathbf{X} \in \Re^{n \times n}$ such that

$$\mathbf{E}^T\mathbf{X} = \mathbf{X}^T\mathbf{E} \geq 0 \tag{4a}$$

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}^T\mathbf{A} + \frac{1}{\gamma^2}\mathbf{X}^T\mathbf{B}_1\mathbf{B}_1^T\mathbf{X} + \mathbf{C}_1^T\mathbf{C}_1 < 0 \tag{4b}$$

or, equivalently
(ii) there exists a solution $\mathbf{Y} \in \Re^{n \times n}$ such that

$$\mathbf{Y}\mathbf{E}^T = \mathbf{E}\mathbf{Y}^T \geq 0 \tag{5a}$$

$$\mathbf{A}\mathbf{Y}^T + \mathbf{Y}\mathbf{A}^T + \mathbf{B}_1\mathbf{B}_1^T + \frac{1}{\gamma^2}\mathbf{Y}\mathbf{C}_1^T\mathbf{C}_1\mathbf{Y}^T < 0 \tag{5b}$$

Herein, we are concerned with the following notions of admissibility for the unforced uncertain descriptor system:

$$\mathbf{E}\dot{\mathbf{x}} = (\mathbf{A} + \Delta\mathbf{A}(\sigma))\mathbf{x} + \mathbf{B}_1\mathbf{w}, \quad \mathbf{z} = \mathbf{C}_1\mathbf{x} \tag{6}$$

Definition 2.4: The unforced system in eqn. 6 is stated to be *quadratically admissible* for all parameter uncertainties $\Delta\mathbf{A}(\sigma)$ if

(i) there exists a matrix $\mathbf{X} \in \Re^{n \times n}$ such that

$$\mathbf{E}^T\mathbf{X} = \mathbf{X}^T\mathbf{E} \geq 0 \tag{7a}$$

$$(\mathbf{A} + \Delta\mathbf{A}(\sigma))^T\mathbf{X} + \mathbf{X}^T(\mathbf{A} + \Delta\mathbf{A}(\sigma)) < 0 \tag{7b}$$

or, equivalently
(ii) there exists a matrix $\mathbf{Y} \in \Re^{n \times n}$ such that

$$\mathbf{Y}\mathbf{E}^T = \mathbf{E}\mathbf{Y}^T \geq 0 \tag{8a}$$

$$\mathbf{Y}(\mathbf{A} + \Delta\mathbf{A}(\sigma))^T + (\mathbf{A} + \Delta\mathbf{A}(\sigma))\mathbf{Y}^T < 0 \tag{8b}$$

Definition 2.5: Given a scalar $\gamma > 0$, the unforced system described by eqn. 6 is stated to be *quadratically admissible with disturbance attenuation* γ for all parameter uncertainties $\Delta\mathbf{A}(\sigma)$ if

(i) there exists a constant matrix $\mathbf{X} \in \Re^{n \times n}$ such that

$$\mathbf{E}^T\mathbf{X} = \mathbf{X}^T\mathbf{E} \geq 0 \tag{9a}$$

$$(\mathbf{A} + \Delta\mathbf{A}(\sigma))^T\mathbf{X} + \mathbf{X}^T(\mathbf{A} + \Delta\mathbf{A}(\sigma)) + \frac{1}{\gamma^2}\mathbf{X}^T\mathbf{B}_1\mathbf{B}_1^T\mathbf{X} + \mathbf{C}_1^T\mathbf{C}_1 < 0 \tag{9b}$$

or, equivalently
(ii) there exists a constant matrix $\mathbf{Y} \in \Re^{n \times n}$ such that

$$\mathbf{Y}\mathbf{E}^T = \mathbf{E}\mathbf{Y}^T \geq 0 \tag{10a}$$

$$(\mathbf{A} + \Delta\mathbf{A}(\sigma))\mathbf{Y}^T + \mathbf{Y}(\mathbf{A} + \Delta\mathbf{A}(\sigma))^T + \mathbf{B}_1\mathbf{B}_1^T + \frac{1}{\gamma^2}\mathbf{Y}\mathbf{C}_1^T\mathbf{C}_1\mathbf{Y}^T < 0 \tag{10b}$$

The following lemma is useful for the subsequent proofs.

Lemma 2.6: Consider the system in eqn. 6 and a prescribed scalar $\gamma > 0$. For all $\Delta\mathbf{A}(\sigma)$ satisfying eqn. 2:

(i) eqn. 9 holds if and only if there exists a solution $\mathbf{X} \in \Re^{n \times n}$ independent of $\Delta\mathbf{A}(\sigma)$ such that

$$\mathbf{E}^T\mathbf{X} = \mathbf{X}^T\mathbf{E} \geq 0 \tag{11a}$$

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}^T\mathbf{A} + \frac{1}{\gamma^2}\mathbf{X}^T[\mathbf{B}_1\gamma\mathbf{G}][\mathbf{B}_1\gamma\mathbf{G}]^T\mathbf{X} + \begin{bmatrix}\mathbf{C}_1\\ \rho\mathbf{H}\end{bmatrix}^T\begin{bmatrix}\mathbf{C}_1\\ \rho\mathbf{H}\end{bmatrix} < 0 \tag{11b}$$

and
(ii) eqn. 10 holds if and only if there exists a solution $\mathbf{Y} \in \Re^{n \times n}$ independent of $\Delta\mathbf{A}(\sigma)$ such that

$$\mathbf{Y}\mathbf{E}^T = \mathbf{E}\mathbf{Y}^T \geq 0 \tag{12a}$$

$$\mathbf{A}\mathbf{Y}^T + \mathbf{Y}\mathbf{A}^T + [\mathbf{B}_1\gamma\mathbf{G}][\mathbf{B}_1\gamma\mathbf{G}]^T + \frac{1}{\gamma^2}\mathbf{Y}\begin{bmatrix}\mathbf{C}_1\\ \rho\mathbf{H}\end{bmatrix}^T\begin{bmatrix}\mathbf{C}_1\\ \rho\mathbf{H}\end{bmatrix}\mathbf{Y}^T < 0 \tag{12b}$$

Proof: We prove only (i). The proof of (ii) is technically similar.

(*Sufficiency*) Rewrite eqn. 11 as

$$\mathbf{E}^T\mathbf{X} = \mathbf{X}^T\mathbf{E} \geq 0 \tag{13a}$$

$$(\mathbf{A} + \Delta\mathbf{A}(\sigma))^T\mathbf{X} + \mathbf{X}^T(\mathbf{A} + \Delta\mathbf{A}(\sigma)) + \frac{1}{\gamma^2}\mathbf{X}^T\mathbf{B}_1\mathbf{B}_1^T\mathbf{X} + \mathbf{C}_1^T\mathbf{C}_1 + \mathbf{X}^T\mathbf{G}\mathbf{G}^T\mathbf{X} + \rho^2\mathbf{H}^T\mathbf{H} - \Delta\mathbf{A}(\sigma)^T\mathbf{X} - \mathbf{X}^T\Delta\mathbf{A}(\sigma) < 0 \tag{13b}$$

For any matrices $\mathbf{M}$ and $\mathbf{N}$ with appropriate dimensions

$$\mathbf{M}^T\mathbf{N} + \mathbf{N}^T\mathbf{M} \leq \mathbf{M}^T\mathbf{M} + \mathbf{N}^T\mathbf{N} \tag{14}$$

