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Decoupling of continuous-time linear time-invariant systems using generalized sampled-data hold functions

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Abstract: We investigate the possibility of input–output decoupling without stability a square continuous-time LTI plant (C, A, B) by designing discrete devices, the generalized sampled-data hold functions. By adopting sequential design procedures for the resulting diagonalizing problem, two cases are found to be solvable and yield nontrivial solutions: (1) CB nonsingular; (2) $m \leqslant n \leqslant 2m$, $\mathrm{rk}(C^{\perp}B) = n - m$ where n, m are the number of states and inputs (outputs) respectively; $C^{\perp} = I_n - C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1}C$.

Keywords: Generalized sampled-data hold function; decoupling; periodic control; digital control.

1. Introduction

The use of generalized sampled-data hold functions (GSHF), as a means of periodic control of sampled continuous-time linear systems, has been investigated in various control system design problems, e.g. pole/zero assignment [2,4,6], simultaneous stabilization and decoupling [6] in recent years. The GSHF controller can be viewed as a sampler with a prescribed periodically switching hold function which is tailored to the particular plant and the problem under investigation. Despite some promising results for discrete-time systems using this idea, the potential of GSHF control method for continuous-time LTI systems has not yet been fully understood. This paper will investigate one such possibility of GSHF control method for decoupling continuous-time LTI systems.

The plant considered here is a continuous-time linear time-invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t)$, $y(t) \in \mathbb{R}^m$, and B, C are of full rank $m \leqslant n$. The GSHF control law

$$u(t) = F(t)x(kT) + G(t)r(kT), \quad kT \leqslant t < (k+1)T, \tag{2}$$
$$F(t+T) = F(t) \in \mathbb{R}^{m \times n}, \qquad G(t+T) = G(t) \in \mathbb{R}^{m \times m}$$

is applied to system (1), where T is the sampling period and $F(t)$, $G(t)$ are piecewise continuous, bounded, T-periodic matrices. The overall block diagram is shown in Figure 1. A motivating question for our study is: Is it possible to use digital controller (2) like an analog control device so as to control, in

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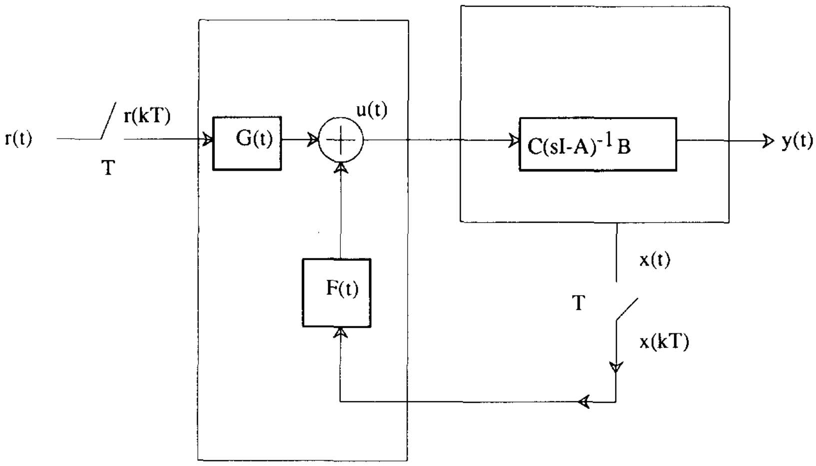

Fig. 1.

particular decouple continuous plant (1) for all time instants, not just at the sampling instants. Specifically, our objective is to design $F(t)$, $G(t)$ such that the closed-loop continuous system (1)–(2) is input–output decoupled, where stability of the closed-loop decoupled system is not taken into account in this study.

The organization of this paper is outlined as follows. Section 2 contains the formulation of the decoupling problem by GSHF control into a pair of diagonalizing problems. Two sequential design procedures for diagonalization are provided and situations under which nontrivial solutions exist are revealed in Section 3. Section 4 provides examples to illustrate the findings. Conclusions are made in Section 5. Throughout this paper, we use the following notations: N for null space, R for range space, rk for rank, Ker for kernel, $\hat{M}(s)$ for the Laplace transform of the time function $M(t)$, and I_n for the $n \times n$ identity matrix. The rank of a transfer matrix, rk($W(s)$), is defined as the normal rank of $W(s)$ [5].

