IV. An Illustrative Example and Conclusion

We consider inner–outer factorization for

$$G(s) = \begin{bmatrix} s(s-2) \\ s(s-2) \end{bmatrix} \frac{1}{(s+1)^3}. \quad (40)$$

This example is from [12]. A minimal realization with identity observability gramian is found for $G(s)$ (with four decimal points)

$$A = \begin{bmatrix} -0.1948 & -1.1264 & 2.1535 \\ 0.7476 & -0.1841 & 0.9221 \\ -0.7242 & 0.4672 & -2.6210 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.5458 \\ -0.3521 \\ 0.6732 \end{bmatrix} \quad C^T = \begin{bmatrix} -0.4414 & -0.4414 \\ -0.4291 & -0.4291 \\ 1.6190 & 1.6190 \end{bmatrix}.$$

Following Step 4 of the algorithm, an orthogonal matrix U can be easily obtained which is listed as follows, together with the results of Step 3 of the algorithm:

$$U = \begin{bmatrix} 0.3764 & -0.7196 & 0.5835 \\ 0.9105 & 0.1711 & -0.3764 \\ 0.1711 & 0.6729 & 0.7196 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix} \quad F_2 = 1.5119.$$

Since $F_2 \neq 0$ and $m = 1$, we need compute a stabilizing solution to (32). However, due to the existence of zeros at the origin, the stabilizing solution to (32) does not exist. Thus, we used the same method as in [12] to compute a semistabilizing solution, and F_1, which are given by

$$Z = \begin{bmatrix} 0.042\,12 & -0.200\,72 \\ -0.200\,72 & 0.956\,51 \end{bmatrix}$$

$$F_1 = [\, 0.014\,66 \quad -0.069\,86 \,].$$

The state–space realization for the inner factor is obtained as

$$\hat{A} = \begin{bmatrix} -0.091\,32 & 0.435\,18 \\ 0.400\,54 & -1.908\,68 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 0.812\,50 \\ -1.875\,00 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} -0.290\,25 & 1.383\,12 \\ -0.290\,25 & 1.383\,12 \end{bmatrix} \quad \hat{D} = V = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}.$$

Clearly, the mode corresponding to zero eigenvalue of $\hat{A}$ is not observable. Indeed, by transforming the state–space realization into the transfer matrix, we find that

$$G_i(s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{0.7071s^2 - 1.4146s}{s^2 + 2s} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{s-2}{\sqrt{2}(s+2)}$$

identical to the result in [12].

To conclude this note, we would like to point out that for the case $\det(F_2) = 0$, the matrix pencil method as in [1], [4], and [5] can also be used to compute the solution F_1. However, it is not easy to program, in terms of the accuracy of the stabilizing solution to (29). The method from [11] seems to be more effective in this regard. It should also be pointed out that for the case $\det(F_2) = 0$, the inner factor $G_i(s)$, as in (37), represents a singular system. Interested readers are referred to [9] for a reduction of singular systems to regular systems using the singular perturbation method.

Acknowledgment

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$\mathcal{H}_\infty$ Control for Nonlinear Descriptor Systems

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***Abstract*—In this note, we study the $\mathcal{H}_\infty$ control problem for nonlinear descriptor systems governed by a set of differential-algebraic equations (DAEs) of the form $E\dot{x} = F(x, w, u)$, $z = Z(x, w, u)$, $y = Y(x, w, u)$, where E is, in general, a singular matrix. Necessary and sufficient conditions are derived for the existence of a controller solving the problem. We first give various sufficient conditions for the solvability of $\mathcal{H}_\infty$ control problem for DAEs. Both state-feedback and output-feedback cases are considered. Then, necessary conditions for the output feedback control problem to be solvable are obtained in terms of two Hamilton–Jacobi inequalities plus a weak coupling condition. Moreover, a parameterization of a family of output feedback controllers solving the problem is also provided.**

***Index Terms*—Descriptor Systems, differential games, differential-algebraic equations (DAEs), dissipation inequalities.**

I. Introduction

For the purpose of control, nonlinear descriptor systems are frequently described by a set of differential-algebraic equations (DAEs)

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of the form

$$\dot{\boldsymbol{x}}_1 = f(\boldsymbol{x}_1,\, \boldsymbol{x}_2,\, u) \tag{1}$$

$$0 = g(\boldsymbol{x}_1,\, \boldsymbol{x}_2,\, u) \tag{2}$$

or in a more compact form $E\dot{\boldsymbol{x}} = F(\boldsymbol{x},\, u)$, where $E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{x} = [\boldsymbol{x}_1^T \quad \boldsymbol{x}_2^T]^T \triangleq \mathrm{col}(x_1, \ldots, x_n)$ are local coordinates for an n-dimensional state–space manifold $\mathcal{X}$. In the state–space $\mathcal{X}$, dynamic state variables $\boldsymbol{x}_1$ and instantaneous state variables $\boldsymbol{x}_2$ are distinguished. The dynamics of the states $\boldsymbol{x}_1$ is directly defined by (1), while the dynamics of $\boldsymbol{x}_2$ is such that the system satisfies the constraint (2). In many cases, the algebraic constraint (2) of the full DAEs can be eliminated (usually due to the consistence of initial conditions). As a consequence, the DAEs reduce to a well-known state-variable system. Nevertheless, in some cases this kind of elimination is not possible (often due to inconsistent initial conditions), since it may result in loss of accuracy or loss of necessary information. A large class of physical systems can be modeled by this kind of DAEs. The paper by Newcomb and Dziurla [6] gives many practical examples, including circuit and system design, robotics, neural network, etc., and presents an excellent review on nonlinear DAEs. Many other applications of DAEs, as well as numerical treatment, can be found in [2].

