Recursive Digital Filter Design in the Complex Domain Using an Efficient Method

Ju-Hong Lee and Charng-Kann Chen

Department of Electrical Engineering
National Taiwan University
Taipei, Taiwan, R. O. C.

Abstract

This paper presents a generalized Ellacott-Williams (GEW) algorithm for the design of recursive digital filters in the complex domain. Two design procedures which are two modified versions of the original Ellacott-Williams algorithm are first developed. At each iteration, linear complex Chebyshev approximation problems are formulated based on the least square error criterion to find the solutions for the numerator and denominator. The approximation problems can be solved by using a weighted least squares (WLS) algorithm recently proposed by the authors. This provides an efficient design technique since the heavy computational burden of using linear programming algorithm can be solved.

The main differences between these two design procedures are as follows. At each iteration, the first procedure finds the numerator and denominator simultaneously. It ensures the stability of the designed filter by adjusting the increments of the numerator and denominator simultaneously to locate the poles of the filter inside the unit circle. The second procedure finds the numerator and denominator separately and only adjusts the increment of the denominator to ensure all the poles of designed filter inside the unit circle. The proposed GEW design algorithm is then a combination of the two procedures. Computer simulation results show that the proposed GEW algorithm has better design capabilities than existing methods.

I. Introduction

In many applications of digital filters, we consider not only the filter's magnitude response, but the filter's phase response. For example, digital filters with constant group delay phase other than exactly linear phase are usually required in the design of digital phase network. Therefore, a complex approximation problem arises when we design digital filters to approximate arbitrary magnitude, phase, or group delay. Several algorithms have been proposed to solve this problem for recursive digital filter design in the complex domain. The most successful one among them is proposed by Chen and Parks in [2]. However, they employ a complex linear approximation tool presented in [3] to modify the Ellacott-Williams (EW) algorithm [1] and then use a conventional linear programming method to solve the resulting linear complex approximation problem. This method does not guarantee that the best approximation solution can be obtained even in the local optimal sense. Moreover, it is not an efficient way to use any linear programming method to solve the resulting linear complex Chebyshev approximation problem.

In this paper, we present a generalized Ellacott-Williams (GEW) algorithm to enhance the original EW algorithm's capability for this design problem. The novel WLS approach proposed in [4] is utilized in conjunction with the GEW algorithm to provide a systematic way for iteratively adjusting the required error weighting function during the design process. It has been reported in the literature that the WLS technique will produce equiripple design if a suitable least squares frequency response weighting function is used. Therefore, the linear programming method originally required in [2] for solving the resulting linear Chebyshev approximation problem can be avoided by using the proposed algorithm.

II. The Proposed Design Method

Let the transfer function of the recursive filter with order M/N (i.e., M zeros and N poles) be given as

$$H(z) = \frac{p(z)}{q(z)} = \frac{\sum_{m=0}^{M} a_m z^{-m}}{\sum_{n=0}^{N} b_n z^{-n}}, \quad \text{with } b_0 = 1. \tag{1}$$

Then the design problem is to find the filter coefficients $\{a_m, b_n\}$, such that the stable filter $H(z)|_{z=e^{j\omega}}$ will optimally approximate a desired complex frequency response $D(e^{j\omega})$ in the Chebyshev sense. That is, we want to find $p(z)$, $q(z)$, such that

$$\|D(e^{j\omega}) - \frac{p(e^{j\omega})}{q(e^{j\omega})}\| \tag{2}$$

will be minimized, where $\|\cdot\|$ denotes the Chebyshev norm. In the literature, there is no systematic approach for finding the best approximation solution for (2). Several authors resorted to find a local best approximation (LBA) solution instead of the global best approximation (GBA) solution. In the following, we describe the proposed design method briefly. The detailed theoretical results will be presented in a forthcoming paper. It has been shown that the Ellacott-Williams (EW) algorithm of [1] is useful for solving the rational Chebyshev approximation problem in the complex domain. By using this algorithm, an LBA solution can always be guaranteed. We modify the EW algorithm to enhance its capability for solving the above design problem.