2. Problem formulation

The state response of the closed-loop system (1)–(2) is

$$x(kT+\sigma) = \mathrm{e}^{A\sigma}x(kT) + \int_{kT}^{kT+\sigma} \mathrm{e}^{A(kT+\sigma-\tau)}B[F(\tau)x(kT) + G(\tau)r(kT)]\,\mathrm{d}\tau$$
$$:= M(\sigma)x(kT) + N(\sigma)r(kT), \quad \sigma \in [0, T], \tag{3}$$

where

$$M(\sigma) := \mathrm{e}^{A\sigma} + \int_0^{\sigma} \mathrm{e}^{A(\sigma-\tau)}BF(\tau)\,\mathrm{d}\tau, \tag{4}$$

$$N(\sigma) := \int_0^{\sigma} \mathrm{e}^{A(\sigma-\tau)}BG(\tau)\,\mathrm{d}\tau. \tag{5}$$

$M(\cdot)$ is T-periodic, piecewise continuous and bounded. The output response is thus given by

$$\begin{aligned} y(kT+\sigma) &= Cx(kT+\sigma) \\ &= CM(\sigma)\big[M^k(T)x(0) + M^{k-1}(T)N(T)r(0) \\ &\qquad + M^{k-2}(T)N(T)r(T) + \cdots + N(T)r((k-1)T)\big] \\ &\quad + CN(\sigma)r(kT). \end{aligned} \tag{6}$$

The decoupling matrix of plant (1) is [3]

$$B^* := \begin{bmatrix} C_1 A^{d_1} B \\ \vdots \\ C_m A^{d_m} B \end{bmatrix} \tag{7}$$

where C_i is the i-th row of C and the decoupling index d_i is defined by

$$d_i := \begin{cases} \min\{k: C_i A^k B \neq 0\}, \\ n-1, \quad \text{if } C_i A^k B = 0 \text{ for all } k = 0, 1, \cdots. \end{cases}$$

Throughout this paper, we assume that the plant (1) is decouplable by static state feedback, or B^* is nonsingular.

From the expression (6) and arbitrariness of reference $r(\cdot)$, we see that the input–output response is decoupled for all time if

$$CM(\sigma)M^k(T)N(T) = \bar{\Lambda}(kT+\sigma), \quad k = 0, 1, 2, \ldots, \tag{8}$$

$$CN(\sigma) = \Lambda(\sigma) \tag{9}$$

where $\bar{\Lambda}$, Λ are $m \times m$ diagonal matrices. Observe that the left-hand-side of (8) can be generated from the output equation of the following matrix differential equation, in view of (4),

$$\begin{aligned} \dot{M}(\sigma) &= AM(\sigma) + BF(\sigma), \quad M(0) = I_n, \\ Y(kT+\sigma) &= CM(\sigma)M^k(T)N(T). \end{aligned} \tag{10}$$

One of our aims is to diagonalize output Y of (10) by designing $F(\sigma)$, which corresponds to the requirement (8). In the light of standard linear decoupling theory [3], we then choose $F(\sigma)$ of the constant gain form

$$F(\sigma) = KM(\sigma), \quad \sigma \in [0, T). \tag{11}$$

With this choice, the solution of (10) is

$$M(\sigma) = \exp((A+BK)\sigma), \quad \sigma \in [0, T), \tag{12}$$

and

$$Y(kT+\sigma) = C \exp((A+BK)(kT+\sigma))N(T). \tag{13}$$

The decoupling objective (8)–(9) by GSHF control is then transformed into the following pair of diagonalizing problems:

Find hold functions $F(t)$ (of the form given by (11)) and $G(t)$ such that

$$C \exp((A+BK)t)N(T) = \bar{\Lambda}(t) = \operatorname{diag}(\bar{\lambda}_i(t)), \tag{14}$$

$$CN(t) = \Lambda(t) = \operatorname{diag}(\lambda_i(t)), \quad t \geqslant 0. \tag{15}$$

If case (14)–(15) can be achieved, the decoupled input–output response is, with zero initial state,

$$\begin{aligned} y_i(kT+\sigma) = {} & \bar{\lambda}_i((k-1)T+\sigma)r_i(0) + \cdots + \bar{\lambda}_i(\sigma)r_i((k-1)T) \\ & + \lambda_i(\sigma)r_i(kT), \quad i = 1, \ldots, m. \end{aligned} \tag{16}$$

3. Two sequential design procedures

Depending on which equation of diagonalizing requirements (14)–(15) is to be tackled at first, two sequential design procedures for hold functions $F(t)$ and $G(t)$ are presented in this section.