In this note, we investigate the contractive property of DAEs, namely the $\mathcal{H}_\infty$ control problem. Our note is mainly divided into two parts. The first part concerns various sufficient conditions for the solvability of the $\mathcal{H}_\infty$ control problem. Both state feedback and output feedback cases are considered. We seek sufficient conditions under which a given DAE has an $\mathcal{L}_2$ gain no greater than a prescribed positive number γ with internal stability, and in the mean time, eliminates possible impulse dynamics and other singularity-induced nonlinearity of the system. We will derive a family of output feedback controllers solving the $\mathcal{H}_\infty$ control problem. The underlying ideas are differential games and dissipation inequalities. These ideas were also used by Isidori and Kang [3] and Yung *et al.* [11], in which they have given the central controller and a family of controllers, respectively, solving the $\mathcal{H}_\infty$ output feedback control problem for general nonlinear systems in usual state-variable form, i.e., the E matrix is nonsingular.

The second part of this note is devoted to a converse result, namely the derivation of necessary conditions for solutions of local disturbance attenuation to exist. We obtain necessary conditions given in terms of the existence of nonnegative solutions to two Hamilton–Jacobi inequalities, together with a weak coupling condition. A similar result has previously been published in [1] for (W-)input affine nonlinear systems with nonsingular E matrix. In a recent monograph [7], among many other important contributions, van der Schaft addressed a number of issues related to necessary conditions for solutions of local disturbance attenuation to exist (see also [3] for some related work). Our results can be thought of as a parallel extension of the results of [1] and [7] to the DAEs case. As a matter of fact, the results in this note reduce to the ones given in [1] for state–space systems.

This note is organized as follows. In Section II, we will review some notions of DAEs together with some preliminary results for the theory of DAEs, including stability theory and dissipativity. Our main results will be summarized in Sections III and IV. We will concentrate on the output feedback case. However, for the sake of completeness, we will first investigate the state feedback control. We also give a parameterizaton of a family of output feedback controllers. In Section IV, we will give a necessary condition for the $\mathcal{H}_\infty$ output feedback control problem to be solvable.

II. Elements for Nonlinear Descriptor Systems

Consider the following DAE:

$$E\dot{\boldsymbol{x}}(t) = F(\boldsymbol{x},\, u), \qquad u \in \mathcal{U} \subset \mathbb{R}^m \tag{3}$$

where $\boldsymbol{x} \triangleq \mathrm{col}(x_1, \ldots, x_n)$ are local coordinates for an n-dimensional state-space manifold $\mathcal{X}$. E is a constant matrix and $F(0,\, 0) = 0$. The constant matrix $E \in \mathbb{R}^{n \times n}$ is, in general, a singular matrix with rank $E = r < n$. Without loss of generality, we can assume that

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

The following definition will be used (see [2]).

Definition 1: [2] The DAE (3) is said to be of (uniform) index one if the constant coefficient system

$$E\dot{\mathbf{w}}(t) - F_{\boldsymbol{x}}(\hat{\boldsymbol{x}},\, 0)\mathbf{w}(t) = g(t)$$

is impulse free for all $\hat{\boldsymbol{x}}$ in a neighborhood of the graph of the solution, where $F_{\boldsymbol{x}}$ denotes the Jacobian matrix $\partial F / \partial \boldsymbol{x}$.

The index of a DAE can be thought of as the generalization of the nilpotent index [2] of a linear time-invariant descriptor system. The notion of index provides an easy way to guarantee the solvability of a given DAE. Rewrite the DAE (3) in the following form:

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} F_1(\boldsymbol{x}_1,\, \boldsymbol{x}_2,\, u) \\ F_2(\boldsymbol{x}_1,\, \boldsymbol{x}_2,\, u) \end{bmatrix}.$$

Suppose that the aforementioned DAE is of index one. Then, from Definition 1, it is necessary that $(\partial/\partial \boldsymbol{x}_2) F_2(\boldsymbol{x}_1,\, \boldsymbol{x}_2,\, 0)$ is nonsingular around the equilibrium point $\boldsymbol{x} = 0$. Consequently, by the implicit function theorem, there exists a function $h(\bullet)$ so that the DAE reduces to an ODE

$$\dot{\boldsymbol{x}}_1 = F_1(\boldsymbol{x}_1,\, h(\boldsymbol{x}_1), u)$$

which is always solvable provided that F_1 is smooth enough. This implies that DAE (3) is solvable.

