Resolution threshold for coherent sources using smoothed eigenstructure methods

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Abstract: The paper presents a performance analysis of certain eigenstructure based methods (MUSIC type) with suitable forward-only and forward/backward smoothing schemes employed to decorrelated coherent sources. Bias and resolution thresholds are derived for two coherent sources with different phase and unequal power. The effect of phase difference and power ratio on the resolution threshold is discussed. For the case of forward/backward smoothing and two coherent sources with same phase and equal power, the resolution threshold reduces to that obtained by Pillai et al. Detailed computer simulations which confirm the analysis are also presented.

1 Introduction

High-resolution eigenstructure based techniques [1–6] can estimate the directions of arrival (DOAs) of incoherent and partially coherent sources. For coherent sources, these techniques may be modified by preprocessing smoothing techniques [7–9], which can decorrelate the coherence between sources. The eigenstructure-based techniques can resolve any two closely spaced sources when the ensemble covariance matrix is available. However, in reality, the covariance matrix is estimated from finite array sample data. The deviation between the estimated and the ensemble covariance matrices produces a biased estimation, and these techniques have different performances.

Recently, Kaveh et al. [10] derived the expressions of the bias and resolution threshold, the signal-to-noise ratio (SNR) at which two closely spaced sources can be resolved, of the MUSIC method for two uncorrelated sources. Later Pillai et al. [11, 12] presented the bias and resolution threshold of the MUSIC method with a forward/backward smoothing scheme for two coherent sources of equal power and the same phase. Since two coherent sources usually have different phase and power, we derive in this paper the bias and resolution thresholds of the MUSIC method for such two coherent sources. Instead of solving for eigencomponents of the covariance matrix, the derivation involves the sum and product of the two largest eigenvalues. The bias and resolution

thresholds are obtained for both the forward/backward and forward-only smoothing schemes.

2 Problem formulation and previous results

Consider a uniform linear array illuminated by two coherent narrow-band sources with complex waveforms $u(t)$ and $|\rho|e^{j\phi}u(t)$ and DOAs θ_1 and θ_2 with respect to the array broadside, where $|\rho|$ and ϕ represent the relative amplitude attenuation and phase delay of the second source with respect to the first source. Without loss of generality, let us assume that $|\rho| \leqslant 1$. To estimate the DOAs of the two coherent sources, the array is grouped into K subarrays, each with M sensors. With the assumption that the signals and noises are stationary, zero mean circular Gaussian independent random processes and, further, the noises are IID (independent and identically distributed) with common variance σ^2, the ensemble forward smoothed covariance matrix of the K subarrays can be written as

$$\bar{R}_f = A\left[\frac{1}{K}\sum_{k=1}^{K} D^{k-1}R_u(D^{k-1})^H\right]A^H + \sigma^2 I \tag{1}$$

where

$$A = \sqrt{(M)}[a(\omega_1), a(\omega_2)] \tag{2}$$

$$D = \text{diag } [e^{-j\omega_1}, e^{-j\omega_2}] \tag{3}$$

$$R_u = E[|u(t)|^2]\begin{pmatrix} 1 & |\rho|e^{-j\phi} \\ |\rho|e^{j\phi} & |\rho|^2 \end{pmatrix} \tag{4}$$

In the above equations, E and H denote expectation and hermitian, respectively, and $a(\omega_i)$ is the normalised direction vector associated with θ_i,

$$a(\omega_i) = \frac{1}{\sqrt{M}} [1, e^{-j\omega_i}, e^{-j2\omega_i}, \ldots, e^{-j(M-1)\omega_i}]^T \tag{5}$$

where

$$\omega_i = \frac{2\pi d \sin \theta_i}{\lambda} \quad i = 1, 2 \tag{6}$$

d = the space between sensors and λ = the wavelength of the signal sources.

It is known that in $\bar{R}_f$ the two coherent sources are decorrelated when $K \geqslant 2$. To reduce the number of extra sensors required for smoothing, the ensemble forward/backward smoothed covariance matrix can be formed as

$$\bar{R} = \tfrac{1}{2}\{\bar{R}_f + \tilde{I}\bar{R}_f^*\tilde{I}\}$$
$$= A\bar{R}_u A^H + \sigma^2 I \tag{7}$$

where $\bar{I}$ is the reverse permutation matrix

$$\bar{I} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} \tag{8}$$

and

$$\bar{R}_u = \frac{1}{2K}\left[\sum_{k=1}^{K} D^{k-1} R_u (D^{k-1})^H \right.$$

$$\left. + \sum_{k=1}^{K} D^{-(K+M-k-1)} R_u^* (D^{-(K+M-k-1)})^H \right] \tag{9}$$

with '*' representing conjugation. In $\bar{R}$, the two coherent sources are decorrelated with $K \geqslant 1$ except when the phase difference between the two coherent sources is either $0°$ or $180°$ at the array aperture centre in the case of $K = 1$.

The MUSIC method can estimate the DOAs of the two coherent sources using either the forward-only smoothed or the forward/backward smoothed covariance matrices. With the ensemble covariance matrices $\bar{R}_f$ and $\bar{R}$ described as above, the MUSIC method can resolve any two closely spaced coherent sources. Practically, the covariance matrix is estimated from array sample data. The difference between the estimated covariance matrix and the ensemble covariance matrix results in a biased estimation of DOAs. Recently, Pillai et al. [12] derived the general expressions for the mean and variance of the null spectrum of the MUSIC method. For two coherent sources, the mean and variance are given by

$$E[\hat{Q}(\omega)]$$

$$= Q(\omega) + \frac{1}{N}\sum_{i=1}^{2}\left[\sum_{\substack{k=1 \\ k\neq i}}^{M}\frac{\Gamma_{iikk}}{(\lambda_i - \lambda_k)^2}\,|\beta_i^H a(\omega)|^2 \right.$$

$$\left. - \sum_{\substack{k=1 \\ k\neq i}}^{M}\sum_{\substack{l=1 \\ l\neq i}}^{M}\frac{\Gamma_{iikl}}{(\lambda_i - \lambda_k)(\lambda_i - \lambda_l)}\,a^H(\omega)\beta_k\beta_l^H a(\omega)\right]$$

$$+ O\left(\frac{1}{N^2}\right) \tag{10}$$

and

$$\text{Var}\,[\hat{Q}(\omega)]$$

$$= \frac{2}{N}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{\substack{k=1 \\ k\neq i}}^{M}\sum_{\substack{l=1 \\ l\neq j}}^{M}a^H\beta_k\beta_l^H a$$

$$\times \frac{\text{Re}\,[\Gamma_{klji}\,a^H(\omega)\beta_j\beta_l^H a(\omega) + \Gamma_{kjli}\,a^H(\omega)\beta_i\beta_j^H a(\omega)]}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)}$$

$$+ O\left(\frac{1}{N^2}\right) \tag{11}$$

In eqns. 10 and 11, N denotes the number of snapshots, λ_i and β_i, $i = 1, 2, \ldots, M$, are the eigenvalues in descending order and associated eigenvectors, respectively, of the ensemble smoothed covariance matrix, $\text{Re}(\cdot)$ stands for the real part of $(\cdot)$, and $Q(\omega)$ is the null spectrum computed from the ensemble smoothed covariance matrix, i.e.