At the initial step, we use the denominator of an Nth order Butterworth filter as the initial polynomial $q_0(e^{j\omega})$ for $q(e^{j\omega})$. The minimization problem (2) can be turned into the following problem

$$\min \sum_{\omega} W(\omega)|D(e^{j\omega})q_0(e^{j\omega}) - p_0(e^{j\omega})|^2. \tag{3}$$

Let $D(e^{j\omega})q_0(e^{j\omega}) = R_0(\omega) + jI_0(\omega)$, then the coefficient vector $\mathbf{a}=[a_0, a_1, \cdots, a_M]^T$ of p_0 can be found by solving the following Toeplitz symmetry linear matrix equation

$$\mathbf{T}\ \mathbf{a} = \mathbf{Z} \tag{4}$$

where the elements of $\mathbf{T}$ and $\mathbf{Z}$ are given as

$$\begin{aligned} t_{ij} &= \sum_{\omega} W(\omega)\cos((i-j)\omega) \ , 0 \le i,j \le M \\ z_i &= \sum_{\omega} W(\omega)(R_0(\omega)\cos i\omega \ - I_0(\omega)\sin i\omega) \ , 0 \le i \le M. \end{aligned} \tag{5}$$

0-7803-0593-0/92 $3.00 1992 IEEE

The WLS algorithm of [3] can be used to iteratively adjust the required error weighting function $W(\omega)$ and provide a good approximation to the minimax solution. Next, we present the design procedures for finding the coefficient vector of $p(e^{j\omega})$ and $q(e^{j\omega})$ at the kth iteration.

A. Procedure 1:

At the kth iteration, the original EW algorithm finds the increment polynomials δp_k and δq_k such the Chebyshev error

$$\|D - \frac{p_k}{q_k} - \frac{q_k \delta p_k - p_k \delta q_k}{q_k^2}\| \quad (6)$$

is minimized. We reformulate this problem as the equivalent least-square problem

$$\min_{a'_m, b'_n} \sum_\omega W(\omega)|D - \frac{p_k}{q_k} + \frac{p_k}{q_k^2}(b'_1 e^{-j\omega} + \cdots + b'_N e^{-jN\omega}) - \frac{1}{q_k}(a'_0 + a'_1 e^{-j\omega} + \cdots + a'_M e^{-jM\omega})|^2 \quad (7)$$

where $W(\omega)$ is the required error weighting function. p_k and q_k denote the p and q polynomials at the kth iteration, respectively. The $\{a'_m, b'_n\}$ represent the coefficients of the increment ploynomials δp_k and δq_k, respectively. Let $W(\omega)/|q_k^2| = W'(\omega)$, $E = Dq_k - p_k = Er + jEi$, $p_k/q_k = R + jI$, the above problem becomes

$$\min_{a'_m, b'_m} \sum_\omega W'(\omega)|Er + jEi + (R + jI)(\sum_{n=1}^{N} b'_n \cos n\omega - j \sum_{n=1}^{N} b'_n \sin n\omega) - (\sum_{n=0}^{M} a'_n \cos n\omega - j \sum_{n=0}^{M} a'_n \sin n\omega)|^2 \quad (8)$$

Taking the derivatives of the object function of (8) with respect to a'_m and b'_n, respectively, and letting the derivative to be zero, we obtain:

$$\sum_{i=0}^{M} a'_i (\sum_\omega W'(\omega)\cos(m-i)\omega) - \sum_{i=1}^{N} b'_i (\sum_\omega W'(\omega)(R\cos(m-i)\omega - I\sin(m-i)\omega)) = \sum_\omega W'(\omega)(Er\cos m\omega - Ei \sin m\omega) \quad \text{for } m = 0, 1, \cdots, M \quad (9)$$

and

$$\sum_{i=0}^{M} a'_i (\sum_\omega (W'(\omega) R\cos(n-i)\omega + I\sin(n-i)\omega)) - \sum_{i=1}^{N} b'_i (\sum_\omega W'(\omega)(R^2 + I^2)\cos(n-i)\omega) = \sum_\omega W'(\omega)((ErR + EiI)\cos n\omega - (ErI - EiR)\sin n\omega) \quad \text{for } n = 1, \cdots, N. \quad (10)$$

