Inversion of Unimodular Matrices via State-Space Approach

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Abstract

Using the inversion of singular systems, we propose a state-space approach to find the inverse of unimodular matrices. As a byproduct, a simpler computation method is developed to solve the generalized polynomial Bezout identity.

1 Introduction

The problem of unimodular matrix inversion is usually encountered in the analysis and synthesis of multivariable systems. For example, to solve the generalized polynomial Bezout identity [1] is an important technique in studying linear systems. The computational algorithms for unimodular matrix inversion can be found in the literature, such as iterative elementary operation methods [1] and some other algorithms [2] , which had solved this problem in frequency domain. A pencil approach for embedding a polynomial matrix into a unimodular matrix is given in [3]. To solve the inversion problem of the unimodular pencil, [3] transforms unimodular pencil into the staircase form using unitary transformations. In this note, a state-space approach to find the inverse of unimodular matrices are proposed. We provide a numericallly stable method, which need not any transformations, to find the inverse of unimodular matrices. In our results, the degrees of inversed unimodular matrices are related to the nilpotent index of a state-space matrix. Furthermore, the generalized polynomial Bezout identity problems are solved by our method more easily than that of [4].

2 Inversion of unimodular matrices

Consider a real coefficient $m \times m$ unimodular polynomial matrix $U(s)$ of degree d:

$$U(s) = U_0 + U_1 s + U_2 s^2 + \cdots + U_d s^d \tag{1}$$

(i.e. det $U(s)$ = nonzero constant=U_0). Let the inverse of $U(s)$ be $V(s)$, then

$$U(s)V(s) = V(s)U(s) = I. \tag{2}$$

Let $\delta_{r_i}U(s)$ denotes the highest degree of the i-th row in $U(s)$. Set $\alpha_i \triangleq \delta_{r_i}U(s)+1$ and $n = \sum_{i=1}^{m}\alpha_i$. In the following, we will realize $U(s)$ to a singular system in the form of $U(s) = C(sE-I)^{-1}B + D$. Set

$$E \triangleq \left\{ \begin{array}{c} \text{Block Diag.} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{\alpha_i \times \alpha_i}, \\ i = 1, 2, \dots, m \end{array} \right\} \tag{3}$$

$$\text{and} \qquad M(s) \triangleq (sE - I). \tag{4}$$

Then $M^{-1}(s)$ is a unimodular matrix, since

$$(sE-I)^{-1} = \left\{ \begin{array}{c} \text{Block Diag.} \; \{- \begin{bmatrix} 1 & s & \dots & s^{\alpha_i - 1} \\ 0 & 1 & \dots & s^{\alpha_i - 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s \\ 0 & 0 & \dots & 1 \end{bmatrix}_{\alpha_i \times \alpha_i} \}, \\ i = 1, 2, \dots, m \end{array} \right\}. \tag{5}$$

Choose

$$C \triangleq \left\{ \begin{array}{c} \text{Block Diag. } [1 \;\; 0 \;\; \cdots \;\; 0 \;\; 0]_{1\times\alpha_i}, \\ i = 1, 2, \ldots, m \end{array} \right\} \tag{6}$$

then

$$C(sE-I)^{-1} = \left\{ \begin{array}{c} \text{Block Diag. } \{-[1 \;\; s \;\; \cdots \;\; s^{\alpha_i - 1}]_{1\times\alpha_i}\}, \\ i = 1, 2, \ldots, m \end{array} \right\}. \tag{7}$$

If we select an arbitrary $m \times m$ nonsingular matrix D, then the $n \times m$ matrix B can be read directly from coefficients of $U(s) - D$ such that $U(s) = C(sE - I)^{-1}B + D$ since the form in (7) is so simple.