$$Q(\omega) = \sum_{i=3}^{M}|\beta_i^H a(\omega)|^2$$

$$= 1 - \sum_{i=1}^{2}|\beta_i^H a(\omega)|^2 \tag{12}$$

Here, Γ_{iklj} is given by

$$\Gamma_{iklj} = \begin{cases} \dfrac{1}{4K^2}\displaystyle\sum_{p=1}^{K}\sum_{q=1}^{K}\beta_i^H R_{pq}^f\beta_k\beta_l^H R_{qp}^f\beta_j & \\[2mm] \quad + \beta_i^H R_{pq}^b\beta_k\beta_l^H R_{qp}^b\beta_j & \\[1mm] \quad + \beta_i^H R_{pq_0}^f\gamma_l\gamma_j^H R_{p_0q}^b\beta_k & \text{for forward/} \\ & \text{backward} \\ \quad + \beta_i^H R_{pq_0}^b\gamma_l\gamma_j^H R_{p_0q}^f\beta_k & \text{smoothing} \\[2mm] \dfrac{1}{K^2}\displaystyle\sum_{p=1}^{K}\sum_{q=1}^{K}\beta_i^H R_{pq}^f & \text{for forward-} \\ & \text{only} \\ \quad \times\,\beta_k\beta_l^H R_{qp}^f\beta_j & \text{smoothing} \end{cases} \tag{13}$$

In eqn. 13, $p_0 = K - p + 1$, $q_0 = K - q + 1$ and γ_j is the backward version of the vector β_j^*. Further,

$$R_{pq}^f = E[X_p^f(X_q^f)^H]$$

$$R_{pq}^b = E[X_p^b(X_q^b)^H] \tag{14}$$

where X_i^f (or X_i^b) is the output vector of the ith forward (or backward) subarray. For the forward/backward smoothing with $K = 1$, eqn. 10 reduces to

$$E[\hat{Q}(\omega)] = Q(\omega) + \frac{1}{2N}\sum_{i=1}^{2}\frac{\lambda_i\sigma^2}{(\lambda_i - \sigma^2)^2}$$

$$\times\,[(M-2)\,|\beta_i^H a(\omega)|^2 - Q(\omega)]$$

$$+ O\left(\frac{1}{N^2}\right) \tag{15}$$

Since $\text{Var}\,[\hat{Q}(\omega_k)]$ is much less than $E[\hat{Q}(\omega_k)]$ for $k = 1, 2$ [12], the resolution threshold for two closely spaced sources is defined as the smallest SNR for which the following inequality is satisfied [10]

$$E[\hat{Q}(\omega_m)] \geqslant E[\hat{Q}(\omega_i)] \quad i = 1, 2 \tag{16}$$

where $\omega_m = (\omega_1 + \omega_2)/2$. From eqns. 15 and 16, Pillai et al. have derived the resolution threshold $\tilde{\xi}_T$ for two coherent sources of equal power and same phase, to be [12]

$$\tilde{\xi}_T \approx \left(\frac{1}{3}\Delta^{-2} - \frac{1}{20} - \frac{1}{16}\Delta^2\right)\xi_T \tag{17}$$

where ξ_T [10] is the normalised SNR (resolution threshold) required to resolve two equipowered uncorrelated sources separated by the normalised 'angle' $\Delta^2 = (M^2\omega_d^2)/3$. $2\omega_d = (\omega_2 - \omega_1)$ represents the actual separation.

In reality, two coherent sources usually have different phase and power. We analyse in this paper the performance of the MUSIC method with forward/backward and forward-only smoothing schemes for such coherent sources. The number of subarrays K is chosen to be one for forward/backward smoothing and two for forward-only smoothing (i.e. the minimum required).

3 Performance of MUSIC with forward/backward smoothing

The expressions of the bias and resolution threshold for the MUSIC scheme with forward/backward smoothing and $K = 1$ for two coherent sources with different phase and power are derived below. $E[\hat{Q}(\omega_i)]$, $i = 1, 2$, is first evaluated by using eqn. 15 which is the general expression for the MUSIC scheme with forward/backward smoothing and $K = 1$ in the case of two coherent

sources. Eqn. 15 involves the eigenvalues and eigenvectors associated with the signal subspace of the smoothed ensemble covariance matrix $\bar{R}$. For the case considered here, the expressions of the eigenvalues and eigenvectors are too complex to be used for evaluating the bias. In the analysis shown below, that problem is circumvented by using the sum and product of the two largest eigenvalues.

To evaluate $E[\hat{Q}(\omega_i)]$, $|\beta_j^H a(\omega_i)|$, $i, j = 1, 2$, is first examined. Let μ_i be the eigenvalue of the noise free ensemble covariance matrix $A\bar{R}_u A^H$. μ_i is related to λ_i, the eigenvalue of $\bar{R}$, by

$$\mu_i = \lambda_i - \sigma^2 \qquad (18)$$

Using eqns. 2, 4 and 9, $A\bar{R}_u A^H$ can be written as

$$A\bar{R}_u A^H = ME[|u(t)|^2][a(\omega_1)a(\omega_2)]$$
$$\times \begin{pmatrix} 1 & |\rho|\rho_t \\ |\rho|\rho_t^* & |\rho|^2 \end{pmatrix}[a(\omega_1)a(\omega_2)]^H$$
$$= ME[|u(t)|^2][\beta_1 \quad \beta_2]$$
$$\times \begin{pmatrix} \mu_1' & 0 \\ 0 & \mu_2' \end{pmatrix}[\beta_1 \quad \beta_2]^H \qquad (19)$$

In eqn. 19, ρ_t is the effective correlation coefficient of the two coherent sources for forward/backward smoothing and $K = 1$

$$\rho_t = e^{-j(M-1)\omega_d}\cos[(M-1)\omega_d - \phi] \qquad (20)$$

$$E[\hat{Q}(\omega_m)] = Q(\omega_m) + \frac{M-2}{2N}\left\{\frac{1}{\zeta}|\rho|^2\frac{(\mu_1' + \mu_2') - Si^2[M, (\omega_d/2)]\rho_a - Q(\omega_m)\mu_1'}{\mu_1'\mu_2'}\right.$$
$$\left. + \frac{1}{\zeta^2}|\rho|^4\frac{(\mu_1'^2 + \mu_1'\mu_2' + \mu_2'^2) - Si^2[M, (\omega_d/2)]\rho_a(\mu_1' + \mu_2') - Q(\omega_m)\mu_1'^2}{\mu_1'^2\mu_2'^2}\right\} \qquad (33)$$

and μ_i' is defined as

$$\mu_i' = \frac{\mu_i}{ME[|u(t)|^2]} \qquad (21)$$

Premultiplying and postmultiplying eqn. 19 by $a^H(\omega_i)$ and $a(\omega_i)$, respectively, for $i = 1, 2$, we get the following equations

$$1 + |\rho|^2|\rho_s|^2 + |\rho|\rho_t\rho_s + |\rho|\rho_t^*\rho_s^*$$
$$= \mu_1'|\beta_1^H a(\omega_1)|^2 + \mu_2'|\beta_2^H a(\omega_1)|^2 \qquad (22)$$

and

$$|\rho|^2 + |\rho_s|^2 + |\rho|\rho_t\rho_s + |\rho|\rho_t^*\rho_s^*$$
$$= \mu_1'|\beta_1^H a(\omega_2)|^2 + \mu_2'|\beta_2^H a(\omega_2)|^2 \qquad (23)$$