Thus, for any $\Delta\mathbf{A}(\sigma)$ satisfying eqn. 2, we have

$$\begin{aligned}\Delta\mathbf{A}(\sigma)^T\mathbf{X} + \mathbf{X}^T\Delta\mathbf{A}(\sigma) &= \mathbf{H}^T\mathbf{L}(\sigma)^T\mathbf{G}^T\mathbf{X} + \mathbf{X}^T\mathbf{G}\mathbf{L}(\sigma)\mathbf{H}\\ &\leq \mathbf{X}^T\mathbf{G}\mathbf{G}^T\mathbf{X} + \mathbf{H}^T\mathbf{L}(\sigma)^T\mathbf{L}(\sigma)\mathbf{H}\\ &\leq \mathbf{X}^T\mathbf{G}\mathbf{G}^T\mathbf{X} + \rho^2\mathbf{H}^T\mathbf{H}.\end{aligned} \tag{15}$$

Eqn. 9 then follows immediately by substituting eqn. 15 into eqn. 13.

(*Necessity*) Assume that there exists a matrix $\mathbf{X} \in \Re^{n \times n}$ such that eqn. 9 holds, that is

$$\mathbf{E}^T\mathbf{X} = \mathbf{X}^T\mathbf{E} \geq 0 \tag{16a}$$

$$\Gamma := \mathbf{A}^T\mathbf{X} + \mathbf{X}^T\mathbf{A} + \frac{1}{\gamma^2}\mathbf{X}^T\mathbf{B}_1\mathbf{B}_1^T\mathbf{X} + \mathbf{C}_1^T\mathbf{C}_1 < -\mathbf{H}^T\mathbf{L}(\sigma)^T\mathbf{G}^T\mathbf{X} - \mathbf{X}^T\mathbf{G}\mathbf{L}(\sigma)\mathbf{H} \tag{16b}$$

for all $\Delta\mathbf{A}(\sigma)$ satisfying eqn. 2. Then, for any nonzero $\mathbf{x} \in \Re^n$

$$\mathbf{x}^T\Gamma\mathbf{x} < -\mathbf{x}^T\mathbf{H}^T\mathbf{L}(\sigma)^T\mathbf{G}^T\mathbf{X}\mathbf{x} - \mathbf{x}^T\mathbf{X}^T\mathbf{G}\mathbf{L}(\sigma)\mathbf{H}\mathbf{x} \tag{17a}$$

that is

$$\mathbf{x}^T\Gamma\mathbf{x} \leq \sup\{\mathbf{x}^T\mathbf{H}^T\mathbf{L}(\sigma)^T\mathbf{G}^T\mathbf{X}\mathbf{x} + \mathbf{x}^T\mathbf{X}^T\mathbf{G}\mathbf{L}(\sigma)\mathbf{H}\mathbf{x} : \mathbf{L}(\sigma)^T\mathbf{L}(\sigma) \leq \rho^2\mathbf{I}, \rho > 0\} - \varepsilon \tag{17b}$$

for sufficiently small $\varepsilon > 0$. Now observe that, for any nonzero $\mathbf{x} \in \Re^n$

$$\begin{aligned} 0 &\leq \|\mathbf{G}^T\mathbf{X}\mathbf{x} - \mathbf{L}(\sigma)\mathbf{H}\mathbf{x}\|^2 \\ &\leq \mathbf{x}^T\mathbf{X}^T\mathbf{G}\mathbf{G}^T\mathbf{X}\mathbf{x} + \rho^2\mathbf{x}^T\mathbf{H}^T\mathbf{H}\mathbf{x} \\ &\quad - \mathbf{x}^T\mathbf{X}^T\mathbf{G}\mathbf{L}(\sigma)\mathbf{H}\mathbf{x} - \mathbf{x}^T\mathbf{H}^T\mathbf{L}(\sigma)^T\mathbf{G}^T\mathbf{X}\mathbf{x} \end{aligned} \tag{18}$$

which implies that

$$\mathbf{x}^T[\mathbf{H}^T\mathbf{L}(\sigma)^T\mathbf{G}^T\mathbf{X} + \mathbf{X}^T\mathbf{G}\mathbf{L}(\sigma)\mathbf{H}]\mathbf{x} \leq \mathbf{x}^T[\mathbf{X}^T\mathbf{G}\mathbf{G}^T\mathbf{X} + \rho^2\mathbf{H}^T\mathbf{H}]\mathbf{x} \tag{19}$$

Substituting eqn. 19 into eqn. 17*a*, we have

$$\mathbf{x}^T\Gamma\mathbf{x} + \mathbf{x}^T[\mathbf{X}^T\mathbf{G}\mathbf{G}^T\mathbf{X} + \rho^2\mathbf{H}^T\mathbf{H}]\mathbf{x} < 0 \tag{20}$$

Therefore, eqn. 11 holds. This completes the proof of part (i) of Lemma 2.6. □

Corollary 2.7: The unforced system in eqn. 6 is *quadratically admissible with disturbance attenuation* γ if and only if the unperturbed auxiliary descriptor system

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + [\mathbf{B}_1 \quad \gamma\mathbf{G}]\tilde{\mathbf{w}}, \quad \tilde{\mathbf{z}} = \begin{bmatrix} \mathbf{C}_1 \\ \rho\mathbf{H} \end{bmatrix}\mathbf{x} \tag{21}$$

is *admissible with disturbance attenuation* γ, where $\tilde{\mathbf{w}} \in \Re^{q+i}$ is the disturbance input, $\tilde{\mathbf{z}} \in \Re^{s+j}$ is the auxiliary system output which is to be controlled, and the other variables are defined as in eqn. 1.

Proof: Lemma 2.3 leads us to conclude that if eqn. 11 is valid, then the auxiliary system in eqn. 21 is *admissible with disturbance attenuation* γ. By Definition 2.5, the perturbed system in eqn. 1 is *quadratically admissible with disturbance attenuation* γ if eqn. 9 is valid. However, by Lemma 2.6, eqn. 9 holds if and only if eqn. 11 holds. This simply implies that eqn. 21 is *admissible with disturbance attenuation* γ if and only if eqn. 1 is *quadratically admissible with disturbance attenuation* γ, which result completes the proof. □

3 Full state feedback control

Consider the uncertain descriptor system in eqn. 1. In this Section, perfect descriptor state information is assumed to be available for feedback, that is $\mathbf{y} = \mathbf{x}$. Herein, we are concerned with designing a fixed static descriptor state feedback law that robustly stabilises the system in eqn. 1, while satisfying an H_∞ performance constraint for all of the possible uncertainties given by eqn. 2. Setting $\mathbf{u} = \mathbf{K}\mathbf{x}$ in eqn. 1 yields the following perturbed closed-loop system:

$$\mathbf{E}_s\dot{\mathbf{x}} = (\mathbf{A}_s + \Delta_s(\sigma))\mathbf{x} + \mathbf{B}_s\mathbf{w}, \quad \mathbf{z} = \mathbf{C}_s\mathbf{x} \tag{22}$$

where $\mathbf{E}_s = \mathbf{E}$, $\mathbf{A}_s = \mathbf{A} + \mathbf{B}_2\mathbf{K}$, $\mathbf{B}_s = \mathbf{B}_1$, $\mathbf{C}_s = \mathbf{C}_1 + \mathbf{D}_{12}\mathbf{K}$, and $\Delta\mathbf{A}_s(\sigma) = \Delta\mathbf{A}(\sigma)$. The following theorem is readily obtained. Its proof is a direct consequence of Lemma 2.6, and is therefore omitted.