In [9], some stability definitions and Lyapunov stability theorems for nonlinear descriptor systems have been given. For the sake of brevity, we do not reproduce those results here. Instead, we will derive an improved version of the Lyapunov stability theorem for DAE (3).

Theorem 2: Consider DAE (3) with $u = 0$. Let $E\boldsymbol{x}(0) = E\boldsymbol{x}_0$ be given. Suppose that there exists a C^3 function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ vanishing at the points where $E\boldsymbol{x} = 0$ and positive elsewhere which satisfies the following properties:

i) $(\partial/\partial \boldsymbol{x})V = \tilde{V}^T(\boldsymbol{x})E$ for some C^2 function $\tilde{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$;
ii) $\tilde{V}^T(\boldsymbol{x})F(\boldsymbol{x}, 0) < 0$ for all $\boldsymbol{x} \neq 0$;
iii) $E^T \tilde{V}_{\boldsymbol{x}} = \tilde{V}_{\boldsymbol{x}}^T E \geq 0$, where $\tilde{V}_{\boldsymbol{x}}$ denote the Jacobian of $\tilde{V}$.

Then, the equilibrium point $\boldsymbol{x} = 0$ is locally asymptotically stable and the DAE is of index one.

Proof: We first show that the DAE has index one. Set $\tilde{V}(\boldsymbol{x}) = \begin{bmatrix} \tilde{V}_1(\boldsymbol{x}_1,\, \boldsymbol{x}_2) \\ \tilde{V}_2(\boldsymbol{x}_1,\, \boldsymbol{x}_2) \end{bmatrix}$. It is easy to show that

$$\frac{\partial^2}{\partial \boldsymbol{x}^2} \tilde{V}^T(\boldsymbol{x}) F(\boldsymbol{x}, 0) \,|_{\boldsymbol{x}=0} = \tilde{V}_{\boldsymbol{x}}^T(0) F_{\boldsymbol{x}}(0,0) + F_{\boldsymbol{x}}^T(0,0)\tilde{V}_{\boldsymbol{x}}(0) < 0. \tag{4}$$

Condition iii) implies that $(\partial/\partial \boldsymbol{x}_2)\tilde{V}_1 = 0$; this, in turn, implies that $(\partial/\partial \boldsymbol{x}_2)F_2(0,0)$ is nonsingular. Consequently, by the continuity of F, the DAE (3) is of index one. Now, consider the constant pair $\{E,\, F_{\boldsymbol{x}}(0,0)\}$. From inequality (4) and condition iii), we can conclude that $\tilde{V}_{\boldsymbol{x}}(0)$ is a solution satisfying the generalized Lyapunov inequality

$$\tilde{V}_{\boldsymbol{x}}^T(0)F_{\boldsymbol{x}}(0,0) + F_{\boldsymbol{x}}^T(0,0)\tilde{V}_{\boldsymbol{x}}(0) < 0$$

$$E^T \tilde{V}_{\boldsymbol{x}}(0) = \tilde{V}_{\boldsymbol{x}}^T(0)E \geq 0.$$

Hence, by the standard result on Lyapunov stability of linear descriptor systems [8], the pair $\{E,\, F_{\boldsymbol{x}}(0,0)\}$ is admissible (i.e., regular, asymptotically stable and impulse free). This, in turn, implies that the DAE (3) with $u = 0$ is asymptotically stable by noting that the pair

$\{E, F_{\boldsymbol{x}}(0,0)\}$ is a linearization of the DAE around the equilibrium point $\boldsymbol{x} = 0$. Q.E.D.

Remark: Because the intrinsic property of descriptor systems, the initial values must be given in the form $E\boldsymbol{x}_0$. Its rationale can be better understood by investigating linear descriptor systems. Consider the following linear differential equation:

$$E\dot{\boldsymbol{x}} = A\boldsymbol{x}(t) + Bu(t).$$

Taking Laplace transform of the previous equation yields

$$X(s) = (sE - A)^{-1}[E\boldsymbol{x}(0) + BU(s)].$$

We assume the invertibility of the pencil $(sE - A)$ so that unique solutions of the above equation are obtained for all $E\boldsymbol{x}(0)$ and $U(s)$. In particular, we point out that the initial conditions must be given in the form $E\boldsymbol{x}(0)$. As a matter of fact, given $u(t)$ for $t \geq 0$, the knowledge of $E\boldsymbol{x}(0)$ is necessary and sufficient to completely determine $\boldsymbol{x}(t)$ for $t \geq 0$. It is part of the reason that the candidate Lyapunov function $V(\boldsymbol{x})$ should be vanishing at the points where $E\boldsymbol{x} = 0$ rather than $\boldsymbol{x} = 0$. On the other hand, for an index one descriptor system, (2) is simply an algebraic constraint. Therefore, only the part that $Ex \neq 0$ contributes to the energy function (see also [4], [9], and [10] for more details).

Next, we give an extension of the LaSalle invariance principle.

