Modified Perturbation Theory for Wideband Finite-Element Model Order Reduction in Eigen-Problems

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***Abstract* — Modified perturbation theory based on high order Taylor series expansion and congruence transformation is applied to the finite-element model order reduction to accelerate the analysis of waveguide eigen-mode problems. The limit of Taylor series by poles is overcome and the bandwidth of a single-point reduced-order model is greatly improved. Without transforming a Taylor series to a Padé rational function, as in an AWE process, this method is more stable and has a wider bandwidth.**

***Index Terms* — Perturbation methods, model reduction, finite element methods, numerical analysis, microwave waveguides, microstrip.**

I. Introduction

The characteristic of a waveguide or a transmission line over a wide frequency band is often desired in electromagnetic design. Direct calculation of each frequency point in the desired band is time-consuming and some information may be lost due to insufficient frequency points. Several model order reduction techniques for fast analysis of waveguide eigen-mode problems have been proposed in the past. For example, hyper-perturbation theory [1][2], based on Taylor series expansion, was proposed and applied with the finite element method to obtain an explicit expression for the dispersion curve and the birefringence of an optical fiber with arbitrary index profile.

Due to the limit of Taylor series by poles, the expression is in a segmented form with each segment valid in a narrow frequency range. Asymptotic waveform evaluation (AWE), originally developed for time-domain analysis of electronic circuits, was combined with the TVFEM for the electromagnetic analysis of dielectric waveguides [3]. In an AWE process, the Taylor series is transformed to a Padé rational function to increase the bandwidth of a reduced-order model, over which the best choice of the rational order is usually difficult to determine.

Another way to obtain a reduced-order model can be found in [4], where several modal eigenvectors evaluated at many frequency points in the desired band are arranged column-wise and orthonomalized using the SVD. These singular vectors span a subspace containing approximate frequency-dependent modal eigenvectors, over which the original matrices are projected to yield a reduced-order model (much smaller matrices) by a congruence transformation. Although frequency derivatives of an eigenvector can also be used, as mentioned in [4], multi-point evaluation is preferred. Since the eigen-solution process is the most costly step, evaluation points in the band are as fewer as better.

In this paper, single-point model order reduction is investigated in order to fully exploit the information from a single-point calculation. The name "modified perturbation theory" is derived form [1], since our aim is to improve the reduced-order model obtained from a single-point Taylor expansion. The proposed method will be called modified perturbation for model order reduction (MPMOR) in the following sections.

II. Modified Perturbation Theory

Assume that the original 2D FEM equation for a waveguide problem is given as

$$A(k)x(k) = \lambda(k)B(k)x(k) \tag{1}$$

where both $A(k)$ and $B(k)$ are $N \times N$ finite-element system matrices and k is the wavenumber. The eigenvalue $\lambda(k)$ and the eigenvector $x(k)$ are related to the propagation constant and the field distribution of a mode, respectively.

By treating the wavenumber as a perturbation, $A(k)$, $B(k)$, $\lambda(k)$ and $x(k)$ can all be expanded into Taylor series. The frequency derivatives of $A(k)$ and $B(k)$ are known as soon as the matrix assembly process is complete, while those of $\lambda(k)$ and $x(k)$ are remained to be determined. Suppose that both $A(k)$ and $B(k)$ have linear and quadratic dependence on k, derivatives of $\lambda(k)$ and $x(k)$ over the wavenumber can be found recursively as

$$(A_0 - \lambda_0 B_0)x_1 = \lambda_1 B_0 x_0 - (A_1 - \lambda_0 B_1)x_0$$

$$(A_0 - \lambda_0 B_0)x_2 = \lambda_2 B_0 x_0 - (A_1 - \lambda_0 B_1)x_1 - (A_2 - \lambda_0 B_2)x_0 + \lambda_1(B_0 x_1 + B_1 x_0)$$

$$\vdots$$

$$(A_0 - \lambda_0 B_0)x_P = \lambda_P B_0 x_0 - \sum_{i=1}^{\min(2,P)} (A_i - \lambda_0 B_i)x_{P-i} + \sum_{i=1}^{P-1} \lambda_i \cdot \sum_{j=0}^{\min(2,P-i)} B_j x_{P-i-j} \tag{2}$$

where the subscripts represent the order of perturbation, and P is the maximum perturbation order. Detailed procedures can be found in [1-3].