Theorem 3.1: Suppose that $\{\mathbf{E}, \mathbf{A}\}$ is admissible. Consider a positive real number γ. The perturbed closed-loop system in eqn. 22 is *quadratically admissible via linear state feedback with disturbance attenuation* γ for all $\Delta\mathbf{A}(\sigma)$ if and only if there exists a constant matrix $\mathbf{P} \in \Re^{n \times n}$ independent of $\Delta\mathbf{A}(\sigma)$ such that

$$\mathbf{E}_s^T\mathbf{P} = \mathbf{P}^T\mathbf{E}_s \geq 0 \tag{23a}$$

$$\mathbf{A}_s^T\mathbf{P} + \mathbf{P}^T\mathbf{A}_s + \frac{1}{\gamma^2}\mathbf{P}^T[\mathbf{B}_s \quad \gamma\mathbf{G}][\mathbf{B}_s \quad \gamma\mathbf{G}]^T\mathbf{P} + \begin{bmatrix} \mathbf{C}_s \\ \rho\mathbf{H} \end{bmatrix}^T \begin{bmatrix} \mathbf{C}_s \\ \rho\mathbf{H} \end{bmatrix} < 0 \tag{23b}$$

eqn. 23 is difficult to solve and contains an unknown variable $\mathbf{K}$ and a matrix $\mathbf{P}$ yet to be determined. Later, we will seek an equivalent condition which transforms GARI (eqn. 23) to another constant-coefficient GARI and obtain an explicit formula of $\mathbf{K}$. To do this we must propose the following.

Proposition 3.2: Consider eqn. 22. The following statements are equivalent.

(i) $\{\mathbf{E}_s, \mathbf{A}_s + \mathbf{G}\mathbf{L}(\sigma)\mathbf{H}\}$ is admissible and $\|\mathbf{C}_s(s\mathbf{E}_s - \mathbf{A}_s - \mathbf{G}\mathbf{L}(\sigma)\mathbf{H})^{-1}\mathbf{B}_s\|_\infty < \gamma$;

(ii) $\{\mathbf{E}_s, \mathbf{A}_s\}$ is admissible and

$$\left\| \begin{bmatrix} \mathbf{C}_s \\ \rho H \end{bmatrix} (sE_s - A_s)^{-1}[B_s \quad \gamma G] \right\|_\infty < \gamma;$$

(iii) there exists a matrix $\mathbf{P} \in \Re^{n \times n}$ satisfying the following GARI:

$$\mathbf{E}_s^T\mathbf{P} = \mathbf{P}^T\mathbf{E}_s \geq 0 \tag{24a}$$

$$\mathbf{A}_s^T\mathbf{P} + \mathbf{P}^T\mathbf{A}_s + \frac{1}{\gamma^2}\mathbf{P}^T[\mathbf{B}_s \quad \gamma\mathbf{G}][\mathbf{B}_s \quad \gamma\mathbf{G}]^T\mathbf{P} + \begin{bmatrix} \mathbf{C}_s \\ \rho\mathbf{H} \end{bmatrix}^T \begin{bmatrix} \mathbf{C}_s \\ \rho\mathbf{H} \end{bmatrix} < 0 \tag{24b}$$

Proof: By Corollary 2.7, the perturbed closed-loop system in eqn. 22 is admissible with an $\mathbf{L}_2$-gain $< \gamma$ if and only if the uncertainty-free auxiliary system

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}_s\mathbf{x} + [\mathbf{B}_s \quad \gamma\mathbf{G}]\tilde{\mathbf{w}}, \quad \tilde{\mathbf{z}} = \begin{bmatrix} \mathbf{C}_s \\ \rho\mathbf{H} \end{bmatrix}\mathbf{x} \tag{25}$$

is admissible with an $\mathbf{L}_2$-gain $< \gamma$. Therefore, the equivalence of (i) and (ii) follows directly from Definition 2.2. By Theorem 3.1, (i), or equivalently (ii), holds if and only if (iii) is satisfied. □

Eqn. 23*b* can be written as

$$(\mathbf{A} + \mathbf{B}_2\mathbf{K})^T\mathbf{P} + \mathbf{P}^T(\mathbf{A} + \mathbf{B}_2\mathbf{K}) + \frac{1}{\gamma^2}\mathbf{P}^T\mathbf{B}_1\mathbf{B}_1^T\mathbf{P} + \mathbf{P}^T\mathbf{G}\mathbf{G}^T\mathbf{P} + (\mathbf{C}_1 + \mathbf{D}_{12}\mathbf{K})^T(\mathbf{C}_1 + \mathbf{D}_{12}\mathbf{K}) + \rho^2\mathbf{H}^T\mathbf{H} < 0 \tag{26}$$

Let $\mathbf{R}_1 := \mathbf{D}_{12}^T\mathbf{D}_{12} > 0$ and $\mathbf{K} = -\mathbf{R}_1^{-1}(\mathbf{B}_2^T\mathbf{P} + \mathbf{D}_{12}^T\mathbf{C}_1)$ in eqn. 26, and complete the square to obtain

$$\mathbf{R}_1(\mathbf{P}) := \left(\mathbf{A} - \mathbf{B}_2\mathbf{R}_1^{-1}[\mathbf{D}_{12}^T \quad 0]\begin{bmatrix}\mathbf{C}_1\\ \rho\mathbf{H}\end{bmatrix}\right)^T\mathbf{P} + \mathbf{P}^T\left(\mathbf{A} - \mathbf{B}_2\mathbf{R}_1^{-1}[\mathbf{D}_{12}^T \quad 0]\begin{bmatrix}\mathbf{C}_1\\ \rho\mathbf{H}\end{bmatrix}\right) + \mathbf{P}^T\left(\frac{1}{\gamma^2}[\mathbf{B}_1 \quad \gamma\mathbf{G}][\mathbf{B}_1 \quad \gamma\mathbf{G}]^T - \mathbf{B}_2\mathbf{R}_1^{-1}\mathbf{B}_2^T\right)\mathbf{P} + \mathbf{C}_1^T(\mathbf{I} - \mathbf{D}_{12}\mathbf{R}_1^{-1}\mathbf{D}_{12}^T)\mathbf{C}_1 + \rho^2\mathbf{H}^T\mathbf{H} \tag{27}$$

We state the main result of this Section.

Theorem 3.3: Consider eqn. 22 with $\mathbf{y} = \mathbf{x}$. Suppose that $\{\mathbf{E}, \mathbf{A}\}$ is admissible and $\mathbf{R}_1 := \mathbf{D}_{12}^T\mathbf{D}_{12} > 0$.

