Time Domain Electromagnetic Field Computation with a Semi-discrete Scheme as an Alternative to Implicit FDTD Methods

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Abstract

A novel semi-discrete scheme is presented as an alternative to the implicit FDTD methods. This new scheme is also unconditionally stable, with much better numerical dispersion and no truncation errors in time-domain. A preliminary numerical experiment for one-dimensional propagation in free space shows that this semi-discrete scheme is still accurate even when the time step size is equal to or larger than the Nyquist rate, which outperforms the alternate-direction-implicit (ADI) FDTD and the Crank-Nicolson (CN) FDTD methods. The efficiency of this new scheme depends on the algorithm of computing the matrix exponential of a sparse matrix. A MATLAB implementation of this scheme is slow, but can be improved in the future.

Introduction

Implicit methods have been proposed to eliminate the Courant-Friedrich-Levy limit of the finite-difference time-domain (FDTD) methods [1]-[3]. However, as pointed out in [4], the Courant number in these methods is still limited by the numerical dispersion and truncation errors, though the methods are unconditionally stable. In this paper a new computation scheme based on the semi-discrete Maxwell equations in [5] is presented. Since the differentiation in time is not approximated, there will be no time-domain truncation errors. The stability and numerical dispersion analysis for 3D problems will be given. A preliminary 1D numerical experiment is also conducted to compare this new scheme with the ADI and CN FDTD methods.

Semi-discrete Scheme

Assume that the 3D space is subdivided into Yee cells, the Maxwell equations can be approximated as

$$\frac{dE_x(i+1/2,j,k)}{dt}=\frac{1}{\varepsilon}\{\frac{H_z(i+1/2,j+1/2,k)-H_z(i+1/2,j-1/2,k)}{\Delta y}-\frac{H_y(i+1/2,j,k+1/2)-H_y(i+1/2,j,k-1/2)}{\Delta z}\} \tag{1a}$$

$$\frac{dE_y(i,j+1/2,k)}{dt}=\frac{1}{\varepsilon}\{\frac{H_x(i,j+1/2,k+1/2)-H_x(i,j+1/2,k-1/2)}{\Delta z}-\frac{H_z(i+1/2,j+1/2,k)-H_z(i-1/2,j+1/2,k)}{\Delta x}\} \tag{1b}$$

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$$\frac{dE_z(i,j,k+1/2)}{dt}=\frac{1}{\varepsilon}\left\{\frac{H_y(i+1/2,j,k+1/2)-H_y(i-1/2,j,k+1/2)}{\Delta x}-\frac{H_x(i,j+1/2,k+1/2)-H_x(i,j-1/2,k+1/2)}{\Delta y}\right\} \tag{1c}$$

$$\frac{dH_x(i,j+1/2,k+1/2)}{dt}=\frac{1}{\mu}\left\{\frac{E_y(i,j+1/2,k+1)-E_y(i,j+1/2,k)}{\Delta z}-\frac{E_z(i,j+1,k+1/2)-E_z(i,j,k+1/2)}{\Delta y}\right\} \tag{1d}$$

$$\frac{dH_y(i+1/2,j,k+1/2)}{dt}=\frac{1}{\mu}\left\{\frac{E_z(i+1,j,k+1/2)-E_z(i,j,k+1/2)}{\Delta x}-\frac{E_x(i+1/2,j,k+1)-E_x(i+1/2,j,k)}{\Delta z}\right\} \tag{1e}$$

$$\frac{dH_z(i+1/2,j+1/2,k)}{dt}=\frac{1}{\mu}\left\{\frac{E_x(i+1/2,j+1,k)-E_x(i+1/2,j,k)}{\Delta y}-\frac{E_y(i+1,j+1/2,k)-E_y(i,j+1/2,k)}{\Delta x}\right\}. \tag{1f}$$

This set of semi-discrete Maxwell equations can be written as a system of ordinary differential equations

$$\frac{d[X]}{dt}=[A][X] \tag{2}$$

Here $[X]$ is the column vector obtained by arranging all field components on the Yee cells, and $[A]$ is a sparse matrix. The solution of (2) is of the form

$$[X]=e^{[A]t}[X]_0, \tag{3}$$

where $[X]_0$ is related to the initial field distribution at $t=0$. If we sample the field at $t=n\Delta t$, we will have the following iterative relationship

$$[X]_n=e^{[A]\Delta t}[X]_{n-1}. \tag{4}$$

Note that both of the E-field and the H-field are sampled at the same time, which is different from common FDTD methods. By (4), we just need to compute the matrix exponential $e^{[A]\Delta t}$ once, and then iterate to find the subsequent field distributions.