Putting (9) and (10) in matrix form yield

$$\begin{bmatrix} \mathbf{C_1} & \mathbf{C_2} \\ \mathbf{C_2^T} & \mathbf{C_3} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ -\mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{F_1} \\ \mathbf{F_2} \end{bmatrix} \quad (11)$$

where $\mathbf{C_1}$ is a Toeplitz symmetric matrix with size $(M+1) \times (M+1)$ and elements given as

$$c^1_{i,j} = \sum_\omega W'(\omega)\cos(i-j)\omega, \quad (12)$$

$\mathbf{C_2}$ is an $(M+1) \times N$ matrix with equal elements given as

$$c^2_{i,j} = \sum_\omega W'(\omega)(R\cos(i-j)\omega - I\sin(i-j)\omega), \quad (13)$$

on each diagonal. $\mathbf{C_3}$ is a Toeplitz symmetric matrix with size $N \times N$ and elements as

$$c^3_{i,j} = \sum_\omega W'(\omega)(R^2 + I^2)\cos(i-j)\omega, \quad (14)$$

$\mathbf{F_1}$ is an $(M+1) \times 1$ vector with elements given as

$$f^1_j = \sum_\omega W'(\omega)(Er\cos j\omega - Ei\sin j\omega), \quad (15)$$

$\mathbf{F_2}$ is an $N \times 1$ vector with elements given as

$$f^2_j = \sum_\omega W'(\omega)((ErR + EiI)\cos j\omega - (ErI - EiR)\sin j\omega), \quad (16)$$

where $\mathbf{A}$ and $\mathbf{B}$ are two vectors containing the coeffiecients $\{a'_0, a'_1, \cdots, a'_M\}$ and $\{b'_1, \cdots, b'_N\}$, respectively. Therefore, only $2M + 2N + 1$ and $M + N + 1$ real numbers must be computed to form the $\mathbf{C}$ matrix and the $\mathbf{F}$ vector, respectively in (11) at each iteration. Again, the WLS algorithm of [3] can be utilized to solve the minimization problem of (8). After obtaining the increment polynomials δp_k and δq_k, we perform a line search to find the best real parameter t_k such that the following Chebyshev error

$$\|D - \frac{p_k + t_k \delta p_k}{q_k + t_k \delta q_k}\| \quad (17)$$

is minimized. To solve the minimization problem of (17) without constraint on the value t_k, we use the Nelder-Meade simplex algorithm [5] which is basically a nonlinear optimization algorithm and performs very well for single variable nonlinear function. From our experience, the Nelder-Meade algorithm can find the optimal t_k efficiently. At this point, the procedure is similar to the original EM algorithm except that the problem of rational Chebyshev approximation problem in the complex domain is reformulated as an equivalent weighted least square approximation problem. Therefore, an LBA solutioncan always be guaranteed as shown in [1]. Moreover, the required suitable error weighting function can be systematically sought by our WLS algorithm. This leads to the advantage of saving computational complexity over the method of using linear programming algorithm, such as [2].

Considering the stability of the designed filter at the kth iteration, we shall find such a t_k that all the zeros of the $q_{k+1} = q_k + t_k \delta q_k$ lie inside the unit circle. A simple approach to achieve this purpose is to test the zeros' positions of $q_{k+1} = q_k + t_k \delta q_k$. Then we set t_k to $t_k/2$ if there exists any zero outside the unit circle. The testing procedure is repeated until all the zeros are inside the unit circle. Next, we consider the descent property of the controlling process of zero's position. From (17), the real parameter t_k is found such that

$$\|D - \frac{p_k + t_k \delta p_k}{q_k + t_k \delta q_k}\| \le \|D - \frac{p_k}{q_k}\| \quad (18)$$