Then the perturbed closed-loop system is *quadratically admissible via linear state feedback with disturbance attenuation* γ for all $\Delta\mathbf{A}(\sigma)$ satisfying eqn. 2 if and only if the GARI

$$\mathbf{E}^T\mathbf{P} = \mathbf{P}^T\mathbf{E} \tag{28a}$$

$$\mathbf{R}_1(\mathbf{P}) < 0 \tag{28b}$$

has a constant solution $\mathbf{P}$ with $\mathbf{E}^T\mathbf{P} = \mathbf{P}^T\mathbf{E} \geq 0$. Moreover, when the above condition holds, one such controller is given by

$$\mathbf{u} = \mathbf{K}\mathbf{x} = -\mathbf{R}_1^{-1}(\mathbf{B}_2^T\mathbf{P} + \mathbf{D}_{12}^T\mathbf{C}_1)\mathbf{x} \tag{29}$$

Proof: (Necessity) The proof of the necessity part of the theorem is similar to that given in Theorem 4.2. We therefore omit the proof here.

(*Sufficiency*) Suppose that eqn. 28 holds. Then, using eqn. 29 to close the loop yields the following perturbed closed-loop system

$$\mathbf{E}_c\dot{\mathbf{x}} = (\mathbf{A}_c + \Delta\mathbf{A}_c(\sigma))\mathbf{x} + \mathbf{B}_c\mathbf{w}, \quad \mathbf{z} = \mathbf{C}_c\mathbf{x} \tag{30}$$

where $\mathbf{E}_c = \mathbf{E}$, $\mathbf{A}_c = \mathbf{A} - \mathbf{B}_2\mathbf{R}_1^{-1}(\mathbf{B}_2^T\mathbf{P} + \mathbf{D}_{12}^T\mathbf{C}_1)$, $\mathbf{B}_c = \mathbf{B}_1$, $\mathbf{C}_c = \mathbf{C}_1 - \mathbf{D}_{12}\mathbf{R}_1^{-1}(\mathbf{B}_2^T\mathbf{P} + \mathbf{D}_{12}^T\mathbf{C}_1)$, and $\Delta\mathbf{A}_c(\sigma) = \Delta\mathbf{A}(\sigma)$. Therefore

$$\mathbf{A}_c^T\mathbf{P} + \mathbf{P}^T\mathbf{A}_c + \frac{1}{\gamma^2}\mathbf{P}^T[\mathbf{B}_c \quad \gamma\mathbf{G}][\mathbf{B}_c \quad \gamma\mathbf{G}]^T\mathbf{P} + \begin{bmatrix}\mathbf{C}_c\\ \rho\mathbf{H}\end{bmatrix}^T\begin{bmatrix}\mathbf{C}_c\\ \rho\mathbf{H}\end{bmatrix} = \mathbf{R}_1(\mathbf{P}) < 0 \tag{31}$$

By Proposition 3.2, eqn. 31 implies that $\{\mathbf{E}_c, \mathbf{A}_c\}$ is admissible and

$$\left\|\begin{bmatrix}\mathbf{C}_c\\ \rho\mathbf{H}\end{bmatrix}(s\mathbf{E}_c - \mathbf{A}_c)^{-1}[\mathbf{B}_c \quad \gamma\mathbf{G}]\right\|_\infty < \gamma.$$

This deduction subsequently implies that the perturbed closed-loop system in eqn. 30 is *quadratically admissible with disturbance attenuation* γ. □

4 Dynamic output feedback control

In this Section, the system in eqn. 1 is assumed to satisfy the following assumptions:

(A1) $\{\mathbf{E}, \mathbf{A}\}$ is admissible.
(A2) $\mathbf{R}_1 := \mathbf{D}_{12}^T\mathbf{D}_{12} > 0$.
(A3) $\mathbf{R}_2 := \mathbf{D}_{21}\mathbf{D}_{21}^T > 0$.
(A4) The matrix pencil

$$\begin{bmatrix}\mathbf{A} - j w\mathbf{E} & \mathbf{B}_2\\ \mathbf{C}_1 & \mathbf{D}_{12}\end{bmatrix}$$

has full column rank for all $\mathbf{w} \in \Re$ and is column reduced.

In fact, the assumptions A1–A3 are by no means restrictive. They can be achieved by the loop shifting method (see [12] for details). We consider a dynamic output feedback controller of the following form:

$$\hat{\mathbf{E}}\dot{\xi} = \hat{\mathbf{A}}\xi + \hat{\mathbf{B}}\mathbf{y}, \quad \mathbf{u} = \hat{\mathbf{C}}\xi \tag{32}$$

where $\xi \in \Re^n$, $\hat{\mathbf{E}} \in \Re^{n\times n}$ and $\{\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}\}$ have proper dimensions. Note that the structure of the matrix $\hat{\mathbf{E}}$ may be singular or nonsingular, and equal or not equal to $\mathbf{E}$. Define a change of variable, $\mathbf{e} \equiv \mathbf{x} - \xi$. The closed-loop system can now be written as

$$\mathbf{E}_o\dot{\mathbf{x}}_o = (\mathbf{A}_o + \Delta\mathbf{A}_o(\sigma))\mathbf{x}_o + \mathbf{B}_o\mathbf{w}, \mathbf{z} = \mathbf{C}_o\mathbf{x}_o \tag{33}$$

where $\mathbf{x}_o = \begin{bmatrix}\mathbf{x}\\ \mathbf{x} - \xi\end{bmatrix} \equiv \begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}$, $\mathbf{E}_o = \begin{bmatrix}\mathbf{E} & 0\\ \mathbf{E} - \hat{\mathbf{E}} & \hat{\mathbf{E}}\end{bmatrix}$,

$\mathbf{B}_o = \begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_1 - \hat{\mathbf{B}}\mathbf{D}_{21}\end{bmatrix}$, $\mathbf{C}_o = [\mathbf{C}_1 + \mathbf{D}_{12}\hat{\mathbf{C}} \quad -\mathbf{D}_{12}\hat{\mathbf{C}}]$,

$\mathbf{A}_o = \begin{bmatrix}\mathbf{A} + \mathbf{B}_2\hat{\mathbf{C}} & -\mathbf{B}_2\hat{\mathbf{C}}\\ \mathbf{A} - \hat{\mathbf{A}} + \mathbf{B}_2\hat{\mathbf{C}} - \hat{\mathbf{B}}\mathbf{C}_2 & \hat{\mathbf{A}} - \mathbf{B}_2\hat{\mathbf{C}}\end{bmatrix}$, and

$\Delta\mathbf{A}_o(\sigma) = \begin{bmatrix}\Delta\mathbf{A}(\sigma) & 0\\ \Delta\mathbf{A}(\sigma) & 0\end{bmatrix} = \mathbf{G}_o\mathbf{L}(\sigma)\mathbf{H}_o$, with $\mathbf{G}_o = \begin{bmatrix}\mathbf{G}\\ \mathbf{G}\end{bmatrix}$

and $\mathbf{H}_o = [\mathbf{H} \ 0]$. We obtain the following preliminary result similar to that in the state feedback case.