To analyze the stability, we assume that the elements in $[X]$ are in the form of $E_x(i+1/2,j,k)=E_{x0}(t)e^{-j(ik_x\Delta x+k_y\Delta y+k_z\Delta z)}e^{-jk_x\Delta x/2}$. The equation (2) then become

$\frac{d[X_0]}{dt} = [A_0][X_0]$ with $[X_0] = [E_{x0}\ E_{y0}\ E_{z0}\ H_{x0}\ H_{y0}\ H_{z0}]^T$. We can prove that all eigenvalues of $[A_0]$ are imaginary, thus the solution (3) will be always stable. The numerical dispersion can be obtained by assuming $[X_0] = [\mathrm{X}_0]e^{jn\omega\Delta t}$, and transforming (4) to a matrix eigenvalue equation. With the help of Mathematica, the eigenvalues are solved out to find the numerical dispersion relationship as

$$\left(\frac{\omega}{c}\right)^2 = k_x^2 \sin^2\left(\frac{k_x \Delta x}{2}\right) + k_y^2 \sin^2\left(\frac{k_y \Delta y}{2}\right) + k_z^2 \sin^2\left(\frac{k_z \Delta z}{2}\right) \tag{5}$$

which is independent of the time step size Δt. In other words, we are free to choose the time step size for the iterative scheme (4). Note that $c = 1/\sqrt{\varepsilon\mu}$ in (5).

The efficiency of this scheme depends on the computation of the Matrix exponential. Many algorithms have been developed [6]. The most promising one could be the one based on the Krylov subspace approximations. The authors of [7] reported that their Krylov approximate solution to a 2D parabolic equation is much faster than the Crank-Nicolson method.

To compare the semi-discrete scheme with implicit FDTD methods, let's consider a one-dimensional free space propagation problem. A 1GHz sine wave is excited from time $t = 0$ at $z = 0$ and the wave propagates to $z > 0$, which is discretized according to the mesh density $N = \lambda_0 / \Delta z = 20$, where λ_0 is the free space wavelength, and Δz is the grid cell size. Figure 1 shows the numerical dispersions of the semi-discrete (SD) scheme and the ADI-FDTD (ADI) as well as the Crank-Nicolson (CN) methods for different Courant numbers $s = c_0 \Delta t / \Delta z$, with c_0 the light speed in free space. Note that the numerical dispersion relationship of the ADI and the CN methods are identical in the one-dimensional case. From Fig. 1 we see that the semi-discrete scheme, as the limiting case of the implicit FDTD methods, is with the smallest numerical dispersion. Figure 2 exhibits the E_x and $\eta_0 H_y$ field distribution at $t = 12$ ns by three methods implemented in MATLAB 5 with the Courant number $s = 10$, which corresponds to the Nyquist rate. Here η_0 is the intrinsic impedance of the free space. We can find that the curves obtained by the semi-discrete scheme are still accurate, while the implicit FDTD methods deviate from the exact solution significantly. We also compared the average absolute errors of 250 cells under different Courant numbers. As expected, the error of the SD scheme is almost invariant with respect to s, which would be advantageous if embedded in an explicit FDTD scheme. One of the few disadvantages of this semi-discrete solution is that it is slow, because we used MATLAB function *expm* to compute the matrix exponential. The whole scheme can be improved if the sparsity of the matrix and more efficient matrix exponential computation algorithms are adopted in the future.

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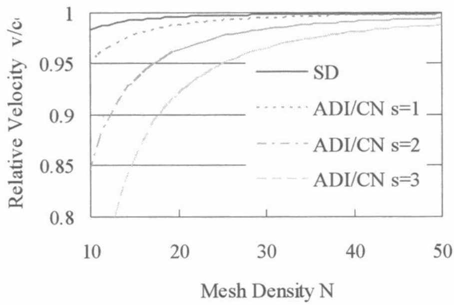

Fig. 1. Numerical dispersions

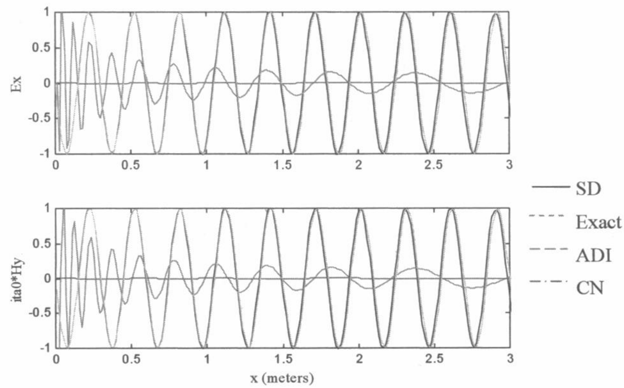

Fig. 2. One-dimensional field distribution computed by four methods