Since the fact that

$$\frac{p_k + t_k \delta p_k}{q_k + t_k \delta q_k} = \frac{p_k}{q_k} + t_k \frac{q_k \delta p_k - p_k \delta q_k}{q_k^2} + o(t_k) \text{ as } t_k \to 0, \quad (19)$$

we have from (18) that

$$\|D - \frac{p_k}{q_k} - t_k \frac{q_k \delta p_k - p_k \delta q_k}{q_k^2} - o(t_k)\| \leq \|D - \frac{p_k}{q_k}\|. \quad (20)$$

This leads to the result that

$$\|D - \frac{p_k}{q_k} - \frac{t_k}{2} \frac{q_k \delta p_k - p_k \delta q_k}{q_k^2} - o(\frac{t_k}{2})\| \leq \|D - \frac{p_k}{q_k}\| \quad (21)$$

That is,

$$\|D - \frac{p_k + (t_k/2)\delta p_k}{q_k + (t_k/2)\delta q_k}\| \leq \|D - \frac{p_k}{q_k}\|. \quad (22)$$

(22) reveals that the controlling process of zero's position does not affect the descent property of the original EW algorithm.

Since an extremely small t_k receals that there is nearly no change between the solutions of the kth iteration and the (k+1)th iteration, The Procedure 1 can be stopped if the value of t_k is less than a preset number. In our computer simulation, we use 10^{-5} as the preset number. From the design process of Procedure 1, it should be noted that the controlling process of zero's position adjusts the value of t_k. Therefore, the value of t_k which provides a stable designed filter would not equal the optimal value of t_k obtained from the line search. This affects the optimality of the designed filter. To alleviate this effect, we propose the Procedure 2 as follows.

B. Procedure 2:

At the kth iteration, we set the δp_k to 0 in the minimization problem of (6) and reformulate this problem as the following equivalent least-square problem

$$\min_{b'_n} \sum_\omega W(\omega)|D - \frac{p_k}{q_k} + \frac{p_k}{q_k^2}(b'_1 e^{-j\omega} + \cdots + b'_N e^{-jN\omega})|^2. \quad (23)$$

to find the optimal δq_k first. Let $W(\omega)/|q_k^2| = W'(\omega)$, $E = Dq_k - p_k = Er + jEi$, $p_k/q_k = R + jI$, the above problem becomes

$$\min_{b'_n} \quad \sum_\omega W'(\omega)|Er + jEi + (R + jI)(\sum_{n=1}^{N} b'_n \cos n\omega - j\sum_{n=1}^{N} b'_n \sin n\omega)|^2 \quad (24)$$

Taking the derivatives of (24) with respect to b'_n and setting the derivative to zero, we obtain the following equations

$$\begin{aligned} &- \sum_{i=1}^{N} b'_i(\sum_\omega W'(\omega)(R^2 + I^2)\cos(n-i)\omega) \\ &= \sum_\omega W'(\omega)((ErR + EiI)\cos n\omega - (ErI - EiR)\sin n\omega) \\ &\qquad \text{for } n = 1, \cdots, N. \end{aligned} \quad (25)$$

In matrix form, (25) can be written as

$$-\mathbf{C_3 B} = \mathbf{F_2} \quad (26)$$

where $\mathbf{C_3}, \mathbf{B}, \mathbf{F_2}$ are the same as those of (11). Therefore, the minimization problem can also be solved by using the WLS technique of [3]. After obtaining the increment polynomials δq_k, we perform line search using the Nelder-Meade algorithm to find the optimal parameter t_k such that $q_{k+1} = q_k + t_k \delta q_k$ satisfies that

$$\|D - \frac{p_k}{q_k + t_k \delta q_k}\| \quad (27)$$

is minimized. This procedure proceeds with solving the following least-square problem to find the required δp_k with fixed q_{k+1}

$$\min_{a'_n} \sum_\omega W(\omega)|D - \frac{p_k}{q_{k+1}} - \frac{1}{q_{k+1}}(a'_0 + a'_1 e^{-j\omega} + \cdots + a'_M e^{-jM\omega})|^2. \quad (28)$$