Proposition 4.1: Consider eqn. 33. The following statements are equivalent:

(i) $\{\mathbf{E}_o, \mathbf{A}_o + \mathbf{G}_o\mathbf{L}(\sigma)\mathbf{H}_o\}$ is admissible and $\|\mathbf{C}_o(s\mathbf{E}_o - \mathbf{A}_o - \mathbf{G}_o\mathbf{L}(\sigma)\mathbf{H}_o)^{-1}\mathbf{B}_o\|_\infty < \gamma$;
(ii) $\{\mathbf{E}_o, \mathbf{A}_o\}$ is admissible and

$$\left\|\begin{bmatrix}\mathbf{C}_o\\ \rho\mathbf{H}_o\end{bmatrix}(s\mathbf{E}_o - \mathbf{A}_o)^{-1}[\mathbf{B}_o \quad \gamma\mathbf{G}_o]\right\|_\infty < \gamma;$$

(iii) there exists a matrix $\mathbf{P} \in \Re^{n\times n}$ such that

$$\mathbf{E}_o^T\mathbf{P} = \mathbf{P}^T\mathbf{E}_o \geq 0 \tag{34a}$$

$$\mathbf{A}_o^T\mathbf{P} + \mathbf{P}^T\mathbf{A}_o + \frac{1}{\gamma^2}\mathbf{P}^T[\mathbf{B}_o \quad \gamma\mathbf{G}_o][\mathbf{B}_o \quad \gamma\mathbf{G}_o]^T\mathbf{P} + \begin{bmatrix}\mathbf{C}_o\\ \rho\mathbf{H}_o\end{bmatrix}^T\begin{bmatrix}\mathbf{C}_o\\ \rho\mathbf{H}_o\end{bmatrix} < 0 \tag{34b}$$

Proof: Follows directly from Proposition 3.2.

The main result of this Section is stated.

Theorem 4.2: Suppose that A1–A4 hold. Consider the system in eqn. 1. The following statements are equivalent:

(i) Given $\gamma > 0$, there exists a dynamic output feedback controller such that the closed-loop system in eqn. 33 is *quadratically admissible with disturbance attenuation* γ.

(ii) (*a*) There exists a matrix $\mathbf{X} \in \Re^{n \times n}$ such that

$$\mathbf{E}^T\mathbf{X} = \mathbf{X}^T\mathbf{E} \geq 0 \tag{35a}$$

$$\begin{aligned}\Delta_X \equiv & \left(\mathbf{A} - \mathbf{B}_2\mathbf{R}_1^{-1}[\mathbf{D}_{12}^T \quad 0]\begin{bmatrix}\mathbf{C}_1 \\ \rho\mathbf{H}\end{bmatrix}\right)^T \mathbf{X} \\ & + \mathbf{X}^T\left(\mathbf{A} - \mathbf{B}_2\mathbf{R}_1^{-1}[\mathbf{D}_{12}^T \quad 0]\begin{bmatrix}\mathbf{C}_1 \\ \rho\mathbf{H}\end{bmatrix}\right) \\ & + \frac{1}{\gamma^2}\mathbf{X}^T([\mathbf{B}_1 \quad \gamma\mathbf{G}][\mathbf{B}_1 \quad \gamma\mathbf{G}]^T - \gamma^2\mathbf{B}_2\mathbf{R}_1^{-1}\mathbf{B}_2^T)\mathbf{X} \\ & + \begin{bmatrix}(\mathbf{I} - \mathbf{D}_{12}\mathbf{R}_1^{-1}\mathbf{D}_{12}^T)\mathbf{C}_1 \\ \rho\mathbf{H}\end{bmatrix}^T \begin{bmatrix}(\mathbf{I} - \mathbf{D}_{12}\mathbf{R}_1^{-1}\mathbf{D}_{12}^T)\mathbf{C}_1 \\ \rho\mathbf{H}\end{bmatrix} \\ & < 0\end{aligned} \tag{35b}$$

(*b*) There exists a matrix $\mathbf{Y} \in \Re^{n \times n}$ such that

$$\mathbf{E}\mathbf{Y}^T = \mathbf{Y}\mathbf{E}^T \geq 0 \tag{36a}$$

$$\begin{aligned}\Delta_Y \equiv & \left(\mathbf{A} - [\mathbf{B}_1 \quad \gamma\mathbf{G}]\begin{bmatrix}\mathbf{D}_{21}^T \\ 0\end{bmatrix}\mathbf{R}_2^{-1}\mathbf{C}_2\right)\mathbf{Y}^T \\ & + \mathbf{Y}\left(\mathbf{A} - [\mathbf{B}_1 \quad \gamma\mathbf{G}]\begin{bmatrix}\mathbf{D}_{21}^T \\ 0\end{bmatrix}\mathbf{R}_2^{-1}\mathbf{C}_2\right)^T \\ & + [\mathbf{B}_1(\mathbf{I} - \mathbf{D}_{21}^T\mathbf{R}_2^{-1}\mathbf{D}_{21}) \quad \gamma\mathbf{G}][\mathbf{B}_1(\mathbf{I} - \mathbf{D}_{21}^T\mathbf{R}_2^{-1}\mathbf{D}_{21}) \quad \gamma\mathbf{G}]^T \\ & + \frac{1}{\gamma^2}\mathbf{Y}\left(\begin{bmatrix}\mathbf{C}_1 \\ \rho\mathbf{H}\end{bmatrix}^T\begin{bmatrix}\mathbf{C}_1 \\ \rho\mathbf{H}\end{bmatrix} - \gamma^2\mathbf{C}_2^T\mathbf{R}_2^{-1}\mathbf{C}_2\right)\mathbf{Y}^T < 0\end{aligned} \tag{36b}$$

(*c*) $\rho(\mathbf{Y}\mathbf{X}^T) < \gamma^2$, where $\rho(\cdot)$ denotes the spectral radius. (37)

Moreover, when the conditions are satisfied, one such controller of the form of eqn 32 is given by

$$\hat{\mathbf{E}} = \mathbf{E}, \hat{\mathbf{B}} = \left(\mathbf{Z}\mathbf{C}_2^T + \left(\mathbf{I} + \frac{1}{\gamma^2}\mathbf{Z}\mathbf{X}^T\right)\mathbf{B}_1\mathbf{D}_{21}^T\right)\mathbf{R}_2^{-1} \tag{38a}$$

$$\begin{aligned}\hat{\mathbf{C}} = \mathbf{F} &= -\mathbf{R}_1^{-1}\left(\mathbf{B}_2^T\mathbf{X} + [\mathbf{D}_{12}^T \quad 0]\begin{bmatrix}\mathbf{C}_1 \\ \rho\mathbf{H}\end{bmatrix}\right) \\ &= -\mathbf{R}_1^{-1}(\mathbf{B}_2^T\mathbf{X} + \mathbf{D}_{12}^T\mathbf{C}_1)\end{aligned} \tag{38b}$$

$$\begin{aligned}\hat{\mathbf{A}} &= \mathbf{A} + \mathbf{B}_2\hat{\mathbf{C}} - \hat{\mathbf{B}}\mathbf{C}_2 \\ & \quad + \frac{1}{\gamma^2}([\mathbf{B}_1\gamma\mathbf{G}] - \hat{\mathbf{B}}[\mathbf{D}_{21} \quad 0])[\mathbf{B}_1 \quad \gamma\mathbf{G}]^T\mathbf{X} \\ &= \mathbf{A} + \mathbf{B}_2\hat{\mathbf{C}} - \hat{\mathbf{B}}\mathbf{C}_2 + \frac{1}{\gamma^2}((\mathbf{B}_1 - \hat{\mathbf{B}}\mathbf{D}_{21})\mathbf{B}_1^T + \gamma^2\mathbf{G}\mathbf{G}^T)\mathbf{X}\end{aligned} \tag{38c}$$

where

$$\mathbf{Z} = \left(\mathbf{I} - \frac{1}{\gamma^2}\mathbf{Y}\mathbf{X}^T\right)^{-1}\mathbf{Y} = \mathbf{Y}\left(\mathbf{I} - \frac{1}{\gamma^2}\mathbf{X}^T\mathbf{Y}\right)^{-1} \tag{39}$$

Proof: See Appendix (Section 8).