Theorem 3: Consider the DAE

$$\dot{x} = f_1(x, y) \tag{5a}$$

$$0 = f_2(x, y) \tag{5b}$$

where f_1, f_2 are continuously differentiable functions. Suppose the DAE is of index one. Let $(x, y) = (0, 0)$ be an equilibrium point for the DAE (5a), (5b). Let $V(x, y) : D \longrightarrow \mathbb{R}^+ = [0, \infty)$ be a smooth positive–definite function on a neighborhood D of $(x, y) = (0, 0)$, such that $\dot{V}(x, y) \leq 0$. Let $S = \{(x, y) \in D \mid \dot{V} = 0\}$, and suppose that no solution can stay forever in S, other than the trivial solution. Then, the origin is locally asymptotically stable.

Proof: The proof is straightforward. Since the DAE is of index one, as far as the (5b) is concerned, there exists a unique solution $y = g(x)$ such that $f_2(x, g(x)) = 0$, with $g(0) = 0$, provided by the implicit function theorem. In this case, if $\lim_{t\to\infty} x(t) = 0$, then $\lim_{t\to\infty} y(t) = 0$. The condition $\lim_{t\to\infty} x(t) = 0$ is a direct consequence of the usual LaSalle invariance principle. Q.E.D.

Remark: In the aforementioned theorem, $\dot{V}$ denotes differentiation with respect to t along the solution trajectory of (5a) and (5b). Because the descriptor system is of index one, it possesses a solution which is impulse free.

For the remainder of this section, we will investigate dissipative property of a given DAE. Consider the following DAE:

$$\begin{aligned} E\dot{\boldsymbol{x}} &= F(\boldsymbol{x}, u), \qquad u \in \mathcal{U} \subset \mathbb{R}^m, \quad F(0,0) = 0 \\ y &= H(\boldsymbol{x}, u), \qquad y \in \mathcal{Y} \subset \mathbb{R}^p, \quad H(0,0) = 0 \end{aligned} \tag{6}$$

where $\boldsymbol{x} \in \mathcal{X}$, together with a function

$$s : \mathcal{U} \times \mathcal{Y} \longrightarrow \mathbb{R}$$

called the supply rate. It is well known [7] that a usual state–space system (i.e., $E \equiv I$, the identity matrix) has $\mathcal{L}_2$ gain $\leq \gamma$ if it is dissipative with respect to the supply rate

$$s(u, y) \triangleq \gamma^2\|u\|^2 - \|y\|^2, \qquad \gamma > 0.$$

This result can also be applied to DAE (6). In fact, we have the following very important result.

Theorem 4: Consider DAE (6) with $E\boldsymbol{x}(0) = E\boldsymbol{x}_0$ given. Suppose that the matrix $D^T D - \gamma^2 I$ is negative definite and $\{E, A, G\}$ is impulse observable (the triple $\{E, A, G\}$ is called impulse observable if there exists a constant matrix L such that $\{E, A + LG\}$ is impulse free), where

$$D = \left(\frac{\partial}{\partial u}H\right)_{(\boldsymbol{x},u)=(0,0)}$$

$$A = \left(\frac{\partial}{\partial \boldsymbol{x}}F\right)_{(\boldsymbol{x},u)=(0,0)}$$

$$G = \left(\frac{\partial}{\partial \boldsymbol{x}}H\right)_{(\boldsymbol{x},u)=(0,0)}.$$

Suppose that any bounded trajectory $\boldsymbol{x}(t)$ of the system $E\dot{\boldsymbol{x}} = F(\boldsymbol{x}(t), 0)$ satisfying $H(\boldsymbol{x}(t), 0) = 0$ for all $t \geq 0$ is such that $\lim_{t\to\infty} \boldsymbol{x}(t) = 0$. Suppose also that there exists a C^3 function $V : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ vanishing at the points where $E\boldsymbol{x} = 0$ and positive elsewhere which satisfies the following properties:

i) $(\partial/\partial\boldsymbol{x})V = \tilde{V}^T(\boldsymbol{x})E$ for some C^2 function $\tilde{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$;

ii) $Y_0 \triangleq \tilde{V}^T(\boldsymbol{x})F(\boldsymbol{x}, u) + \|y\|^2 - \gamma^2\|u\|^2 \leq 0$, for all $u \in \mathcal{U}$;

iii) $E^T\tilde{V}_{\boldsymbol{x}} = \tilde{V}_{\boldsymbol{x}}^T E$.

Then, the DAE has an $\mathcal{L}_2$ gain less than or equal to γ and the equilibrium point $\boldsymbol{x} = 0$ is locally asymptotically stable. Moreover, the DAE is of index one.