Again, we let $W(\omega)/|q_{k+1}^2| = W'(\omega)$, $E = Dq_{k+1} - p_k = Er + jEi$, then (28) becomes

$$\mathbf{C_1 A} = \mathbf{F_1}, \quad (29)$$

where $\mathbf{C_1}, \mathbf{B}$, and $\mathbf{F_1}$ are the same as those of (11). Therefore, the minimization problem of (28) can also be solved by using the novel WLS technique of [3]. After obtaining δp_k, we set $p_{k+1} = p_k + \delta p_k$. Next, the t_k is adjusted as in the Procedure 1 to ensure that all the zeros of $q_{k+1} = q_k + t_k \delta q_k$ lie inside the unit disk. Finally, we compute the Chebyshev error e(k+1) associated with the polynomials p_{k+1} and q_{k+1}. The procedure 2 will be stopped if (e(k)-e(k+1))/e(k) is less than a preset number. In our computer simulation, the preset small number is 10^{-3}.

Considering the descent property of this design procedure. We note that

$$\begin{aligned} \frac{p_k + t'_k \delta p_k}{q_k + t_k \delta q_k} &= \frac{p_k}{q_k} + \frac{\delta p_k}{q_k} t'_k + \frac{-p_k \delta q_k}{q_k^2} t_k + O(t_k t'_k) + \\ &\quad o(t'_k) + o(t_k) \text{ as } t_k \text{ and } t'_k \to 0. \end{aligned} \quad (30)$$

Hence, the descent property will not affected by controlling process of the zero's position. Moreover, we neglect the term $O(t_k t'_k)$ to obtain a linear approximation during the optimization process of finding the increment polynomials δp_k and δq_k. As a result, the interaction between the numerator and denominator during the optimization process is also eliminated. This affects the optimality of the designed filter.

To enhance the capabilities of the above two design procedures, we present a generalized Ellacott-Williams (GEW) algorithm for designing recursive digital filters in the complex domain. The proposed GEW algorithm is summarized as follows.

Step 1: Using Procedure 1 to find a soluion.

Step 2: Using the solution of *Step 1* as an initial guess and performing Procedure 2 to obtain a design solution.

Step 3: If the solution of *Step 2* satisfies the stopping criterion of Procedure 1, then the design process terminates. Otherwise, using the solution of *Step 2* as an initial guess and performing Procedure 1 again to obtain an improved design solution.

Step 4: If the solution of *Step 3* satisfies the stopping criterion of Procedure 2, then the design process terminates. Otherwise, using the solution of *Step 3* as an initial guess and performing Procedure 2 again to obtain an improved design solution.

Step 5: Go to *Step 3.*

The proposed GEW design algorithm can be viewed as a combination of Procedure 1 and Procedure 2. The descent property of both design procedures ensures the convergence of the GEW algorithm. Moreover, our computer simulations show that satisfactory design results can be obtained after several iterations.

III. Computer Simulations

In this section, a simulation example is presented for illustration and comparison. The design example is the same as the Example 1 of [2]. A low-pass filter of degree 4/4 with a passband $[0, 0.1]$, a stopband $[0.2, 0.5]$, and a desired group delay of five is designed using the proposed algorithm. The denominator of a forth order Butterworth filter is chosen as the initial denominator q_0. It takes five iterations for convergence (on a 117 Mhz 80486 personal computer). The Chebyshev error at each iteration are in Table I. Figure 1 shows the magnitude and group delay response of the designed filter. The resultusing the proposed algorithm and algorithm of [2] are listed in Table 2 for comparison. We note from the simulation results that the propose algorithm outperforms the algorithm of [2].

IV. Conclusion

In this paper, we have presented a generalized Ellacott-Williams (GEW) algorithm for the design of recursive digital filters in the complex domain. The GEW algorithm is basically a combination of two design procedures which are two modified versions of the original Ellacott-Williams algorithm. At each iterations, linear complex Chebyshev approximation problems are formulated based on the least square error criterion to find the solutions for the numerator and the denominator. The approximation problems can be solved by using a weighted least square (WLS) algorithm recently proposed by the authors. This provides an efficient design technique since the heavy computational burden of using linear programming algorithm can be avoided. Considering the stability of the designed recursive filter. We employ a pole controlling mechanism to ensure all the poles of the filter inside the unit circle. Finally, the computer simulations show the effectiveness of the proposed design method.