Remark 4.3: The results in [1, 2] can be regarded as special cases of theorem 4.2 if we let $\Delta\mathbf{A}(\sigma) \equiv 0$. Our method is more transparent than the method of [1], since the controller is explicitly formulated by the solutions of two GARIs. The present formulation involves only two variables to be determined. It contrasts with the condition given in [1], which involves two unknown parameters and two variables to be determined. Furthermore, our result has wider application than the results of [2], since our approach does not depend on the assumptions A2–A4 which were made in that paper.

5 Conclusions

This investigation has considered the problem of robust H_∞ control for a class of uncertain linear time-invariant descriptor systems. Algebraic conditions that characterise quadratic admissibility with disturbance attenuation conditions for uncertain descriptor systems are presented. Descriptor state feedback control and descriptor dynamic output feedback control designs are then proposed. According to our results, the robust H_∞ control design problem is equivalent to a standard H_∞ control problem for an auxiliary descriptor system. In both feedback configurations, necessary and sufficient conditions are obtained such that the closed-loop systems satisfy a prescribed H_∞-norm disturbance attenuation constraint. An efficient method for computing solutions to the Riccati inequalities is being investigated.

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7 References

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8 Appendix

Proof of Theorem 4.2: In view of Proposition 4.1, we need only show that the auxiliary system

$$\mathbf{E}_o\dot{\mathbf{x}}_o = \mathbf{A}_o\mathbf{x}_o + [\mathbf{B}_o \quad \gamma\mathbf{G}_o]\tilde{\mathbf{w}}, \quad \tilde{\mathbf{z}} = \begin{bmatrix} \mathbf{C}_o \\ \rho\mathbf{H}_o \end{bmatrix}\mathbf{x}_o \tag{40}$$

is internally stable with an $\mathbf{L}_2$ gain $<\gamma$, where $\{\mathbf{E}_o, \mathbf{A}_o, \mathbf{B}_o, \mathbf{C}_o\}$ and $\mathbf{G}_o$, $\mathbf{H}_o$ are defined as in eqn. 33.

(*Necessity*) We require the following lemma in the intermediate stage (see [1] for proof).

Lemma 8.1: Consider eqn. 33. Assume that there exists a controller of the form of eqn. 32 such that $\{\mathbf{E}_o, \mathbf{A}_o\}$ is admissible and

$$\left\| \begin{bmatrix} C_o \\ \rho H_o \end{bmatrix} (sE_o - A_o)^{-1}[B_o \quad \gamma G_o] \right\|_\infty < \gamma.$$

Then the following conditions hold:

(*a*) There exist a descriptor state feedback matrix $\mathbf{K}$ and a matrix $\mathbf{X}$ such that

$$\mathbf{E}^T\mathbf{X} = \mathbf{X}^T\mathbf{E} \geq 0 \tag{41a}$$

$$(\mathbf{A}+\mathbf{B}_2\mathbf{K})^T\mathbf{X} + \mathbf{X}^T(\mathbf{A}+\mathbf{B}_2\mathbf{K}) + \frac{1}{\gamma^2}\mathbf{X}^T[\mathbf{B}_1 \quad \gamma\mathbf{G}][\mathbf{B}_1 \quad \gamma\mathbf{G}]^T\mathbf{X} + \begin{bmatrix} \mathbf{C}_1+\mathbf{D}_{12}\mathbf{K} \\ \rho\mathbf{H} \end{bmatrix}^T \begin{bmatrix} \mathbf{C}_1+\mathbf{D}_{12}\mathbf{K} \\ \rho\mathbf{H} \end{bmatrix} < 0 \tag{41b}$$

(*b*) There exist an output injection matrix $\mathbf{L}$ and a matrix $\mathbf{Y}$ such that

$$\mathbf{E}\mathbf{Y}^T = \mathbf{Y}\mathbf{E}^T \geq 0 \tag{42a}$$

$$(\mathbf{A}+\mathbf{L}\mathbf{C}_2)\mathbf{Y}^T + \mathbf{Y}(\mathbf{A}+\mathbf{L}\mathbf{C}_2)^T + [\mathbf{B}_1+\mathbf{L}\mathbf{D}_{21} \quad \gamma\mathbf{G}][\mathbf{B}_1+\mathbf{L}\mathbf{D}_{21} \quad \gamma\mathbf{G}]^T + \frac{1}{\gamma^2}\mathbf{Y}\begin{bmatrix} \mathbf{C}_1 \\ \rho\mathbf{H} \end{bmatrix}^T \begin{bmatrix} \mathbf{C}_1 \\ \rho\mathbf{H} \end{bmatrix}\mathbf{Y}^T < 0 \tag{42b}$$

(*c*) $$\rho(\mathbf{Y}\mathbf{X}^T) < \gamma^2 \tag{42c}$$

In (1), the condition (*c*) is $\mathbf{E}^T(\gamma^2\mathbf{Y}^{-T} - \mathbf{X}) \geq 0$, which implies $(\gamma^2\mathbf{Y}^{-T} - \mathbf{X})$ being nonsingular in their work. Here we imply the nonsingularity of $(\gamma^2\mathbf{Y}^{-T} - \mathbf{X})$ by an alternative expression $\rho(\mathbf{Y}\mathbf{X}^T) < \gamma^2$. We can now prove the necessary. Assume that (i) holds. By Lemma 8.1, eqn. 41 has a solution $\mathbf{X}$. Rewrite eqn. 41 as

$$\mathbf{E}^T\mathbf{X} = \mathbf{X}^T\mathbf{E} \geq 0 \tag{43a}$$

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}^T\mathbf{A} + \frac{1}{\gamma^2}\mathbf{X}^T\mathbf{B}_1\mathbf{B}_1^T\mathbf{X} + \mathbf{X}^T\mathbf{G}\mathbf{G}^T\mathbf{X} + \rho^2\mathbf{H}^T\mathbf{H} + \mathbf{C}_1^T\mathbf{C}_1$$
$$\mathbf{K}^T\mathbf{B}_2^T\mathbf{X} + \mathbf{X}^T\mathbf{B}_2\mathbf{K} + \mathbf{K}^T\mathbf{D}_{12}^T\mathbf{C}_1 + \mathbf{C}_1^T\mathbf{D}_{12}\mathbf{K} + \mathbf{K}^T\mathbf{R}_1\mathbf{K} < 0 \tag{43b}$$