In eqns. 22 and 23, ρ_s is defined as

$$\rho_s = a^H(\omega_2)a(\omega_1) = e^{j(M-1)\omega_d}Si(M, \omega_d) \qquad (24)$$

where

$$Si(x, y) \triangleq \frac{\sin(xy)}{x\sin y} \qquad (25)$$

Since β_1 and β_2 span the signal subspace, we have

$$|\beta_1^H a(\omega_i)|^2 + |\beta_2^H a(\omega_i)|^2 = 1 \quad \text{for } i = 1, 2 \qquad (26)$$

Solving eqns. 22, 23 and 26, we get

$$|\beta_j^H a(\omega_i)|^2 = (-1)^j\frac{|\rho|^{2(2-i)}(1 - |\rho_s|^2) - \mu_j'}{\mu_1' - \mu_2'}$$
$$\text{for } i, j = 1, 2 \qquad (27)$$

Substituting eqn. 27 into eqn. 15, the mean of the null spectrum in the directions of ω_1 and ω_2 can be obtained within the first-order approximation, as

$$E[\hat{Q}(\omega_i)] = \frac{M-2}{2N}\left\{-\frac{|\rho|^4}{\zeta^2\mu_1'\mu_2'} + \frac{|\rho|^{2(3-i)}(1 - |\rho_s|^2)}{\zeta\mu_1'\mu_2'}\right.$$
$$\left.\times\left[1 + \frac{|\rho|^2(\mu_1' + \mu_2')}{\zeta\mu_1'\mu_2'}\right]\right\} \quad \text{for } i = 1, 2 \qquad (28)$$

where ζ is defined as the array output SNR of the weaker source,

$$\zeta \triangleq \frac{ME[|u(t)|^2]|\rho|^2}{\sigma^2} \qquad (29)$$

Similarly, to find the bias in the direction of ω_m, the factor $|\beta_i^H a(\omega_m)|^2$ can be evaluated by premultiplying and postmultiplying eqn. 19 by $a^H(\omega_m)$ and $a(\omega_m)$, respectively, and this results in

$$|\beta_1^H a(\omega_m)|^2 \simeq \frac{-\mu_2' + Si^2[M, (\omega_d/2)]\rho_a}{\mu_1' - \mu_2'} \qquad (30)$$

$$|\beta_2^H a(\omega_m)|^2 \simeq \frac{\mu_1' - Si^2[M, (\omega_d/2)]\rho_a}{\mu_1' - \mu_2'} - Q(\omega_m) \qquad (31)$$

where ρ_a is given by

$$\rho_a = 1 + |\rho|^2 + |\rho|\rho_t e^{j(M-1)\omega_d} + |\rho|\rho_t^* e^{-j(M-1)\omega_d} \qquad (32)$$

From eqns. 30 and 31, we may obtain the $E[\hat{Q}(\omega_m)]$ with the first-order approximation

Since $Q(\omega_m)$ is much less than unity for two closely spaced sources and μ_2' is smaller than μ_1', $Q(\omega_m)\mu_1'$ and $Q(\omega_m)(\mu_1')^2$ in eqn. 33 can be approximated by $Q(\omega_m)(\mu_1' + \mu_2')$ and $Q(\omega_m)(\mu_1' + \mu_2')^2$, respectively. Therefore, from eqns. 28 and 33, $E[\hat{Q}(\omega)]$, $\omega = \omega_1$, ω_2 and ω_m, is only related to the sum and product of μ_i', $i = 1, 2$. Since μ_1 and μ_2 are eigenvalues of $A\bar{R}_u A^H$, the following equation holds

$$\mu_i^2 - \text{Tr}(A\bar{R}_u A^H)\mu_i + |\bar{R}_u A^H A| = 0 \qquad (34)$$

where $\text{Tr}(\cdot)$ denotes the trace of $(\cdot)$. From eqns. 21 and 34, the sum and product of μ_1' and μ_2' are given by

$$\mu_1' + \mu_2' = \frac{\text{Tr}(A\bar{R}_u A^H)}{ME[|u(t)|^2]}$$
$$= 1 + |\rho|^2 + |\rho|(\rho_s\rho_t + \rho_s^*\rho_t^*) \qquad (35)$$

$$\mu_1'\mu_2' = \frac{|\bar{R}_u A^H A|}{\{ME[|u(t)|^2]\}^2}$$
$$= |\rho|^2(1 - |\rho_s|^2)(1 - |\rho_t|^2) \qquad (36)$$

One can see from eqn. 28 that $E[\hat{Q}(\omega_2)] \geq E[\hat{Q}(\omega_1)]$ for $|\rho|^2 \leq 1$. Therefore, the resolution threshold is determined by the weaker source,

$$E[\hat{Q}(\omega_m)] \geq E[\hat{Q}(\omega_2)] \qquad (37)$$

which is the same as the conclusion drawn by Lee [13] for two uncorrelated sources. Substituting eqns. 28 and 33 into eqn. 37 and using $Q(\omega_m) \simeq \frac{1}{80}\Delta^4$ [10, 13], we may

get the resolution threshold,

$$\zeta = \frac{20(M-2)}{N\Delta^4} A \left\{ 1 + \sqrt{\left[1 + \frac{N\Delta^4}{10(M-2)} B \right]} \right\} \qquad (38)$$

where A and B are given by

$$A \simeq \frac{1 - |\rho_s|^2 + Si^2[M, (\omega_d/2)]\rho_a + (Q(\omega_m) - 1)(\mu_1' + \mu_2')}{(1 - |\rho_s|^2)(1 - |\rho_t|^2)} \qquad (39)$$

and

$$B \simeq \frac{\mu_1' + \mu_2'}{1 - |\rho_s|^2 + Si^2[M, (\omega_d/2)]\rho_a + (Q(\omega_m) - 1)(\mu_1' + \mu_2')} \qquad (40)$$

For two closely spaced sources, it can be shown that the resolution threshold may be approximated as follows.