After the modal eigenvector x_0 and its frequency derivatives $x_i\ (i = 1,2,\cdots,P)$ are obtained, they are orthonormalized to form an $N \times L$ matrix E where $L = P+1$. Taking E as a congruence transformation matrix, we get a reduced-order model as

$$\widetilde{A} = E^* AE \tag{3}$$

$$\widetilde{B} = E^* BE \tag{4}$$

where both $\widetilde{A}$ and $\widetilde{B}$ are $L \times L$ matrices, much smaller than the original ones, and the symbol * represents the Hermitian operator. Note that $\widetilde{A}$ and $\widetilde{B}$ are still frequency-dependent. We can introduce a new frequency k into $\widetilde{A}$ and $\widetilde{B}$, and obtain a new eigen-problem. These reduced eigen-problems can be solved very quickly over the frequency band. If high order modes are desired, the above procedure can be repeated. We do not arrange all of the desired modal eigenvectors and their frequency derivatives column-wise together, since this does not improve the bandwidth of a model significantly but costs more time instead. Thus each dispersion curve is traced respectively during the MPMOR process.

It is worthy to mention the special cases of a metallic waveguide with homogenous material. In those cases, the eigenvector for a mode is theoretically frequency-independent and hence its frequency derivative will be zero. The reduced matrices become of rank one only and are ideal to extract the wideband characteristic of that mode. This contradicts the common conjectures that it will be better to construct the congruence matrix E by using the eigenvectors of more modes and at more frequencies.

It is not a trivial task to determine which eigenpair $(\widetilde{\lambda}, \widetilde{x})$ of $(\widetilde{A}, \widetilde{B})$ is the desired mode, especially for higher order perturbation. The adopted scheme is to check both eigenvalues and eigenvectors at each frequency points. Starting from the expansion frequency point, where the solution can be viewed as an exact one, we compare the current eigenpairs with the one determined at the previous frequency point. Eigenvalues very different from the previous one are eliminated. It has been observed that some nonphysical spurious DC modes are also contained in the subspace spanned by E. They may lead to failure in tracing a desired mode near cutoff and must be filtered out. Since the form of these nonphysical solutions was described in [5], we can calculate the inner products of current eigenvectors with

$$\widetilde{q} = E^* \begin{bmatrix} q_t \\ q_z \end{bmatrix} \tag{5}$$

where

$$q_t = [1,1,\cdots,1]^T / \sqrt{N} \quad \text{(transverse unknowns)}$$

$$\text{and } q_z = 0 \quad \text{(axial unknowns)}. \tag{6}$$

Eigenvectors whose inner products with $\widetilde{q}$ below a certain criterion, say, 10^{-6} are identified as spurious DC modes and are eliminated. Moreover, inner products of the current eigenvectors with the previous one are calculated, and the largest is selected. Note that it is not necessary to recover the approximate eigenvector of the original model by

$$x = E\widetilde{x} \tag{7}$$

to compute the inner products since E is an ortho-normalized matrix.

III. Numerical Results

Three examples have been analyzed to validate the proposed method. An FEM solver is written in MATLAB™ and implemented on a PC with Pentium III 1-GHz CPU and 256-MB RAM.

A. Dielectric-Loaded Metallic Rectangular Waveguide

A dielectric-loaded metallic rectangular waveguide as shown in Fig. 1 is simulated from k_0b = 1 to 6. The expansion point is selected at the central frequency k_0b = 3.5, where the first five modes are calculated. Perturbations of each modal eigenvector are computed up to P = 10. The dispersion curves (Fig. 1) obtained by MPMOR are quite consistent with the direct solutions (circles), even below the cutoff frequency of each mode.

To analyze the accuracy of MPMOR, we define the relative error as

$$error = \frac{\|A(k)x(k) - \lambda(k)B(k)x(k)\|}{\|x(k)\|} \tag{8}$$

where $A(k)$ and $B(k)$ are the original matrices at frequency k, and $x(k)$ and $\lambda(k)$ are the approximate solutions by MPMOR.