Notably, eqn. 43*b* can be rewritten as

$$\begin{aligned} &\mathbf{A}^T\mathbf{X} + \mathbf{X}^T\mathbf{A} + \frac{1}{\gamma^2}\mathbf{X}^T\mathbf{B}_1\mathbf{B}_1^T\mathbf{X} + \mathbf{X}^T\mathbf{G}\mathbf{G}^T\mathbf{X} + \rho^2\mathbf{H}^T\mathbf{H} + \mathbf{C}_1^T\mathbf{C}_1 \\ &\quad - (\mathbf{X}^T\mathbf{B}_2 + \mathbf{C}_1^T\mathbf{D}_{12})\mathbf{R}_1^{-1}(\mathbf{X}^T\mathbf{B}_2 + \mathbf{C}_1^T\mathbf{D}_{12})^T \\ &\quad < -(\mathbf{X}^T\mathbf{B}_2 + \mathbf{C}_1^T\mathbf{D}_{12} + \mathbf{K}^T\mathbf{R}_1)\mathbf{R}_1^{-1} \\ &\qquad (\mathbf{X}^T\mathbf{B}_2 + \mathbf{C}_1^T\mathbf{D}_{12} + \mathbf{K}^T\mathbf{R}_1)^T \leq 0 \end{aligned} \tag{44}$$

The last inequality in eqn. 44 relies on Assumption A2, $\mathbf{R}_1 > 0$. Thus, matrix $\mathbf{X}$ satisfies eqn. 35. By a similar argument, it can be shown that eqn. 36 has a solution $\mathbf{Y}$. Again, by Lemma 8.1, the spectral radius condition $\rho(\mathbf{Y}\mathbf{X}^T) < \gamma^2$ holds.

(*Sufficiency*) By assuming that (ii) holds. The parameters given by eqn. 38 are well defined so we can use them to construct a controller. We must show that a controller thus constructed makes $\{\mathbf{E}_0, \mathbf{A}_0\}$ admissible and

$$\left\| \begin{bmatrix} C_0 \\ \rho H_0 \end{bmatrix} (sE_0 - A_0)^{-1}[B_0 \quad \gamma G_0] \right\|_\infty < \gamma$$

simultaneously. We require the following theorem (see (2) for proof).

Theorem 8.2: Assume that Assumptions A1, A2, and A4 hold and GARI (eqn. 35) has a solution $\mathbf{X}$. Then, there exist a real number ε_0, and a family of symmetric matrices $S_\varepsilon < 0$, such that

$$\mathbf{E}^T\mathbf{X}_\varepsilon = \mathbf{X}_\varepsilon^T\mathbf{E} \geq 0 \tag{45a}$$

$$\begin{aligned} \mathbf{S}_\varepsilon = &(\mathbf{A} - \mathbf{B}_2\mathbf{R}_1^{-1}\mathbf{D}_{12}^T\mathbf{C}_1)^T\mathbf{X}_\varepsilon \\ &+ \mathbf{X}_\varepsilon^T(\mathbf{A} - \mathbf{B}_2\mathbf{R}_1^{-1}\mathbf{D}_{12}^T\mathbf{C}_1) + \rho^2\mathbf{H}^T\mathbf{H} \\ &+ \frac{1}{\gamma^2}\mathbf{X}_\varepsilon^T([\mathbf{B}_1 \quad \gamma\mathbf{G}][\mathbf{B}_1 \quad \gamma\mathbf{G}]^T - \gamma^2\mathbf{B}_2\mathbf{R}_1^{-1}\mathbf{B}_2^T)\mathbf{X}_\varepsilon \\ &+ \mathbf{C}_1^T(\mathbf{I} - \mathbf{D}_{12}\mathbf{R}_1^{-1}\mathbf{D}_{12}^T)\mathbf{C}_1 < 0 \end{aligned} \tag{45b}$$

has a solution $\mathbf{X}_\varepsilon$. Moreover, $lim_{\varepsilon\to 0}\mathbf{S}_\varepsilon = 0$, and in this case, $\mathbf{X}_\varepsilon = \mathbf{X}_\infty$. $\mathbf{X}_\infty$ is an admissible solution of the GARE

$$\mathbf{E}^T\mathbf{X} = \mathbf{X}^T\mathbf{E} \geq 0 \tag{46a}$$

$$\begin{aligned} \Delta_x \equiv &\left(\mathbf{A} - \mathbf{B}_2\mathbf{R}_1^{-1}[\mathbf{D}_{12}^T \quad 0]\begin{bmatrix} \mathbf{C}_1 \\ \rho\mathbf{H} \end{bmatrix}\right)^T \mathbf{X} \\ &+ \mathbf{X}^T\left(\mathbf{A} - \mathbf{B}_2\mathbf{R}_1^{-1}[\mathbf{D}_{12}^T \quad 0]\begin{bmatrix} \mathbf{C}_1 \\ \rho\mathbf{H} \end{bmatrix}\right) \\ &+ \frac{1}{\gamma^2}\mathbf{X}^T([\mathbf{B}_1 \quad \gamma\mathbf{G}][\mathbf{B}_1 \quad \gamma\mathbf{G}]^T - \gamma^2\mathbf{B}_2\mathbf{R}_1^{-1}\mathbf{B}_2^T)\mathbf{X} \end{aligned} \tag{46b}$$

$$+ \begin{bmatrix} (\mathbf{I} - \mathbf{D}_{12}\mathbf{R}_1^{-1}\mathbf{D}_{12}^T)\mathbf{C}_1 \\ \rho\mathbf{H} \end{bmatrix}^T \begin{bmatrix} (\mathbf{I} - \mathbf{D}_{12}\mathbf{R}_1^{-1}\mathbf{D}_{12}^T)\mathbf{C}_1 \\ \rho\mathbf{H} \end{bmatrix} = 0$$

Remark 8.3: The above theorem is a modified version of the one given in [2]. However, the proof is essentially the same. Since $\rho(\mathbf{Y}\mathbf{X}^T) < \gamma^2$, the matrix $\mathbf{Z}$ in eqn. 39 is well-defined and satisfies $\mathbf{E}\mathbf{Z}^T = \mathbf{Z}\mathbf{E}^T \geq 0$. Now, by theorem 8.2, we can choose a matrix $\mathbf{X}_\varepsilon$ such that Δ_X is arbitrarily small.