Proof: The proof of the dissipative property is standard, hence omitted. We now prove asymptotical stability and index one property. We first show that the DAE (6) is of index one. Setting $u \equiv 0$ in Y_0 and taking the second-order partial derivative of Y_0 with respect to $\boldsymbol{x}$ at $(\boldsymbol{x}, u) = (0, 0)$ yields

$$A^T\tilde{V}_{\boldsymbol{x}}(0) + \tilde{V}_{\boldsymbol{x}}^T(0)A + G^T G \leq 0. \tag{7}$$

Since $\{E, A, G\}$ is impulse observable, inequality (7) along with condition iii) implies that DAE (6) is of index one. To prove asymptotical stability, observe that along any trajectory $\boldsymbol{x}(\bullet)$ of the DAE with $u \equiv 0$ is such that

$$\frac{dV(\boldsymbol{x}(t))}{dt} \leq -\|y\|^2 \leq 0.$$

This shows that the equilibrium point $\boldsymbol{x} = 0$ of the DAE (6) is stable. In addition, observe that any trajectory $\boldsymbol{x}(\bullet)$ such that $\dot{V}(\boldsymbol{x}(t)) = 0$ for all $t \geq 0$ is necessarily a trajectory of $E\dot{\boldsymbol{x}} = f(\boldsymbol{x}, 0)$ such that $\boldsymbol{x}(t)$ is bounded and $H(\boldsymbol{x}(t), 0) = 0$ for all $t \geq 0$. Hence, by hypothesis, it is concluded that $\lim_{t\to\infty} \boldsymbol{x}(t) = 0$ by using Theorem 3. Q.E.D.

III. The $\mathcal{H}_\infty$ Control Problem

Let Σ be a nonlinear system described by the following DAE:

$$\begin{aligned} E\dot{\boldsymbol{x}} &= F(\boldsymbol{x}, w, u), \qquad w \in \mathcal{W} \subset \mathbb{R}^l, \ u \in \mathcal{U} \subset \mathbb{R}^m \\ z &= Z(\boldsymbol{x}, w, u), \qquad z \in \mathcal{Z} \subset \mathbb{R}^s \\ y &= Y(\boldsymbol{x}, w, u), \qquad y \in \mathcal{Y} \subset \mathbb{R}^p \end{aligned} \tag{8}$$

where $\boldsymbol{x} \in \mathcal{X}$. Here u stands for the vector of control inputs, w is the exogenous input (disturbances to-be-rejected or signals to-be-tracked), y is the measured output, and finally z denotes the to-be-controlled outputs (tracking errors, cost variables). It is assumed throughout that $F(0,0,0) = 0$, $Z(0,0,0) = 0$ and $Y(0,0,0) = 0$. The standard $\mathcal{H}_\infty$ control problem consists of finding, if possible, a controller Γ such that the resulting closed-loop system has a locally asymptotically stable equilibrium point at the origin, is of index one, and has $\mathcal{L}_2$ gain (from w to z) less than or equal to γ. In the state feedback $\mathcal{H}_\infty$ control problem we assume that $y = \boldsymbol{x}$ in (8), i.e., that the whole state is available for measurement. We suppose the following.

A1) The matrix D_{12} has rank m and the matrix $D_{11}^T D_{11} - \gamma^2 I$ is negative definite, where $D_{12} = (\partial Z/\partial u)_{(\boldsymbol{x},w,u)=(0,0,0)}$ and $D_{11} = (\partial Z/\partial w)_{(\boldsymbol{x},w,u)=(0,0,0)}$.

A2) Any bounded trajectory $\boldsymbol{x}(t)$ of the system $E\dot{\boldsymbol{x}}(t) = F(\boldsymbol{x}(t), 0, u(t))$ satisfying $Z(\boldsymbol{x}(t), 0, u(t)) = 0$ for all $t \geq 0$ is such that $\lim_{t\to\infty} \boldsymbol{x}(t) = 0$.

A3) The matrix pencil $\begin{bmatrix} A - j\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R} \cup \{\infty\}$, where $A = (\partial F/\partial \boldsymbol{x})_{(\boldsymbol{x},w,u)=(0,0,0)}$, $B_2 = (\partial F/\partial u)_{(\boldsymbol{x},w,u)=(0,0,0)}$, and $C_1 = (\partial Z/\partial \boldsymbol{x})_{(\boldsymbol{x},w,u)=(0,0,0)}$.

Two preliminary lemmas will be needed in the sequel.

Lemma 5: Consider the DAE (8). Assume that assumptions A1)–A3) are satisfied. Suppose the following hypothesis also holds.

H1: There exists a smooth real-valued function $V(\boldsymbol{x})$, locally defined on a neighborhood of the equilibrium point $\boldsymbol{x} = 0$ in $\mathcal{X}$, which is vanishing at the points where $E\boldsymbol{x} = 0$ and positive elsewhere such that the function

$$Y_1(\boldsymbol{x}) = H(\boldsymbol{x}, \tilde{V}(\boldsymbol{x}), \alpha_1(\boldsymbol{x}), \alpha_2(\boldsymbol{x}))$$

is negative semidefinite near $\boldsymbol{x} = 0$, where the function $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is defined on a neighborhood of $(\boldsymbol{x}, p, w, u) = (0, 0, 0, 0)$ as

$$H(\boldsymbol{x}, p, w, u) = p^T F(\boldsymbol{x}, w, u) + \|Z(\boldsymbol{x}, w, u)\|^2 - \gamma^2 \|w\|^2 \quad (9)$$