(a) For $|\sin \phi| \ll \Delta$:

$$\zeta \simeq \frac{20(M-2)}{N\Delta^6}(a_0 + a_1\Delta^2)$$

$$\times \left(1 + \sqrt{\left\{ 1 + \frac{2N\Delta^2}{5(M-2)} \frac{1 + |\rho|^2 + 2|\rho| \, \mathrm{Sgn} \, (\cos \phi) - 4|\rho| \, \mathrm{Sgn} \, (\cos \phi)\Delta^2}{3 - |\rho|^2 + 2|\rho| \, \mathrm{Sgn} \, (\cos \phi) + \{[3|\rho|^2 - 66|\rho| \, \mathrm{Sgn} \, (\cos \phi) - 29]/20\}\Delta^2} \right\}} \right)$$

$$\simeq 3(a_0 + a_1\Delta^2)\tilde{\xi}_T \qquad (41)$$

where $\tilde{\xi}_T$ is the resolution threshold for two coherent sources of equal power and same phase given in Reference 12, and a_0 and a_1 are given by

$$a_0 = \frac{3 - |\rho|^2 + 2|\rho| \, \mathrm{Sgn} \, (\cos \phi)}{12} \qquad (42)$$

and

$$a_1 = \frac{11 - 5|\rho|^2 - 2|\rho| \, \mathrm{Sgn} \, (\cos \phi)}{48} \qquad (43)$$

with

$$\mathrm{Sgn} \, (a) = \begin{cases} 1 & \text{for } a \geqslant 0 \\ -1 & \text{for } a < 0 \end{cases} \qquad (44)$$

It can be seen from eqn. 41 that for $|\rho| = 1$ and $\phi = 0$, ζ reduces to $\tilde{\xi}_T$.

(b) For $|\sin \phi| \gg \Delta$,

$$\zeta \simeq \frac{20(M-2)}{N\Delta^4} \left(\frac{3 - |\rho|^2 + 2|\rho| \cos \phi}{4 \sin^2 \phi} \right)$$

$$\times \left(1 + \sqrt{\left[1 + \frac{2N\Delta^2}{5(M-2)} \frac{1 + |\rho|^2 + 2|\rho| \cos \phi}{3 - |\rho|^2 + 2|\rho| \cos \phi} \right]} \right)$$

$$\simeq \frac{3 - |\rho|^2 + 2|\rho| \cos \phi}{4 \sin^2 \phi} 3\Delta^2 \tilde{\xi}_T \qquad (45)$$

(c) For $\cos^2(\phi - (M-1)\omega_d) \approx 1$,

$$\zeta \simeq \frac{20(M-2)}{N\Delta^4} \left(\frac{3 - |\rho|^2 + 2|\rho|}{4} \right)$$

$$\times \theta^{-2} \left(1 + \sqrt{\left[1 + \frac{2N\Delta^2}{5(M-2)} \frac{1 + |\rho|^2 + 2|\rho|}{3 - |\rho|^2 + 2|\rho|} \right]} \right)$$

$$\simeq \frac{3 - |\rho|^2 + 2|\rho|}{4} 3\Delta^2 \theta^{-2} \tilde{\xi}_T \qquad (46)$$

where

$$\theta = \begin{cases} \phi - (M-1)\omega_d & \text{for } \phi \simeq (M-1)\omega_d \\ (M-1)\omega_d - \phi - \pi & \text{for } \phi \simeq (M-1)\omega_d - \pi \end{cases} \qquad (47)$$

It can be seen from eqn. 46 that the resolution threshold has a peak value when θ approaches zero. This result comes from the fact that the MUSIC with forward/backward smoothing and $K = 1$ can not decorrelate the two coherent sources with phase difference equal to $(M-1)\omega_d$ or $(M-1)\omega_d - \pi$.

4 Forward-only smoothing

For forward-only smoothing, we analyse the performance of the MUSIC with $K = 2$. In the case, Γ_{iklj} in eqn. 13

reduces to

$$\Gamma_{iklj} = \begin{cases} \lambda_i \lambda_j \delta_{ik} \delta_{jl} - \frac{1}{4}[\beta_i^H R_{11}^f \beta_k \beta_l^H R_{22}^f \beta_j \\ \quad + \beta_i^H R_{22}^f \beta_k \beta_l^H R_{11}^f \beta_j \\ \quad - \beta_i^H R_{12}^f \beta_k \beta_l^H R_{21}^f \beta_j \qquad \text{for } i, j, k, l \leqslant 2 \\ \quad - \beta_i^H R_{21}^f \beta_k \beta_l^H R_{12}^f \beta_j] \\ \frac{1}{2}\lambda_i \lambda_j \delta_{ik} \delta_{jl} \\ \quad + \frac{1}{4}[\beta_i^H R_{12}^f \beta_k \beta_l^H R_{21}^f \beta_j \qquad \text{otherwise} \\ \quad + \beta_i^H R_{21}^f \beta_k \beta_l^H R_{12}^f \beta_j] \end{cases} \qquad (48)$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise} \end{cases} \qquad (49)$$

In eqn. 48, Γ_{iklj} is not only dependent on the covariance matrices of the two subarrays, R_{11}^f and R_{22}^f, but also related to the cross correlation matrices R_{12}^f and R_{21}^f. Concerning R_{21}^f defind by eqn. 14, it can be written as

$$R_{21}^f = ADR_u A^H + \sigma^2 J_1 \qquad (50)$$

where J_1 is

$$J_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \qquad (51)$$

As to R_{12}^f, it is equal to $(R_{21}^f)^H$. Substituting eqn. 48 into eqn. 10 and using the following identity (shown in Appendix 9.1),

$$\sum_{k=3}^{M} \beta_k^H J_1 B_k = -\sum_{k=1}^{2} \beta_k^H J_1 \beta_k \qquad (52)$$

it is proved in Appendix 9.2 that the expression for the mean of the null spectrum in the case of forward-only

smoothing and $K = 2$ is given by

$$E[\hat{Q}(\omega)] = Q(\omega) + \frac{1}{2N} \sum_{i=1}^{2} \frac{\sigma^2}{(\lambda_i - \sigma^2)^2}$$

$$\times \left\{ \left[\lambda_i(M - 2) - \text{Re}\left(\beta_i^H R_{12}^f \beta_i \sum_{k=1}^{2} \beta_k^H J_1 \beta_k \right) \right] \right.$$

$$\times |\beta_i^H a(\omega)|^2 - \lambda_i Q(\omega) \right\}$$

$$+ O\!\left(\frac{1}{N^2}\right) \quad \text{for } \omega = \omega_1, \omega_2 \text{ and } \omega_m \quad (53)$$

Then, using $|\beta_i^H R_{12}^f \beta_i| \leqslant \beta_i^H \bar{R}_f \beta_i = \lambda_i$ and $|\beta_k^H J_1 \beta_k| \leqslant \beta_k^H \beta_k = 1$, it can be easily shown that

$$\left| \text{Re}\left\{ \beta_i^H R_{12}^f \beta_i \left(\sum_{k=1}^{2} \beta_k^H J_1 \beta_k \right) \right\} \right| \leqslant 2\lambda_i \quad (54)$$