3.1. Approach 1: solving (14) first by designing F(t)

To meet the requirement (14), we choose, motivated from continuous-time linear decoupling theory [3], matrices K and $N(T)$ as the standard decoupling gain matrices

$$K = -(B^*)^{-1} \begin{bmatrix} C_1 A^{d_1+1} & + & \sum_{j=0}^{d_1} p_{1j} C_1 A^j \\ & \vdots & \\ C_m A^{d_m+1} & + & \sum_{j=0}^{d_m} p_{mj} C_m A^j \end{bmatrix}, \quad p_{ij} > 0, \tag{17}$$

and

$$N(T) = B(B^*)^{-1} \Gamma \tag{18}$$

where $\Gamma = \text{diag}(r_i)$ is any $m \times m$ diagonal matrix. Then (14) is accomplished. Indeed, the Laplace transform of (14) is immediately seen to be a diagonal transfer matrix of the form

$$C(sI_n - (A+BK))^{-1} N(T) = \text{diag}\left(\frac{r_i}{s^{d_i+1} + p_{id_i} s^{d_i} + \cdots + p_{i0}} \right)$$
$$=: \text{diag}\left(\hat{\lambda}_i(s) \right).$$

Finally the required hold function $F(t)$ can be calculated by using (11), (12), (17). This completes the design of $F(t)$.

Next we should meet the requirement (15) under the boundary constraint (18). The overall requirements of $G(t)$ are collected in the following equation:

$$CN(t) = \Lambda(t) = \text{diag}(\lambda_i(t)), \tag{19}$$

$$N(0) = 0, \qquad N(T) = B(B^*)^{-1} \Gamma. \tag{20}$$

Note that by (20) we have

$$\Lambda(0) = CN(0) = 0,$$
$$\Lambda(T) = CN(T) = CB(B^*)^{-1} \Gamma = \text{diag}(0, \ldots, r_i, 0, \ldots, 0)$$

where the r_i's appear in those i-th positions with $i \in I$ where

$$I = \{k \colon d_k = 0\}. \tag{21}$$

In summary, we have:

Lemma 3.1. *The admissible diagonal matrix $\Lambda(t)$ must satisfy the constraints*

$$\lambda_i(0) = 0, \qquad \lambda_i(T) = \begin{cases} 0, & i \notin I, \\ r_i, & i \in I. \end{cases} \tag{22}$$

A possible candidate solution of $N(t)$ to (19)–(20) is provided in the following lemma.

Lemma 3.2. *A family of solutions $N(t)$ to (19)–(20) has the form*

$$N(t)=C^{+}\Lambda(t)+C^{\perp}B(B^{*})^{-1}\Gamma Q(t)$$

where $C^{+}=C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1}$, $C^{\perp}=I_n-C^{+}C$, $Q(t)$ is an $m\times m$ square matrix with $Q(0)=0$, $Q(T)=I_m$.

Proof. The general solution to the algebraic equation (19) is

$$N(t)=C^{+}\Lambda(t)+H(t)$$

where $H(t)\in \mathrm{Ker}(C)$, i.e. $CH(t)=0$. The constraints (20) on $N(t)$ are then transfered to H:

$$H(0)=0,\qquad H(T)=N(T)-C^{+}\Lambda(T)=C^{\perp}B(B^{*})^{-1}\Gamma\in \mathrm{Ker}(C).$$

So we can choose

$$H(t)=C^{\perp}B(B^{*})^{-1}\Gamma Q(t)$$

with $Q(0)=0$, $Q(T)=I_m$. □

Remark. It is easily shown that $\mathrm{rk}(C^{\perp})=n-m$.

In the sequel, we need the following.

Fact 3.3 [3]. Let U, V be $q\times n$ and $n\times m$ matrices, then

$$\mathrm{rk}(UV)=\mathrm{rk}(V)-d$$

where $d=\dim(N(U)\cap R(V))$.

Given $N(t)$, the hold function $G(t)$ is related to $N(t)$ via the overconstrained equations

$$B\hat{G}(s)=(sI_n-A)\hat{N}(s)=(sI_n-A)\left[C^{+}\hat{\Lambda}(s)+C^{\perp}B(B^{*})^{-1}\Gamma\hat{Q}(s)\right] \tag{23}$$

by taking the Laplace transform of (5). Since B is of full column rank, we can premultiply the above equation by

$$\begin{bmatrix} B^{+} \\ B^{\perp} \end{bmatrix},\qquad B^{+}B=I_m,\qquad B^{\perp}B=0 \tag{24}$$

so that (23) is transformed into

$$\begin{bmatrix} I_m \\ 0 \end{bmatrix}\hat{G}(s)=\begin{bmatrix} \overline{T}(s) \\ T(s) \end{bmatrix}\hat{\Lambda}(s)+\begin{bmatrix} \overline{W}(s) \\ W(s) \end{bmatrix}\hat{Q}(s). \tag{25}$$

An important property of equation (25) is revealed in the following result.