References

[1] S. Ellacott and J. Williams, "Rational Chebyshev approximation in the complex plane," *SIAM J. Numer. Anal.* , vol. 13, no. 3, pp. 310-323, 1976.

[2] X. Chen and T. W. Parks, " Design of IIR filters in the complex domain," *IEEE Trans. Acoust., Speech, Signal Processing.*, vol. ASSP-38, pp. 910-920, Jun. 1990.

[3] X. Chen and T. W. Parks, " Design of FIR filters in the complex domain," *IEEE Trans. Acoust., Speech, Signal Processing.*, vol. ASSP-35, pp. 144-153, Jun. 1987.

[4] Y. C. Lim, J. H. Lee, C. K. Chen and R. H. Yang, "A weighted least squares algorithm for quasi-equiripple FIR and IIR digital filter design," to appear in *IEEE Trans. Signal Processing.* vol. 40, March 1992.

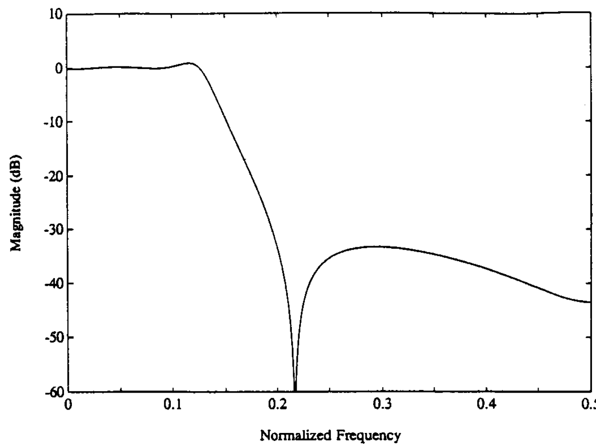
[5] J. E. Dennis,Jr. and D. J. Woods, "*New Computing Environments: Microcomputers in Large-Scale Computing,*" edited by A. Wouk, SIAM, 1987, pp. 116-122.

TABLE I The Chebyshev Error at Iteration for the Design Example

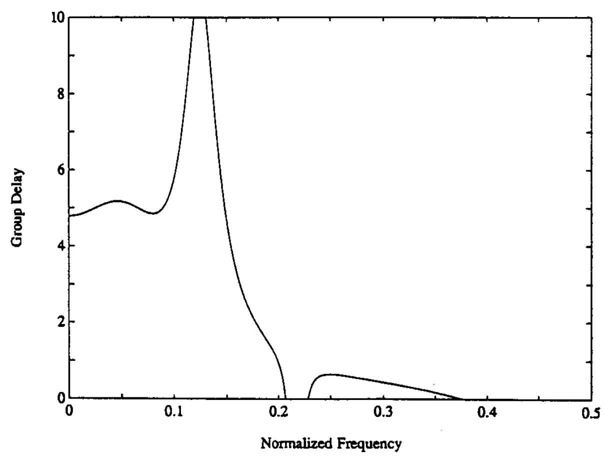
Iteration Number	Chebyshev error
0 (Initial point)	0.1264
1	0.0690
2	0.0352
3	0.0222
4	0.0216
5	0.0216

TABLE II Simulation Result for Comparison

	Proposed Algorithm	The Algorithm of [2]
Chebyshev Error	0.0216	0.0420
Chebyshev Error in dB	-33.3 dB	-27.5 dB
Group delay in Passbnad	Between 4.79 and 5.64	Between 4.65 and 6.34
Maximum Deviation of Passband Group Delay	0.64	1.34



1(a) The Magnitude Response



1(b) The Group Delay Response

Figure 1 The Magnitude and Group Delay Response of The Designed Filter