Therefore, for large M, eqn. 53 may be approximated by

$$E[\hat{Q}(\omega)] \simeq Q(\omega) + \frac{1}{2N} \sum_{i=1}^{2} \frac{\sigma^2 \lambda_i}{(\lambda_i - \sigma^2)^2}$$

$$\times [(M - 2)|\beta_i^H a(\omega)|^2 - Q(\omega)]$$

$$+ O\!\left(\frac{1}{N^2}\right) \quad (55)$$

Note that eqn. 55 is the same as eqn. 15 which represents the forward/backward smoothing case. Therefore, the expressions for the mean and resolution threshold derived from eqn. 55 have forms similar to eqns. 28, 33 and 38, except that the effective correlation coefficient is different. For forward-only smoothing with $K = 2$, the effective correlation coefficient is

$$\rho_i^f = e^{j(\omega_d - \phi)} \cos(\omega_d) \quad (56)$$

Since $|\rho_i^f|$ is independent of ϕ, the bias and resolution threshold of the forward-only smoothing are less sensitive to ϕ than those of the forward/backward smoothing. Therefore, using $\Delta \ll 1$, the resolution threshold may be approximated by

$$\zeta \simeq \frac{20(M - 2)}{N\omega_d^2} \Delta^{-4}(k_0 + k_1 \Delta + k_2 \Delta^2)$$

$$\times \left(1 + \sqrt{\left[1 + \frac{2N\Delta^2}{5(M - 2)} C\right]}\right)$$

$$\simeq M^2(k_0 + k_1 \Delta + k_2 \Delta^2)\xi_T \quad (57)$$

where

$$k_0 = \frac{3 - |\rho|^2 + 2|\rho| \cos \phi}{4} \quad (58)$$

$$k_1 = \frac{\sqrt{3}}{2} |\rho| \sin \phi \quad (59)$$

$$k_2 = -\frac{|\rho|^2 + 1 + 10|\rho| \cos \phi}{16}$$

$$+ \frac{3 - |\rho|^2 - |\rho| \cos \phi}{4M^2} \quad (60)$$

$$C = \frac{1 + |\rho|^2 + 2|\rho| \cos \phi}{3 - |\rho|^2 + 2|\rho| \cos \phi + 2\sqrt{(3)}|\rho| \sin \phi \Delta} $$

$$+ \{[3|\rho|^2 - 66|\rho| \cos \phi - 29]/20\}\Delta^2$$

$$\quad (61)$$

It can be seen from eqn. 57 that for $|\rho| = 1$ and $\phi = 0$, $\zeta \simeq M^2 \xi_T$, where ξ_T is the resolution threshold for the forward/backward smoothing with $K = 1$ in similar source situations.

5 Computer simulations

Computer simulation results are presented in this section. We first simulated the bias and resolution threshold for the forward/backward smoothing with $K = 1$. Fig. 1 presents the bias computed from eqn. 28 and the bias averaged from 30 independent simulation trials with $M = 10$, $|\rho| = 1$, SNR = 30 dB, $N = 100$, and $\phi = -60°$, $0°$ and $30°$. Fig. 2 shows the bias for $|\rho| = 1$ and $|\rho| = 0.1$ with $M = 10$, SNR of the second source $= 20$ dB, $\theta_1 = 69°$ and $\theta_2 = 75°$. Results show that the computed and averaged biases match very well. The resolution threshold was simulated with parameters the same as those used in Fig. 1 except that $M = 7$. Table 1 shows the probability of resolution averaged from a hundred independent trials for different SNR and angular separation. The resolution threshold is chosen to be the SNR where the probability of resolution is 0.5.
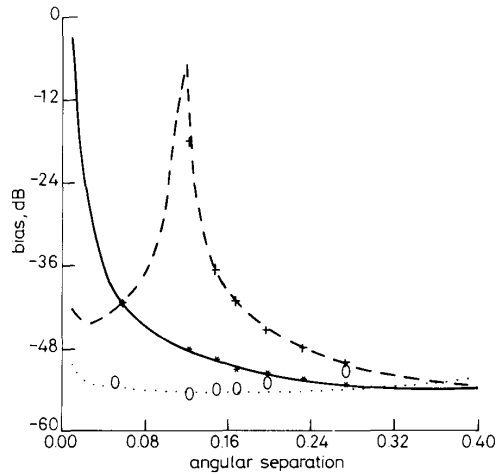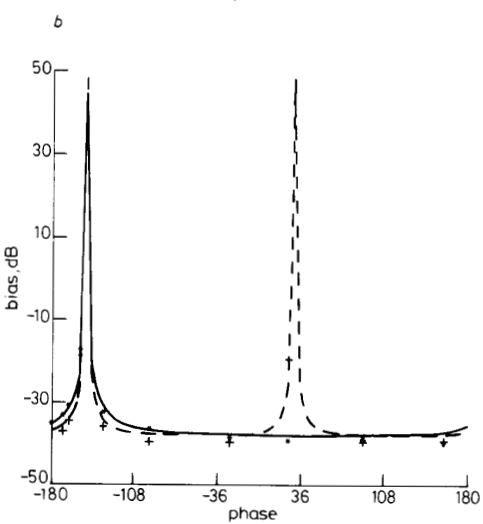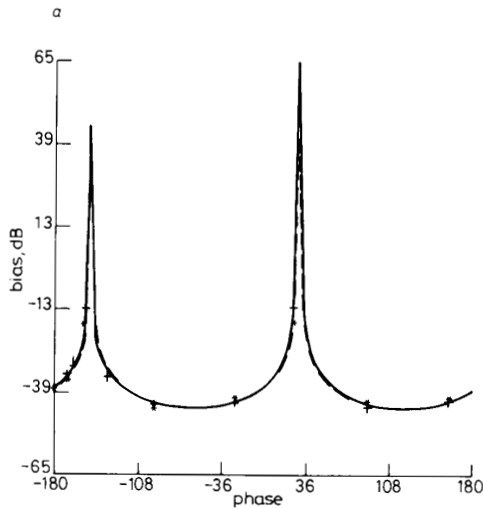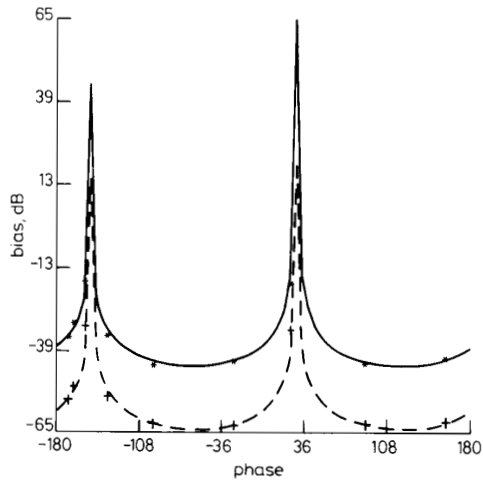

**Fig. 1** *Bias at one of the arrival angles for forward/backward smoothing*

$K = 1$, ten sensors, $|\rho| = 1$, SNR = 30 dB and 100 snapshots; —— computed bias for $\phi = 0°$; – – – computed bias for $\phi = 30°$; ⋯⋯ computed bias for $\phi = -60°$; • averaged bias for $\phi = 0°$; + averaged bias for $\phi = 30°$; ○ averaged bias for $\phi = -60°$