We have just mentioned an efficient method to realize a time-invariant singular system whose frequency domain input-output relationship is the given $m \times m$ unimodular matrix $U(s)$. The system equation can be written as

$$E\dot{x}(t) = Ix(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t), \tag{8}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input vector, $y(t) \in R^m$ is the output vector with $n > m$. E, I, B, C and D are real constant matrices of appropriate dimensions. Note that rank $E = n - m$ (from (3)) and D is nonsingular. The associated frequency expression of (8) is

$$U(s) = C(sE-I)^{-1}B + D \triangleq \begin{bmatrix} sE - I & \vdots & B \\ \cdots & \cdot & \cdots \\ C & \vdots & D \end{bmatrix}. \tag{9}$$

The square singular system (8) is invertible if the determinant of (9) does not vanish identically [5]. The explicit formulas for inverting unimodular matrices $U(s)$ are stated in the following Theorem.

Theorem 1 : The inverse of the unimodular matrix $U(s)$ in (9) is

$$\begin{aligned} V(s) &= U^{-1}(s) \\ &= (-D^{-1}C)(sE - (I - BD^{-1}C))^{-1}BD^{-1} + D^{-1} \\ &\triangleq \begin{bmatrix} sE - (I - BD^{-1}C) & \vdots & BD^{-1} \\ \cdots & \cdot & \cdots \\ -D^{-1}C & \vdots & D^{-1} \end{bmatrix}. \end{aligned}$$

Proof: Since D is nonsingular, from (8), we have $u(t) = -D^{-1}Cx(t) + D^{-1}y(t)$, substituting into (8) gives $E\dot{x}(t) = (I - BD^{-1}C)x(t) + BD^{-1}y(t)$. Note that the combination of above is the inversed singular system of (8) since its input and output are $y(t)$ and $u(t)$ respectively. Hence, $V(s)$ is the inverse of $U(s)$, where

$$\begin{aligned} V(s) &\triangleq U^{-1}(s) \\ &= (-D^{-1}C)(sE - (I - BD^{-1}C))^{-1}BD^{-1} + D^{-1} \end{aligned} \tag{10}$$

□

For convenience, we set $A_c \triangleq I - BD^{-1}C$. It is well known that, the inverse of a unimodular matrix is unimodular also. Thus, $\det[sE - A_c]$ should be a nonzero constant. In other words, A_c is nonsingular. We can use this result to verify $U(s)$ in (9), whether it is a unimodular matrix or not.

Corollary 1 : If $\det(sE - A_c)$ is not a nonzero constant then the polynomial matrix $U(s)$ is not a unimodular matrix. □

Remark 1 :

(i) $(sE - A_c)^{-1}$ is a unimodular matrix and can be represented as

$$\begin{aligned} &(sE - A_c)^{-1} \\ &= (-1)^n\{I + [(A_c)^{-1}E]s + [(A_c)^{-1}E]^2 s^2 + \cdots \\ &\quad + [(A_c)^{-1}E]^{q_c - 1}s^{q_c - 1}\}[(A_c)^{-1}], \end{aligned}$$

where $q_c =$ nilpotent index of $[(A_c)^{-1}E]$. Hence, the highest degree of $U^{-1}(s)$ will be $q_c - 1$.

(ii) In (9), if D is selected as U_0 on purpose, let $P = BD^{-1}C$, the 1st, $(\alpha_1 + 1)$th, $\cdots$, $(\alpha_1 + \alpha_2 + \cdots + \alpha_{m-1} + 1)$th rows of B have zero coefficients and the columns' coefficients of C are zero values, except for the 1st, $(\alpha_1 + 1)$th, $\cdots$, $(\alpha_1 + \alpha_2 + \cdots + \alpha_{m-1} + 1)$th columns. We can easy to prove $CB = 0$, $P^2 = BD^{-1}CBD^{-1}C = 0$, and $P^i = 0$, where $i \geq 2$, so

$$A_c^{-1} = (I - BD^{-1}C)^{-1} = (I - P)^{-1} = I + P.$$

The computation of $(A_c^{-1}E)^i = ((I+P)E)^i$ meed only multiplication. □

In the above, the problem of unimodular matrix inversion was converted to the problem of singular system inversion. This result can be applied to solving the polynomial generalized Bezout identity as it will be discussed in the next section.