$(\partial V/\partial \boldsymbol{x}) = \tilde{V}^T(\boldsymbol{x})E$ defined as shown in Theorem 4, $\alpha_1(\boldsymbol{x}) = w^*(\boldsymbol{x}, \tilde{V}(\boldsymbol{x}))$ and $\alpha_2(\boldsymbol{x}) = u^*(\boldsymbol{x}, \tilde{V}(\boldsymbol{x}))$, and $w^*(\boldsymbol{x}, p)$ and $u^*(\boldsymbol{x}, p)$ are defined on a neighborhood of $(\boldsymbol{x}, p) = (0, 0)$ satisfying

$$\frac{\partial H}{\partial w}(\boldsymbol{x}, p, w^*(\boldsymbol{x}, p), u^*(\boldsymbol{x}, p)) = 0$$
$$\frac{\partial H}{\partial u}(\boldsymbol{x}, p, w^*(\boldsymbol{x}, p), u^*(\boldsymbol{x}, p)) = 0$$

with $w^*(0, 0) = 0$ and $u^*(0, 0) = 0$.

Then, the feedback law $u = \alpha_2(\boldsymbol{x})$ solves the $\mathcal{H}_\infty$ state feedback control problem. ◇◇◇

Proof: The result of the lemma is a direct consequence of Theorem 4. The details are thus omitted.

Next, consider the case in which the state $\boldsymbol{x}$ of the DAE (8) is not available for direct measurement. Motivated by the work of Isidori and Kang [3] and Yung *et al.* [11], we consider a dynamic controller of the form

$$\hat{E}\dot{\xi} = F(\xi, \alpha_1(\xi), \alpha_2(\xi)) + G(\xi)(y - Y(\xi, \alpha_1(\xi), \alpha_2(\xi)))$$
$$u = \alpha_2(\xi) \quad (10)$$

where $\xi = \mathrm{col}(\xi_1, \ldots, \xi_n)$ are local coordinates for the state-space manifold $\mathcal{X}_c$ of the controller Γ. The matrix $G(\xi)$, called the output injection gain, is to be determined. Substitute the controller (10) in (8) to obtain the corresponding closed-loop system as

$$E^e \dot{\boldsymbol{x}}^e = F^e(\boldsymbol{x}^e, w) \quad z = Z^e(\boldsymbol{x}^e, w) = Z(\boldsymbol{x}, w, \alpha(\xi)) \quad (11)$$

where $E^e = \begin{bmatrix} E & 0 \\ 0 & \hat{E} \end{bmatrix}$, shown in the expressions at the bottom of the page. Again, we try to render the closed-loop system locally dissipative with respect to the supply rate $\gamma^2\|w\|^2 - \|z\|^2$. Clearly, it suffices to show that there exists a smooth nonnegative function $U(\boldsymbol{x}^e)$ with $(\partial U/\partial \boldsymbol{x}^e) = \tilde{U}^T E^e$ and $E^{eT}\tilde{U}_{\boldsymbol{x}} = \tilde{U}_{\boldsymbol{x}}^T E^e$ such that

$$\tilde{U^T} F^e(\boldsymbol{x}^e, w) + \|Z(\boldsymbol{x}^e, w)\|^2 - \gamma^2\|w\|^2 \leq 0, \qquad \text{for all } w \quad (12)$$

and such that the closed-loop system is locally asymptotically stable and is of index one. To state the main result of this section, a further assumption is needed.

A4) The matrix $D_{21} = (\partial Y/\partial w)_{(\boldsymbol{x},w,u)=(0,0,0)}$ has rank p. Define

$$r_{11}(\boldsymbol{x}) = \frac{1}{2}\left(\frac{\partial^2 H(\boldsymbol{x}, \tilde{V}^T(\boldsymbol{x}), w, u)}{\partial w^2}\right)_{w=\alpha_1(\boldsymbol{x}), u=\alpha_2(\boldsymbol{x})}$$
$$r_{12}(\boldsymbol{x}) = \frac{1}{2}\left(\frac{\partial^2 H(\boldsymbol{x}, \tilde{V}^T(\boldsymbol{x}), w, u)}{\partial u \partial w}\right)_{w=\alpha_1(\boldsymbol{x}), u=\alpha_2(\boldsymbol{x})}$$
$$r_{21}(\boldsymbol{x}) = \frac{1}{2}\left(\frac{\partial^2 H(\boldsymbol{x}, \tilde{V}^T(\boldsymbol{x}), w, u)}{\partial w \partial u}\right)_{w=\alpha_1(\boldsymbol{x}), u=\alpha_2(\boldsymbol{x})}$$
$$r_{22}(\boldsymbol{x}) = \frac{1}{2}\left(\frac{\partial^2 H(\boldsymbol{x}, \tilde{V}^T(\boldsymbol{x}), w, u)}{\partial u^2}\right)_{w=\alpha_1(\boldsymbol{x}), u=\alpha_2(\boldsymbol{x})}$$

and set

$$R(\boldsymbol{x}) = \begin{bmatrix} (1-\epsilon_1)r_{11}(\boldsymbol{x}) & r_{12}(\boldsymbol{x}) \\ r_{21}(\boldsymbol{x}) & (1+\epsilon_2)r_{22}(\boldsymbol{x}) \end{bmatrix}$$

where ϵ_1 and ϵ_2 are any real numbers satisfying $0 < \epsilon_1 < 1$ and $\epsilon_2 > 0$, respectively. The following theorem is readily obtained.