Fig. 3 plots the resolution threshold obtained from Table 1 and that computed from eqn. 38. Fig. 4 shows the resolution threshold simulated with the same parameters as used in Fig. 2. Again, the theoretical results agree with the results obtained from Monte Carlo simulations. From Figs. 3 and 4, it may be observed that the resolution threshold has a peak value for ϕ close to $(M - 1)\omega_d$ or $(M - 1)\omega_d - \pi$. This result is consistant with the analysis in eqn. 46. The bias and resolution threshold for the forward-only smoothing with $K = 2$ is presented in Figs. 5–8. Here, M is chosen to be ten in simulating both the bias and resolution threshold. Figs. 5 and 6 show the bias simulation results, and Figs. 7 and 8 present the resolution thresholds. From Figs. 4 and 8, the results show that the resolution threshold of the forward/backward smoothed case with $K = 1$ is lower than that of the forward-only smoothed case with $K = 2$ except when ϕ is close to $(M - 1)\omega_d$ or $(M - 1)\omega_d - \pi$.
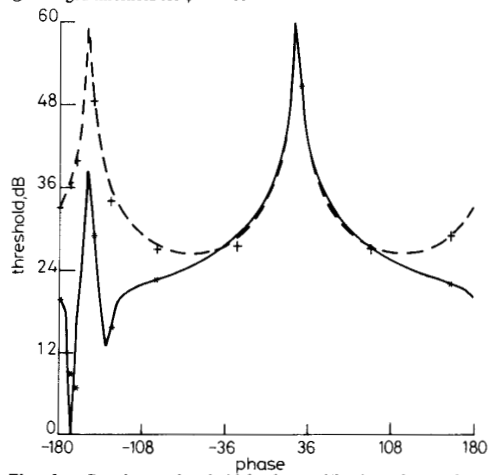
a

b

c

Fig. 2   *Bias for forward/backward smoothing*

$K = 1$, ten sensors, 100 snapshots, SNR of the weaker source equal to 20 dB, $\theta_1 = 69°$ and $\theta_2 = 75°$; ——— computed bias for $|\rho| = 1$; – – – – computed bias for $|\rho| = 0.1$; * averaged bias for $|\rho| = 1$; + averaged bias for $|\rho| = 0.1$

*a* bias at ω_1
b bias at ω_2
c bias at ω_m
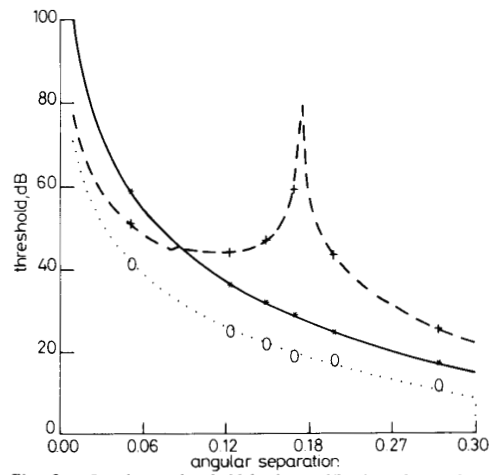
Fig. 3   *Resolution threshold for forward/backward smoothing*

$K = 1$, seven sensors, $|\rho| = 1$ and 100 snapshots; ——— computed threshold for $\phi = 0°$; – – – – computed threshold for $\phi = 30°$; ······· computed threshold for $\phi = -60°$; • averaged threshold for $\phi = 0°$; + averaged threshold for $\phi = 30°$; ○ averaged threshold for $\phi = -60°$
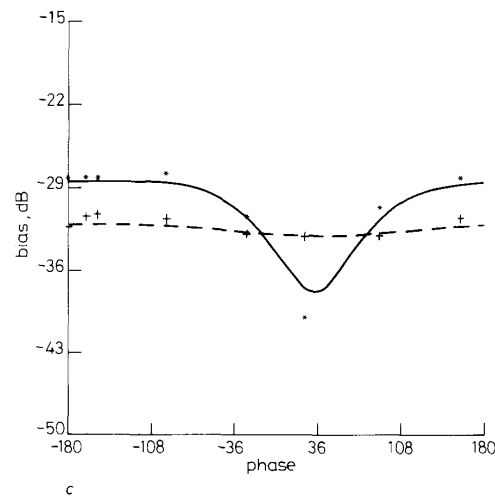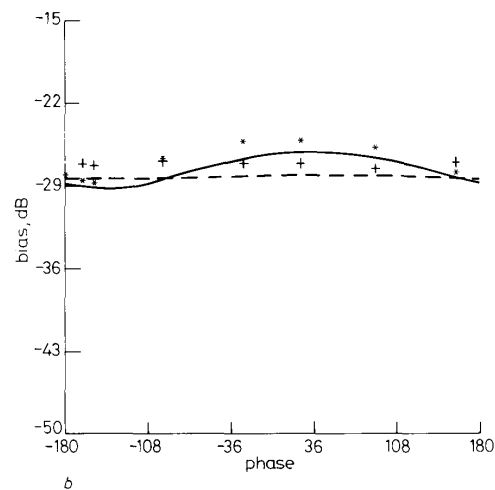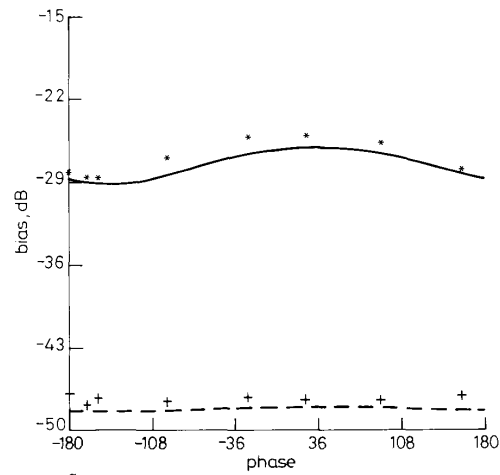

Fig. 4   *Resolution threshold for forward/backward smoothing*

$K = 1$, ten sensors, 100 snapshots, $\theta_1 = 69°$ and $\theta_2 = 75°$; ——— computed thresholds for $|\rho| = 1$; – – – – computed threshold for $|\rho| = 0.1$; * averaged threshold for $|\rho| = 1$; + averaged threshold for $|\rho| = 0.1$
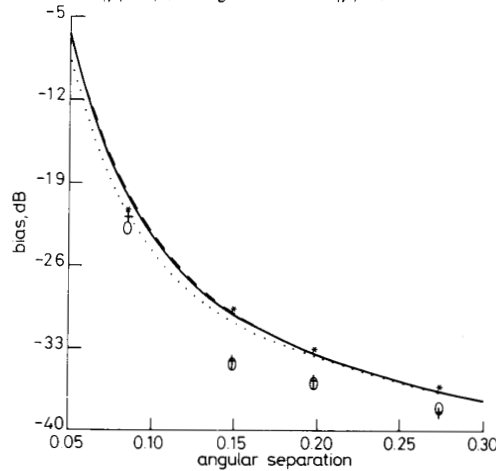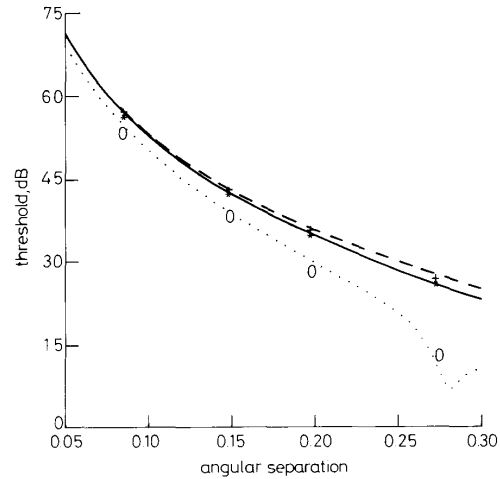