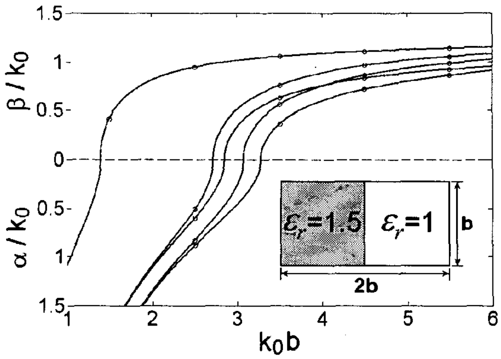

Fig. 1. The dielectric-loaded metallic rectangular waveguide and the dispersion curves of its first five modes. The results of MPMOR with the expansion point at k_0b = 3.5 (solid lines) and direct calculation (circles).

Four curves are shown in Fig. 2. The first one corresponds to the error of the first mode with perturbation order P = 5. The error is lower than 10^{-7} over the frequency band. The second, corresponding to the first mode with P = 10, has even lower error, below 10^{-12} over the entire frequency range! The error of the second mode (dashed line) and the third mode (dotted line) are also quite low.

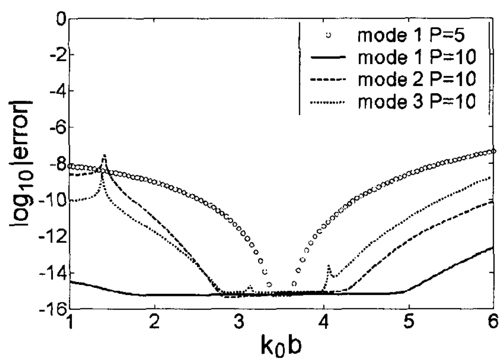

Fig. 2. Error versus frequency of the reduced-order models for the first three modes by MPMOR.

For this structure, the number of unknowns is 10961. The total time for solving the generalized eigen-problem and the generation of the reduced-order model (P = 10, i.e. L = 11) for the first mode is 17 s. Solving the reduced-order model at 501 frequency points (Δk_0b = 0.01) takes only 0.42 s, while solving the original FEM model at 51 frequency points (Δk_0b = 0.1) takes 526 s.

It is natural to ask, in addition to the desired mode, what the other eigenpairs of a reduced-order model are. In fact, it has been observed that some of them are also approximate eigen-modes. Every eigenvalue (transformed to the corresponding propagation constant and normalized to frequency) of the reduced-order model for the first mode in the frequency band are shown in Fig. 3. It is apparent that the third mode and some other high order modes exist in the reduced-order model. We traced the third mode in the reduced-order model for the first mode and found that the relative error is quite low and uniform (10^{-6} ~ 10^{-8}) over the frequency band. This solution is really an eigen-mode of the waveguide.

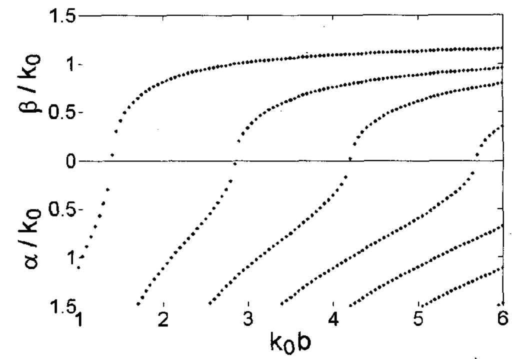

Fig. 3. Eigenvalues (transformed to the corresponding propagation constants and normalized to k_0) of the reduced-order model (P = 10) for the first mode.

B. Shielded Anisotropic Image Waveguide

A shielded anisotropic image waveguide as shown in inset of Fig. 4 is simulated from 23 MHz to 45 MHz and the results are compared with those in [3]. The dielectric is TiO_2, which has a very high permittivity ε_{rx} = 170 and ε_{ry} = ε_{rz} = 85. The expansion point is chosen at f = 36 MHz, where the first two modes are calculated. Perturbations are computed up to P = 15. The dispersion curves (Fig. 4) of the first two modes at 36 MHz obtained by MPMOR are in good agreement with the direct solutions (circles). Note that the two modes cross over around 40 MHz. The proposed method can trace the desired mode through this region successfully without mistaking the first mode for the second or vice versa.