Theorem 6: Consider (11). Suppose assumptions A1)–A4) are satisfied. Suppose hypothesis H1 of Lemma 5 holds. Suppose the following hypothesis also holds.

H2: There exists a smooth real-valued function $Q(\boldsymbol{x})$, locally defined on a neighborhood of $\boldsymbol{x} = 0$, which is vanishing at the points where $E\boldsymbol{x} = 0$ and positive elsewhere such that the function

$$Y_2(\boldsymbol{x}) = K(\boldsymbol{x}, \tilde{Q}(\boldsymbol{x}), \hat{w}(\boldsymbol{x}, \tilde{Q}(\boldsymbol{x}), \hat{y}(\boldsymbol{x}, \tilde{Q}(\boldsymbol{x}))), \hat{y}(\boldsymbol{x}, \tilde{Q}(\boldsymbol{x})))$$

is negative definite near $\boldsymbol{x} = 0$, and its Hessian matrix is nonsingular at $\boldsymbol{x} = 0$. Here $\tilde{Q} : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function defined by $(\partial Q/\partial \boldsymbol{x}) = \tilde{Q}^T E$ with $E^T\tilde{Q}_{\boldsymbol{x}} = \tilde{Q}_{\boldsymbol{x}}E$, the function $K : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^p \to \mathbb{R}$ is defined on a neighborhood of the origin as

$$K(\boldsymbol{x}, p, w, y) = p^T F(\boldsymbol{x}, w + \alpha_1(\boldsymbol{x}), 0) - y^T Y(\boldsymbol{x}, w + \alpha_1(\boldsymbol{x}), 0) + \begin{bmatrix} w \\ -\alpha_2(\boldsymbol{x}) \end{bmatrix}^T R(\boldsymbol{x}) \begin{bmatrix} w \\ -\alpha_2(\boldsymbol{x}) \end{bmatrix}$$

and the function $\hat{w}(\boldsymbol{x}, p, y)$, respectively $\hat{y}(\boldsymbol{x}, p)$, defined on a neighborhood of (0,0,0), respectively (0,0), satisfies

$$\left(\frac{\partial K(\boldsymbol{x}, p, w, y)}{\partial w}\right)_{w=\hat{w}(x,p,y)} = 0 \quad \hat{w}(0,0,0) = 0$$

respectively

$$\left(\frac{\partial K(\boldsymbol{x}, p, \hat{w}(\boldsymbol{x}, p, y), y)}{\partial y}\right)_{y=\hat{y}(x,p)} = 0 \quad \hat{y}(0,0) = 0.$$

Then, if

$$\tilde{Q}(\boldsymbol{x})G(\boldsymbol{x}) = \hat{y}^T(\boldsymbol{x}, \tilde{Q}(\boldsymbol{x})) \quad (13)$$

has a smooth solution $G(\boldsymbol{x})$ near $\boldsymbol{x} = 0$, the nonlinear $\mathcal{H}_\infty$ output feedback control problem is solved by the output feedback controller (10) with $\hat{E} = E$. ◇◇◇

Proof: Since the result of the theorem is a special case of that given in Theorem 8, we omit the proof here for brevity.

A. Parameterization of Output Feedback Controllers

Recently, Yung *et al.* [11] have derived a set of parameterized solutions to the $\mathcal{H}_\infty$ control problem for general nonlinear systems in state-variable form. They have considered both output feedback and

$$\boldsymbol{x}^e = \begin{bmatrix} \boldsymbol{x} \\ \xi \end{bmatrix} \quad \text{and}$$

$$F^e(\boldsymbol{x}^e, w) = \begin{bmatrix} F(\boldsymbol{x}, w, \alpha_2(\xi)) \\ F(\xi, \alpha_1(\xi), \alpha_2(\xi)) + G(\xi)(y - Y(\xi, \alpha_1(\xi), \alpha_2(\xi))) \end{bmatrix}.$$

state feedback cases. Indeed, we can extend the technique developed in [11] to give a family of $\mathcal{H}_\infty$ controllers for nonlinear differential-algebraic systems.