Fig. 5   *Bias at one of the arrival angles for forward-only smoothing*

$K = 2$, ten sensors, $|\rho| = 1$, SNR = 30 dB and 100 snapshots; ——— computed bias for $\phi = 0°$; – – – – computed bias for $\phi = 30°$; ······· computed bias for $\phi = -60°$; * averaged bias for $\phi = 0°$; + averaged bias for $\phi = 30°$; ○ averaged bias for $\phi = -60°$

Fig. 6   Bias for forward only smoothing

$K = 2$, ten sensors, 100 snapshots, SNR of the weaker source equal to 30 dB, $\theta_1 = 69°$ and $\theta_2 = 75°$; ———— computed bias for $|\rho| = 1$; — — — computed bias for $|\rho| = 0.1$; * averaged bias for $|\rho| = 1$; + averaged bias for $|\rho|q0.1$
a Bias at ω_1
b Bias at ω_2
c Bias at ω_3

Fig. 7   Resolution threshold for forward only smoothing

$K = 2$, ten sensors, $|\rho| = 1$ and 100 snapshots; ———— computed threshold for $\phi = 0°$; — — — computed threshold for $\phi = 30°$; ······ computed threshold for $\phi = -60°$; * averaged threshold for $\phi = 0°$; + averaged threshold for $\phi = 30°$; ○, averaged threshold for $\phi = -60°$
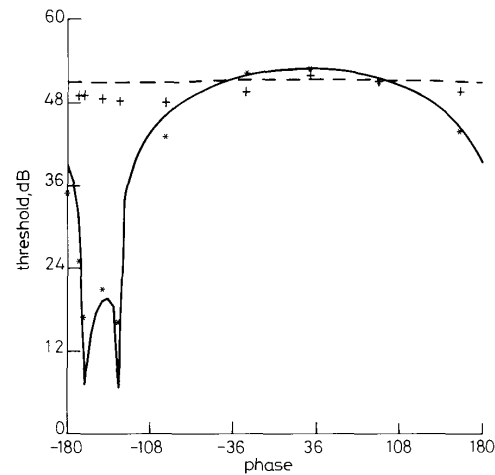
Fig. 8   Resolution threshold for forward only smoothing

$K = 2$, ten sensors, 100 snapshots, $\theta_1 = 69°$ and $\theta_2 = 75°$; ———— computed threshold for $|\rho| = 1$; — — — computed threshold for $|\rho| = 0.1$; * averaged threshold for $|\rho| = 1$; + averaged threshold for $|\rho| = 0.1$

6   Conclusion

We have derived the bias and resolution threshold of the MUSIC method with forward/backward smoothing and forward-only smoothing for two coherent sources with different phase and unequal power. The numbers of sub-arrays used for derivation are one for forward/backward smoothing and two for forward-only smoothing. Results show that the resolution threshold of the forward/backward smoothing scheme is highly dependent on the phase difference between the two coherent sources whereas that of the forward-only smoothing is not, except when the power ratio of the two sources is close to 1. The resolution thresholds associated with both of these smoothing schemes also depend on the power ratio of the two sources and are mainly dictated by the SNR of the weaker source. The simulation results indicate that the resolution threshold of the forward/backward smoothing

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Angles of arrival		Ang. sep.	$\phi = -60°$		$\phi = 0°$		$\phi = 30°$	
θ_1	θ_2	$2\omega_d$	SNR	Prob.	SNR	Prob.	SNR	Prob.
84	79	0.0514	40	0.39	57	0.30	49	0.28
			41	0.46	58	0.37	50	0.35
			42	0.57	59	0.48	51	0.48
			43	0.64	60	0.60	52	0.62
71	65	0.1231	23	0.34	34	0.24	42	0.29
			24	0.41	35	0.37	43	0.39
			25	0.52	36	0.53	44	0.52
			26	0.59	37	0.67	45	0.67
66	60	0.1493	20	0.35	29	0.22	45	0.25
			21	0.40	30	0.33	46	0.38
			22	0.50	31	0.46	47	0.56
			23	0.63	32	0.59	48	0.64
62	56	0.1694	17	0.37	27	0.29	57	0.28
			18	0.45	28	0.41	58	0.41
			19	0.52	29	0.56	59	0.51
			20	0.60	30	0.67	60	0.55
56	50	0.1979	15	0.33	23	0.25	41	0.16
			16	0.40	24	0.38	42	0.26
			17	0.46	25	0.52	43	0.47
			18	0.59	26	0.68	44	0.65
48	41	0.2736	9	0.38	15	0.28	24	0.26
			10	0.41	16	0.40	25	0.45
			11	0.49	17	0.48	26	0.60
			12	0.56	18	0.60	27	0.69

with $K = 1$ is usually smaller than that of the forward-only smoothing with $K = 2$ except when the absolute value of the correlation coefficient is very close to unity, i.e. the phase difference between the two coherent sources is close to $(M - 1)\omega_d$ or $(M - 1)\omega_d - \pi$.

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9 Appendix

9.1 Derivation of eqn. 52

Since $\beta_k, k = 1, \ldots, M$, are a set of orthogonal vectors,

$$[\beta_1 \quad \beta_2 \quad \cdots \quad \beta_M] \begin{bmatrix} \beta_1^H \\ \beta_2^H \\ \vdots \\ \beta_M^H \end{bmatrix} = I \tag{62}$$

Define

$$d_i = [\beta_{i1} \quad \beta_{i2}]^T \tag{63}$$

and

$$r_i = [\beta_{i3} \quad \cdots \quad \beta_{iM}]^T \tag{64}$$

where $\beta_{ik}, i, k = 1, 2, \ldots, M$, is the ith element of the vector $\beta_k, k = 1, 2, \ldots, M$. Using eqns. 63 and 64, eqn. 62 can be written as

$$\begin{bmatrix} d_1^H & r_1^H \\ d_2^H & r_2^H \\ \vdots & \vdots \\ d_M^H & r_M^H \end{bmatrix} \begin{bmatrix} d_1 & d_2 & \cdots & d_M \\ r_1 & r_2 & \cdots & r_M \end{bmatrix} = I \tag{65}$$

In eqn. 65, the elements off the main diagonal are zeros. Therefore, we have

$$d_i^H d_{i+1} + r_i^H r_{i+1} = 0 \tag{66}$$

for $i = 1, 2, \ldots, M - 1$. Using eqn. 64, the right-hand side of eqn. 52 becomes

$$\sum_{k=3}^{M} \beta_k^H J_1 \beta_k = \sum_{k=3}^{M} \left(\sum_{i=1}^{M-1} \beta_{ik}^* \beta_{(i+1)k} \right)$$
$$= \sum_{i=1}^{M-1} \left(\sum_{k=3}^{M} \beta_{ik}^* \beta_{(i+1)k} \right)$$
$$= \sum_{i=1}^{M-1} r_i^H r_{i+1} \tag{67}$$

Using eqn. 66, eqn. 67 can be written as

$$\sum_{k=3}^{M} \beta_k^H J_1 \beta_k = - \sum_{i=1}^{M-1} d_i^H d_{i+1}$$
$$= - \sum_{k=1}^{2} \left(\sum_{i=1}^{M-1} \beta_{ik}^* \beta_{(i+1)k} \right)$$
$$= - \sum_{k=1}^{2} \beta_k^H J_1 \beta_k \tag{68}$$

which is eqn. 52.