Proposition 3.4. $\mathrm{rk}(W(s))=\mathrm{rk}(C^{\perp}B)$.

Proof. $W(s)=B^{\perp}(sI_n-A)C^{\perp}B(B^{*})^{-1}\Gamma$, by Fact 3.3 we have

$$\mathrm{rk}(W(s))=\mathrm{rk}(C^{\perp}B)-d$$

where $d = \dim S$ and

$$S := N(B^{\perp}) \cap R\big((sI_n - A)C^{\perp}B(B^*)^{-1}\Gamma\big)$$
$$= R(B) \cap R\big((sI_n - A)C^{\perp}B(B^*)^{-1}\Gamma\big).$$

We claim that $S = \{0\}$ so $d = 0$. To show this, let v be any element of S, then there exist w, z so that

$$v = Bw = (sI_n - A)C^{\perp}B(B^*)^{-1}\Gamma z$$

or

$$(sI_n - A)^{-1}Bw = C^{\perp}B(B^*)^{-1}\Gamma z.$$

Premultiplying both sides of the above equation by C yields

$$P(s)w = 0 \quad \text{or} \quad w = 0$$

since the plant square transfer matrix $P(s) = C(sI_n - A)^{-1}B$ is of full rank m and thus is invertible. This completes the proof that $v = 0$ and thus $d = 0$. □

The proposition tells us that the upper part of (25)

$$\hat{G}(s) = \overline{T}\hat{A}(s) + \overline{W}(s)\hat{Q}(s) \tag{26}$$

can be expressed in terms of $\hat{A}(s)$ only. In fact, the Laplace transform of (9) yields the unique solution

$$\hat{G}(s) = P^{-1}(s)\hat{A}(s) \tag{27}$$

which basically amounts to extracting from (23) m linearly independent rows.

We say that the solution given by (27) is *nontrivial* if all $\lambda_i(t)$ are nonzero functions. This corresponds to the physical situation where all reference input channels are not blocked. Whether or not we can obtain one, in particular a nontrivial $G(t)$ is determined by the consistency of the constraint equations (cf. the lower part of (25))

$$T(s)\hat{A}(s) + W(s)\hat{Q}(s) = 0. \tag{28}$$

The above matrix equation can be regarded as $n - m$ equations in m unknowns $\hat{Q}_i(s)$. Its i-th column is

$$T_i(s)\hat{\lambda}_i(s) + W(s)\hat{Q}_i(s) = 0$$

where $T_i(s)$, $\hat{Q}_i(s)$ denote the i-th column of $T(s)$, $\hat{Q}(s)$, respectively. For solvability, $n - m \leqslant m$ must hold in general. Two cases are distinct in nature:

(i) $\mathrm{rk}(C^{\perp}B) = n - m$. In this case, $\mathrm{rk}[W(s)] = \mathrm{rk}[W(s) \mid T_i(s)] = n - m$, for all $i = 1, \ldots, m$. All $\hat{\lambda}_i(s)$ are nontrivial, so $G(t)$ is nontrivial.

(ii) $\mathrm{rk}(C^{\perp}B) < n - m$. Two subcases are examined:

(iia) (generic case) $\mathrm{rk}[W(s) \mid T_i(s)] > \mathrm{rk}[W(s)]$ for some i. This gives $\lambda_i(t) = 0$ and thus the i-th column of $G(t)$ is zero.

(iib) Otherwise, $\mathrm{rk}[W(s) \mid T_i(s)] = \mathrm{rk}[W(s)]$ for all $i = 1, \ldots, m$. This yields $\lambda_i(t) \neq 0$ for all i.

3.2. Approach 2: solving (15) first by designing $G(t)$

The hold function $G(t)$ solving (15) is already given in (27). With $\hat{G}(s)$ given as (27), we examine (14) in some detail. First note that each row of (14) has the property

$$C_i(A + BK)^k N(T) = C_i A N(T), \quad k = 0, 1, \ldots, d_i, \tag{29}$$

and

$$C_i(A + BK)^{d_i+1} N(T) = \left(C_i A^{d_i+1} + C_i A^{d_i} BK\right) N(T). \tag{30}$$

Table 1
Applicability of Approach 1 and/or 2

Cases		$B^* = CB$ is nonsingular	$B^* \neq CB$ is nonsingular	B^* is singular
$m \leqslant n \leqslant 2m$	$\mathrm{rk}(C^{\perp}B) = n - m$	1 and 2	1	–
	$\mathrm{rk}(C^{\perp}B) < n - m$	2	–	–
$n > 2m$		2	–	–

From (29)–(30), we see that in general (14) cannot be diagonalized by designing $F(t)$ if $G(t)$ is given by (27). However, when $d_i = 0$ for all $i = 1, \ldots, m$, (14) can be diagonalized (cf. [7]).