3 Solutions of the polynomial Bezout identity

The polynomial matrix fraction descriptions (MFDs) are useful frequency expressions for multivariable systems [1]. One key technique for system analysis and design is to solve the polynomial Bezout identity. The connection between singular system and polynomial Bezout identity was discussed in [4]. Compared to [4], our result is simpler since only proportional feedback is involved and only one pole placement computation is needed.

We first introduce an efficient method to realize a time-invariant singular system whose input-output relationship is given by the $m \times r$ polynomial matrix fraction description $\overline{D}^{-1}(s)\overline{N}(s)$.

Lemma 1 : Let us consider the given $m \times r$ polynomial matrix fraction description $\overline{D}^{-1}(s)\overline{N}(s)$ with $\delta_{ri}\overline{N}(s) \leq \delta_{ri}\overline{D}(s)$. Assume $\overline{D}(s)$ and $\overline{N}(s)$ are left coprime. If we choose L and B satisfying $\overline{D}(s) = I + CM^{-1}(s)L$ and $\overline{N}(s) = CM^{-1}(s)B$ respectioely; where C and $M(s)$ are shown in (6) and (4), then $\overline{D}^{-1}(s)\overline{N}(s) = CW^{-1}(s)B$, where $W(s) \triangleq (sE - A)$ and $A \triangleq I - LC$. Furthermore, the quadruple $\{E, A, B, C\}$ are strongly controllable and observable.

Proof :
$\overline{D}^{-1}(s)\overline{N}(s) = [I + CM^{-1}(s)L]^{-1}CM^{-1}(s)B = C(M(s) + LC)^{-1}B = CW(s)^{-1}B$. Since $\overline{N}(s)$ and $\overline{D}(s)$ are left coprime, the realization of $\overline{D}^{-1}(s)\overline{N}(s)$ can be written as the quadruple $\{E, A, B, C\}$, which are strongly controllable and observable from [6]. □

The condition of strongly controllable (observable) contains the requirements of finite controllable (observable) and nondynamic infinite controllable (observable), which are defined in [7]. The realization of Lemma 1 satisfies finite controllable is thus evident.

Lemma 2 [8] : If $\{E, A, B\}$ is finite controllable and E is singular, then there exists a gain matrix K such that the matrix pencil $sE - (A + BK)$ has no finite poles. □

In [8], a useful algorithm to compute K so that $(sE - (A + BK))$ being a unimodular matrix is provided also. Combining Lemma 1 and 2, we are ready to investigate the solutions of the polynomial Bezout identity.

Theorem 2 : Let $\overline{D}^{-1}(s)\overline{N}(s)$ be the left coprime $m \times r$ polynomial matrix fraction description. Let the strongly controllable and observable realization of $\overline{D}^{-1}(s)\overline{N}(s) = C(sE - A)^{-1}B$ be the same as in Lemma 1. The following eight polynomial matrices satisfy

$$\begin{bmatrix} \overline{D}(s) & \overline{N}(s) \\ -Y(s) & X(s) \end{bmatrix} \begin{bmatrix} \overline{X}(s) & -N(s) \\ \overline{Y}(s) & D(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{11}$$

where

$$\overline{D}(s) = I + CM^{-1}(s)L, \quad \overline{N}(s) = CM^{-1}(s)B,$$

$$D(s) = I + KH^{-1}(s)B, \quad N(s) = CH^{-1}(s)B,$$

$$\overline{X}(s) = I - CH^{-1}(s)L, \quad \overline{Y}(s) = KH^{-1}(s)L,$$

$$X(s) = I - KM^{-1}(s)B, \quad Y(s) = KM^{-1}(s)L,$$

in which $M(s) \triangleq sE - I$, $A \triangleq I - LC$ and $H(s) \triangleq sE - A - BK$, K is obtained from the infinite eigenvalue assignment as stated in Lemma 2.