9.2 Derivation of eqn. 53

Substituting eqn. 48 into eqn. 10 we get

$$E[\hat{Q}(\omega)] = Q(\omega) + \frac{1}{N}\sum_{i=1}^{2}\left[\sum_{\substack{k=1\\k\neq i}}^{M}\frac{\Gamma_{iikk}^{f}}{(\lambda_i-\lambda_k)^2}|\beta_i^H a(\omega)|^2\right.$$

$$\left. -\sum_{\substack{k=1\\k\neq i}}^{M}\sum_{\substack{l=1\\l\neq i}}^{M}\frac{\Gamma_{iikl}^{f}}{(\lambda_i-\lambda_k)(\lambda_i-\lambda_l)}a^H(\omega)\beta_k\beta_l^H a(\omega)\right]$$

$$+ O\left(\frac{1}{N^2}\right)$$

$$= Q(\omega) + \frac{1}{N}$$

$$\times \sum_{i=1}^{2}\{A_i + B_i + C_i + D_i + E_i + F_i + G_i\}$$

$$+ O\left(\frac{1}{N^2}\right) \tag{69}$$

Here,

$$A_i = \sum_{k=3}^{M}\frac{1}{2}\frac{\lambda_i\sigma^2}{(\lambda_i-\sigma^2)^2}(|\beta_i^H a(\omega)|^2 - |\beta_k^H a(\omega)|^2)$$

$$= \frac{1}{2}\frac{\lambda_i\sigma^2}{(\lambda_i-\sigma^2)^2}\{(M-2)|\beta_i^H a(\omega)|^2 - Q(\omega)\} \tag{70}$$

$$B_i = \frac{1}{4}\sum_{k=3}^{M}\frac{\beta_i^H R_{12}^f\beta_i\beta_k^H R_{21}^f\beta_k + \beta_i^H R_{21}^f\beta_i\beta_k^H R_{12}^f\beta_k}{(\lambda_i-\sigma^2)^2}$$

$$\times |\beta_i^H a(\omega)|^2$$

$$= -\text{Re}\left\{\frac{\sigma^2(\beta_i^H R_{12}^f\beta_i)}{2(\lambda_i-\sigma^2)^2}\right.$$

$$\left.\times\left(\sum_{k=1}^{2}\beta_k^H J_1\beta_k\right)|\beta_i^H a(\omega)|^2\right\} \tag{71}$$

where J_1 is defined by eqn. 51, and

$$C_i = \sum_{\substack{k=1\\k\neq i}}^{2}\frac{\lambda_i\lambda_k}{(\lambda_i-\lambda_k)^2}(|\beta_i^H a(\omega)|^2 - |\beta_k^H a(\omega)|^2) \tag{72}$$

$$D_i = \frac{-1}{4}\sum_{k=3}^{M}\sum_{\substack{l=1\\l\neq i}}^{M}\frac{\beta_i^H R_{12}^f\beta_i\beta_k^H R_{21}^f\beta_l + \beta_i^H R_{21}^f\beta_i\beta_k^H R_{12}^f\beta_l}{(\lambda_i-\sigma^2)(\lambda_i-\lambda_l)}$$

$$\times a^H(\omega)\beta_k\beta_l^H a(\omega) \tag{73}$$

$$E_i = \frac{-1}{4}\sum_{\substack{k=1\\k\neq i}}^{2}\frac{|\beta_i^H a(\omega)|^2}{(\lambda_i-\lambda_k)^2}\{\beta_i^H R_{11}^f\beta_i\beta_k^H R_{22}^f\beta_k$$

$$+ \beta_i^H R_{22}^f\beta_i\beta_k^H R_{11}^f\beta_k - \beta_i^H R_{12}^f\beta_i\beta_k^H R_{21}^f\beta_k$$

$$- \beta_i^H R_{21}^f\beta_i\beta_k^H R_{12}^f\beta_k\} \tag{74}$$

$$F_i = \frac{1}{4}\sum_{\substack{k=1\\k\neq i}}^{2}\sum_{\substack{l=1\\l\neq i}}^{2}\frac{a^H(\omega)\beta_k\beta_l^H a(\omega)}{(\lambda_i-\lambda_k)(\lambda_i-\lambda_l)}$$

$$\times \{\beta_i^H R_{11}^f\beta_i\beta_k^H R_{22}^f\beta_l + \beta_i^H R_{22}^f\beta_i\beta_k^H R_{11}^f\beta_l$$

$$- \beta_i^H R_{12}^f\beta_i\beta_k^H R_{21}^f\beta_l - \beta_i^H R_{21}^f\beta_i\beta_k^H R_{12}^f\beta_l\} \tag{75}$$

$$G_i = \frac{-1}{4}$$

$$\times \sum_{\substack{k=1\\k\neq i}}^{2}\sum_{l=3}^{M}\frac{\beta_i^H R_{12}^f\beta_i\beta_k^H R_{21}^f\beta_l + \beta_i^H R_{21}^f\beta_i\beta_k^H R_{12}^f\beta_l}{(\lambda_i-\lambda_k)(\lambda_i-\lambda_l)}$$

$$\times a^H(\omega)\beta_k\beta_l^H a(\omega) \tag{76}$$

Since $\sum C_i = 0$, $\sum(E_i + F_i) = 0$, and D_i and G_i are close to zero for ω equal to ω_1, ω_2 and ω_m, we have from eqn. 69

$$E[\hat{Q}(\omega)] = Q(\omega) + \frac{1}{2N}\sum_{i=1}^{2}\frac{\sigma^2}{(\lambda_i-\sigma^2)^2}$$

$$\times \left\{\left[\lambda_i(M-2) - \text{Re}\left(\beta_i^H R_{12}^f\beta_i\sum_{k=1}^{2}\beta_k^H J_1\beta_k\right)\right]\right.$$

$$\left.\times |\beta_i^H a(\omega)|^2 - \lambda_i Q(\omega)\right\}$$

$$+ O\left(\frac{1}{N^2}\right) \quad \text{for } \omega = \omega_1, \omega_2 \text{ and } \omega_m \tag{77}$$