Proposition 3.5. *Suppose CB is nonsingular. Then the design*

$$F(t) = K \exp((A + BK)t), \quad K = (CB)^{-1}(-CA + JC)$$

where J is any diagonal matrix with negative diagonal elements, can achieve the diagonalizing objective (14).

Proof. We only need to verify (14) is achieved. Note that by the given design $C(A + BK) = JC$ and thus

$$C \exp(A + BK)t = \exp(Jt)C.$$

So the left-hand-side of (14) is equivalent to

$$\exp(Jt)CN(T) = \exp(Jt)\Lambda(t).$$

It is indeed diagonal. □

Remark. It can be shown that when both approaches are applicable, they yield the same designs of $F(t)$ and $G(t)$.

The discussions so far are summarized in Table 1.

4. Examples

(1) In this example, we will illustrate the effect of A upon the solution of $G(t)$ when using approach 1 and $\mathrm{rk}(C^{\perp}B) < n - m$. Let

$$B = \begin{bmatrix} & I_3 & \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad C = [I_3 \quad 0].$$

Then

$$B^* = CB = I_3, \qquad \mathrm{rk}(C^{\perp}B) = 1 < n - m = 2.$$

(a) $A = -I_5$; then (25) is given by

$$\begin{bmatrix} I_3 \\ 0 \end{bmatrix} \hat{G}(s) = (s+1)\left\{\begin{bmatrix} & I_3 & \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \hat{\Lambda}(s) + \begin{bmatrix} & 0 & \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Gamma \hat{Q}(s)\right\}$$

so $\hat{G}$ is solvable and given by $\hat{G}(s) = (s + 1)\hat{\Lambda}(s)$ (cf. (27)).

(b) $A = \mathrm{diag}(-1, -2, -3, -4, -5)$; then (25) is given by

$$\begin{bmatrix} I_3 \\ 0 \end{bmatrix} \hat{G}(s) = \begin{bmatrix} \mathrm{diag}(s+1, & s+1, & s+2) \\ -(s+1) & 0 & 0 \\ -(s+1) & 0 & 0 \end{bmatrix} \hat{\Lambda}(s) + \begin{bmatrix} & 0 & \\ s+3 & 0 & 0 \\ s+4 & 0 & 0 \end{bmatrix} \Gamma \hat{Q}(s)$$

so $\lambda_1(t) = 0$, $\lambda_2(t)$ and $\lambda_3(t)$ nonzero functions and G is trivial.

(2) Let $A = \mathrm{diag}(-1, -1, -2, -3)$,

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\mathrm{rk}(C^{\perp}B) = n - m = 2$, and

$$B^* = \begin{bmatrix} C_1AB \\ C_2B \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix}$$

is nonsingular. Approach 1 can be applied and expected to give nontrivial solutions; (25) is given by

$$\begin{bmatrix} I_2 \\ 0 \end{bmatrix} \hat{G}(s) = \begin{bmatrix} 0 & (s+1) \\ 0.5(s+1) & -0.5(s+1) \\ -(s+1) & (s+1) \\ -s+1 & -2 \end{bmatrix} \hat{\Lambda}(s) + \begin{bmatrix} 0 & 0 \\ -0.25(s+1) & -0.5(s+1) \\ -0.5 & -(s+3) \\ 1 & 2 \end{bmatrix} \Gamma \hat{Q}(s)$$

so $\hat{G}(s)$ is nontrivial and given by (27), that is

$$\hat{G}(s) = (s+1) \begin{bmatrix} 0 & 1 \\ 0.25(-s+3) & -1 \end{bmatrix} \hat{\Lambda}(s)$$

where the constraints (cf. (22)) are $\lambda_1(0) = \lambda_1(T) = \lambda_2(0) = 0$.

5. Conclusions

We have investigated the possibility and limitation of input–output decoupling without stability a continuous-time decouplable plant using GSHF control. Two sequential design procedures for designing hold functions $F(t)$ and $G(t)$ have been presented and situations under which the two approaches are applicable and yield nontrivial solutions have been revealed. From this study, it seems that GSHF based digital control of a continuous-time system is very limited if the hold function $F(t)$ is given by constant-gain form (10), (11), (17). The proposed GSHF decoupling control, in the solvable case, can make the closed loop input–output behavior decoupled for all time instants, not just at the sampling instants, in sharp distinction with earlier decoupling results [1,6]. Since stability analysis was not done in this paper, further study taking into account the stability issue is needed.

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