Proof: Set

$$U(s) \triangleq \begin{bmatrix} \overline{D}(s) & \overline{N}(s) \\ -Y(s) & X(s) \end{bmatrix} = \begin{bmatrix} M(s) & \vdots & L & B \\ \cdots & \cdot & \cdots & \cdots \\ C & \vdots & I & 0 \\ -K & \vdots & 0 & I \end{bmatrix}.$$

From Theorem 1, the inverse of the polynomial matrix $U(s)$ is

$$\begin{aligned} V(s) &\triangleq U^{-1}(s) \\ &= \begin{bmatrix} H(s) & \vdots & L & B \\ \cdots & \cdot & \cdots & \cdots \\ -C & \vdots & I & 0 \\ K & \vdots & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} \overline{X}(s) & -N(s) \\ \overline{Y}(s) & D(s) \end{bmatrix}. \end{aligned}$$

where

$$\begin{aligned} &H(s) \\ &= sE - \left(I - [L \quad B] \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} C \\ -K \end{bmatrix} \right) \\ &= sE - I + LC - BK = sE - A - BK. \end{aligned}$$

□

4 Example

Given the unimodular matrix in [3]

$$U(s) = \begin{bmatrix} 0 & s^2 & 1 \\ 0 & 1 & 0 \\ 1 & s+7 & s^2 + 7s + 3 \end{bmatrix},$$

find its inverse.

Solution : Set $\alpha_1 = 3, \alpha_2 = 1$, $\alpha_3 = 3$, and $n = \alpha_1 + \alpha_2 + \alpha_3 = 7$, from (3) and (6), we obtain

$$E = \left[\begin{array}{ccccccccc} 0 & 1 & 0 & \vdots & 0 & & 0 & 0 & 0 \\ 0 & 0 & 1 & \vdots & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdot & \cdots & \cdot & & & \\ 0 & 0 & 0 & \vdots & 0 & \vdots & 0 & 0 & 0 \\ & & & \cdot & \cdots & \cdot & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & & 0 & \vdots & 0 & 0 & 1 \\ 0 & 0 & 0 & & 0 & \vdots & 0 & 0 & 0 \end{array}\right]$$

and

$$C = \left[\begin{array}{ccccccccc} 1 & 0 & 0 & \vdots & 0 & & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdot & \cdots & \cdot & & & \\ 0 & 0 & 0 & \vdots & 1 & \vdots & 0 & 0 & 0 \\ & & & \cdot & \cdots & \cdot & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & 0 & \vdots & 1 & 0 & 0 \end{array}\right].$$

If we choose $D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 7 & 3 \end{bmatrix}$ then the matrix B can be read from the coefficients of

$$U(s) - D = \begin{bmatrix} 0 & s^2 & 0 \\ 0 & 0 & 0 \\ 0 & s & s^2 + 7s \end{bmatrix},$$

The obtained

$$B = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \\ 0 & -1 & -7 \\ 0 & 0 & -1 \end{bmatrix}.$$

It can be verified that $U(s) = C(sE - I)^{-1}B + D$ in our result. The inverse of $U(s)$ is shown in (10)

$$\begin{aligned} &U^{-1}(s) \\ &= (-D^{-1}C)(sE - (I - BD^{-1}C))^{-1}BD^{-1} + D^{-1} \\ &= \begin{bmatrix} (-s^2 - 7s - 3) & (s^4 + 7s^3 + 3s^2 - s - 7) & 1 \\ 0 & 1 & 0 \\ 1 & -s^2 & 0 \end{bmatrix}. \end{aligned}$$

The coefficient of s^5 is exact zero which is more reliable than [3].

5 Conclusions

The major contributions of this note are as follows:

(i) Corollary 1 can be used to check a square polynomial matrix whether it is unimodular or not.

(ii) Explicit formulas for solving inversion of unimodular matrices are developed. Our proposed method is numerically reliable algorithm.

(iii) A strongly controllable and observable generalized state-space realization method for polynomial matrix fraction descriptions is constructed.

(iv) Solutions of the generalized polynomial Bezout identity are provided in a simple form.

The method presented in this note is straightforward and easy to follow. Our method allows one to use the available software packages (e,g. $MATRIX_X$, $MATLAB$ etc.) to compute the solution.

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