Hence, there exists a pair of matrices $\{\mathbf{X}, \mathbf{Y}\}$ such that $\Delta_Y - 1/\gamma^2 \mathbf{Y}\Delta_X \mathbf{Y}^T < 0$. Observe that

$$\begin{aligned}
&\Delta_Y - \frac{1}{\gamma^2}\mathbf{Y}\Delta_X\mathbf{Y}^T \\
&= \left(\mathbf{I} - \frac{1}{\gamma^2}\mathbf{Y}\mathbf{X}^T\right) \\
&\quad \cdot \Big[(\tilde{\mathbf{A}} - \mathbf{B}_1\mathbf{D}_{21}^T\mathbf{R}_2^{-1}\tilde{\mathbf{C}}_2)\mathbf{Z}^T + \mathbf{Z}(\tilde{\mathbf{A}} - \mathbf{B}_1\mathbf{D}_{21}^T\mathbf{R}_2^{-1}\tilde{\mathbf{C}}_2)^T \\
&\quad + \tilde{\mathbf{B}}_1\tilde{\mathbf{B}}_1^T + \frac{1}{\gamma^2}\mathbf{Z}(\mathbf{F}^T\mathbf{R}_1\mathbf{F} - \gamma^2\tilde{\mathbf{C}}_2^T\mathbf{R}_2^{-1}\tilde{\mathbf{C}}_2)\mathbf{Z}^T\Big] \\
&\quad \cdot \left(\mathbf{I} - \frac{1}{\gamma^2}\mathbf{X}\mathbf{Y}^T\right) < 0
\end{aligned} \tag{47}$$

where

$$\begin{aligned}
\tilde{\mathbf{A}} &= \mathbf{A} + \frac{1}{\gamma^2}(\mathbf{B}_1\mathbf{B}_1^T + \gamma^2\mathbf{G}\mathbf{G}^T)\mathbf{X}, \\
\tilde{\mathbf{B}}_1 &= [\,\mathbf{B}_1(\mathbf{I} - \mathbf{D}_{21}^T\mathbf{R}_2^{-1}\mathbf{D}_{21}) \quad \gamma\mathbf{G}\,] \\
\tilde{\mathbf{C}}_2 &= \mathbf{C}_2 + \frac{1}{\gamma^2}[\,\mathbf{D}_{21} \quad 0\,][\,\mathbf{B}_1 \quad \gamma\mathbf{G}\,]^T\mathbf{X} = \mathbf{C}_2 + \frac{1}{\gamma^2}\mathbf{D}_{21}\mathbf{B}_1^T\mathbf{X} \\
\mathbf{F} &= -\mathbf{R}_1^{-1}\left(\mathbf{B}_2^T\mathbf{X} + [\,\mathbf{D}_{12}^T \quad 0\,]\begin{bmatrix} \mathbf{C}_1 \\ \rho\mathbf{H} \end{bmatrix}\right) \\
&= -\mathbf{R}_1^{-1}(\mathbf{B}_2^T\mathbf{X} + \mathbf{D}_{12}^T\mathbf{C}_1)
\end{aligned}$$

The above discussion shows that condition (ii) implies that there exists a matrix $\mathbf{Z}$ satisfying the following GARI:

$$\mathbf{E}\mathbf{Z}^T = \mathbf{Z}\mathbf{E}^T \geq 0 \tag{48a}$$

$$\begin{aligned}
&(\tilde{\mathbf{A}} - \mathbf{B}_1\mathbf{D}_2^T\mathbf{R}_2^{-1}\tilde{\mathbf{C}}_2)\mathbf{Z}^T + \mathbf{Z}(\tilde{\mathbf{A}} - \mathbf{B}_1\mathbf{D}_{21}^T\mathbf{R}_2^{-1}\tilde{\mathbf{C}}_2)^T + \tilde{\mathbf{B}}_1\tilde{\mathbf{B}}_1^T \\
&+ \frac{1}{\gamma^2}\mathbf{Z}(\mathbf{F}^T\mathbf{R}_1\mathbf{F} - \gamma^2\tilde{\mathbf{C}}_2^T\mathbf{R}_2^{-1}\tilde{\mathbf{C}}_2)\mathbf{Z}^T < 0
\end{aligned} \tag{48b}$$

Rewrite eqn. 48 as

$$\mathbf{E}\mathbf{Z}^T = \mathbf{Z}\mathbf{E}^T \geq 0 \tag{49a}$$

$$\mathbf{A}_\alpha\mathbf{Z}^T + \mathbf{Z}\mathbf{A}_\alpha^T + \mathbf{B}_\alpha\mathbf{B}_\alpha^T + \frac{1}{\gamma^2}\mathbf{Z}\mathbf{C}_\alpha^T\mathbf{C}_\alpha\mathbf{Z}^T < 0 \tag{49b}$$

where $\mathbf{A}_\alpha = \tilde{\mathbf{A}} - \mathbf{B}_1\,\mathbf{D}_{21}^T\,\mathbf{R}_2^{-1}\,\tilde{\mathbf{C}}_2 - \mathbf{Z}\,\tilde{\mathbf{C}}_2^T\,\mathbf{R}_2^{-1}\tilde{\mathbf{C}}_2$,

$$B_\alpha = \lfloor\,\tilde{\mathbf{B}}_1 \quad \mathbf{Z}\tilde{\mathbf{C}}_2^T\mathbf{R}_2^{-1}\mathbf{D}_2\,\rfloor,$$

and $C_\alpha = \mathbf{R}_1^{1/2}\mathbf{F}$.

By Definition 2.2 and Lemma 2.3, we can conclude that $\{\mathbf{E}, \mathbf{A}_\alpha\}$ is admissible and $\|C_\alpha(s\mathbf{E} - \mathbf{A}_\alpha)^{-1}\mathbf{B}_\alpha\|_\infty < \gamma$. By duality, we conclude that $\{\mathbf{E}^T, \mathbf{A}_\alpha^T\}$ is admissible and $\|\mathbf{B}_\alpha^T(s\mathbf{E}^T - \mathbf{A}_\alpha^T)^{-1}\mathbf{C}_\alpha^T\| < \gamma$. This result subsequently implies that the GARI

$$\mathbf{E}^T\mathbf{W} = \mathbf{W}^T\mathbf{E} \geq 0 \tag{50a}$$

$$\mathbf{A}_\alpha^T\mathbf{W} + \mathbf{W}^T\mathbf{A}_\alpha + \frac{1}{\gamma^2}\mathbf{W}^T\mathbf{B}_\alpha\mathbf{B}_\alpha^T\mathbf{W} + \mathbf{C}_\alpha^T\mathbf{C}_\alpha < 0 \tag{50b}$$

has a solution $\mathbf{W}$. Let

$$\mathbf{P}_0 := \begin{bmatrix} \mathbf{X} & 0 \\ 0 & \mathbf{W} \end{bmatrix}, \quad \mathbf{E}_0^T\mathbf{P}_0 = \mathbf{P}_0^T\mathbf{E}_0 \tag{51}$$

A lengthy but routine calculation shows that $\mathbf{P}_0$ is a solution to the GARI

$$\mathbf{E}_0^T\mathbf{P}_0 = \mathbf{P}_0^T\mathbf{E}_0 \geq 0 \tag{52a}$$

$$\begin{aligned}
&\mathbf{A}_0^T\mathbf{P}_0 + \mathbf{P}_0^T\mathbf{A}_0 + \frac{1}{\gamma^2}\mathbf{P}_0^T[\,\mathbf{B}_0 \quad \gamma\mathbf{G}_0\,][\,\mathbf{B}_0 \quad \gamma\mathbf{G}_0\,]^T\mathbf{P}_0 \\
&+ \begin{bmatrix} \mathbf{C}_0 \\ \rho\mathbf{H}_0 \end{bmatrix}^T \begin{bmatrix} \mathbf{C}_0 \\ \rho\mathbf{H}_0 \end{bmatrix} < 0
\end{aligned} \tag{52b}$$

Then, by Proposition 4.1, eqn. 52 implies that the controller given by eqn. 38 is a dynamic output feedback controller that internally stabilises the system in eqn 1 and renders the closed-loop system in eqn. 33 strictly contractile with an $\mathbf{L}_2$-gain $< \gamma$. This result completes the proof of Theorem 4.2. □