The relative error of the reduced-order models as defined in (8) are presented in Fig. 5, where we can see that MPMOR has a much better performance than AWE [3]. The number of unknowns for this structure is 10529. The total time for solving the generalized eigen-problem and the generation of the reduced-order model (P = 15, i.e. L = 16) for the first mode is 42 s. Solving the reduced-order model at 221 frequency points (Δf = 0.01) takes 0.33 s only. However, solving the original FEM model at 23 frequency points (Δf = 0.1) takes 614 s.

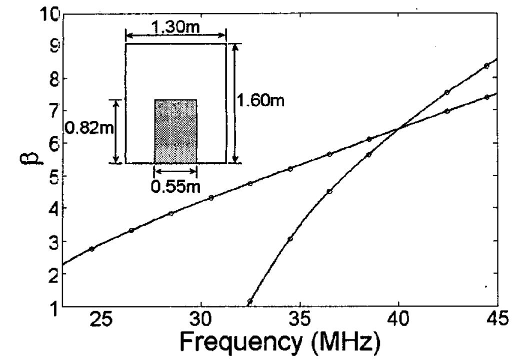

Fig. 4. The shielded anisotropic image waveguide and the dispersion curves of its first two modes at 36 MHz. The results obtained by MPMOR (solid lines) and direct calculation (circles).

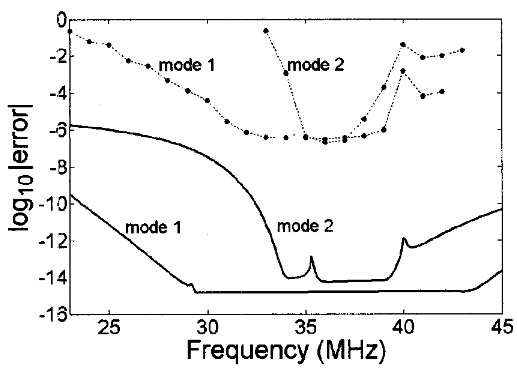

Fig. 5. Error versus frequency of the reduced-order models for the first two modes at 36 MHz. The results obtained by MPMOR (solid lines) and AWE [3] (dots).

C. Shielded Microstrip Line

As a final example, we simulate a shielded microstrip line [4] from 10 GHz to 25 GHz. The expansion point is chosen at 18 GHz. Perturbations are computed up to P = 20. The results obtained by MPMOR are in good agreement with the direct solutions for the first to the fifth mode (Fig. 6). However, for the sixth and seventh modes, the adopted scheme succeeds only in the region of the complex modes, but tends to trace one of two ordinary modes after the complex modes split around 17 GHz and 20 GHz. A more robust scheme is still under study.

IV. Conclusion

The modified perturbation theory has been successfully applied to the finite-element model order reduction to greatly accelerate the analysis of waveguide eigen-mode problems. A reduced-order model produced by MPMOR

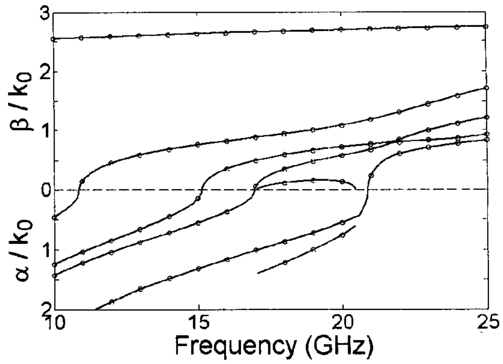

Fig. 6. The dispersion curves of the first seven modes of the microstrip line. The results of MPMOR with the expansion point at 18 GHz (solid lines) and direct calculation (circles).

is not limited by poles and has a much wider bandwidth than an AWE reduced-order model. It has been found that as far as a wideband characteristic of a particular mode is concerned, the eigenvector and its frequency derivatives are enough to construct the congruence matrix.

In addition, a reduced-order model of some mode may contain the characteristics of other modes. Thus if several dispersion curves are needed, we may check at the expansion point whether the desired high order modes (eigenvalues) exist in the already obtained reduced-order model with good enough accuracy. If they do, we can directly use the same model to obtain the dispersion curves of the high order modes without generating reduced-order models for them.

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