Motivated by the work of [11], we consider the family of controllers described by the following DAEs:

$$\begin{aligned}\hat{E}\dot{\xi} =& F(\xi, \alpha_1(\xi), \alpha_2(\xi) + c(\eta)) \\ &+ G(\xi)(y - Y(\xi, \alpha_1(\xi), \alpha_2(\xi) + c(\eta))) \\ &+ \hat{g}_1(\xi)c(\eta) + \hat{g}_2(\xi)d(\eta) \\ E_Q\dot{\eta} =& a(\eta, y - Y(\xi, \alpha_1(\xi), \alpha_2(\xi) + c(\eta))) \\ u =& \alpha_2(\xi) + c(\eta)\end{aligned} \tag{14}$$

where ξ and η are defined on some neighborhoods of the origins in $\mathcal{X}_c$ and $\mathbb{R}^q$, respectively. $G(\bullet)$ satisfies (13). $a(\bullet, \bullet)$ and $c(\bullet)$ are smooth functions with $a(0, 0) = 0$ and $c(0) = 0$. $\hat{g}_1(\bullet)$, $\hat{g}_2(\bullet)$ and $d(\bullet)$ are C^k functions ($k \geq 1$). E_Q is a constant matrix, and, in general, is singular. The functions $a(\bullet, \bullet)$, $c(\bullet)$, $\hat{g}_1(\bullet)$, $\hat{g}_2(\bullet)$, $d(\bullet)$, and the matrix E_Q are to-be-determined variables such that the closed-loop system (8)–(14) is dissipative with respect to the supply rate $\gamma^2\|w\|^2 - \|z\|^2$, and is locally asymptotically stable with index one.

Observe first that the DAEs describing the closed-loop system (8)–(14) can be put in the form

$$\begin{aligned}E_a\dot{\boldsymbol{x}}_a =& F_a(\boldsymbol{x}_a, w) \\ z =& Z(\boldsymbol{x}, w, \alpha_2(\xi) + c(\eta))\end{aligned}$$

where $\boldsymbol{x}_a \triangleq \mathrm{col}(\boldsymbol{x}, \xi, \eta)$, $E_a \triangleq \begin{bmatrix} E & 0 & 0 \\ 0 & \hat{E} & 0 \\ 0 & 0 & E_Q \end{bmatrix}$, and the equation at the bottom of the page holds. In that equation, $\tilde{F}(\xi, \eta) \triangleq F(\xi, \alpha_1(\boldsymbol{x}), \alpha_2(\xi) + c(\eta)) - G(\xi)Y(\xi, \alpha_1(\xi), \alpha_2(\xi) + c(\eta))$. Consider a Hamiltonian function $J : \mathbb{R}^{2n+q} \times \mathbb{R}^{2n+q} \times \mathbb{R}^r \rightarrow \mathbb{R}$ defined as follows:

$$\begin{aligned}J(\boldsymbol{x}_a, p_a, w) =& p_a^T F_a(\boldsymbol{x}_a, w) \\ &+ \begin{bmatrix} w - \alpha_2(\boldsymbol{x}) \\ \alpha_2(\xi) + c(\eta) - \alpha_2(\boldsymbol{x}) \end{bmatrix}^T \\ &\times R(\boldsymbol{x}) \begin{bmatrix} w - \alpha_2(\boldsymbol{x}) \\ \alpha_2(\xi) + c(\eta) - \alpha_2(\boldsymbol{x}) \end{bmatrix}.\end{aligned} \tag{15}$$

It is easy to check that

$$\left(\frac{\partial^2 J(\boldsymbol{x}_a, p_a, w)}{\partial w^2}\right)_{(\boldsymbol{x}_a, p_a, w) = (0,0,0)} = 2(1 - \epsilon_1)(D_{11}^T D_{11} - \gamma^2 I)$$

which is negative definite by A1). Then, by the implicit function theorem, there exists a unique smooth function $\tilde{w}(\boldsymbol{x}_a, p_a)$, defined on a neighborhood of the origin, satisfying

$$\left(\frac{\partial J(\boldsymbol{x}_a, p_a, w)}{\partial w}\right)_{w = \tilde{w}(\boldsymbol{x}_a, p_a)} = 0 \quad \tilde{w}(0, 0) = 0.$$

Lemma 7: Consider (8) and (14). Suppose assumptions A1)–A4) are satisfied. Suppose hypotheses H1 of Lemma 5 and H2 of Theorem 6 hold. Furthermore, suppose that the following hypothesis also holds.

H3: There exists a smooth real-valued function $M(\boldsymbol{x}_a)$, locally defined on a neighborhood of the origin in $\mathbb{R}^{2n+q}$, which vanishes at the points where $\boldsymbol{x}_a = \mathrm{col}(E\boldsymbol{x}, E\boldsymbol{x}, 0) = \mathrm{col}(0, 0, 0)$, is positive elsewhere, satisfies $(\partial M(\boldsymbol{x}_a)/\partial \boldsymbol{x}_a) = \tilde{M}^T(\boldsymbol{x}_a)E_a$ with $\tilde{M}_{\boldsymbol{x}_a}^T(\boldsymbol{x}_a)E_a = E_a^T \tilde{M}_{\boldsymbol{x}_a}(\boldsymbol{x}_a)$, and is such that the function $Y_3(\boldsymbol{x}_a) = J(\boldsymbol{x}_a, \tilde{M}(\boldsymbol{x}_a), \tilde{w}(\boldsymbol{x}_a, \tilde{M}(\boldsymbol{x}_a)))$ vanishes at the points where $\boldsymbol{x}_a = \mathrm{col}(E\boldsymbol{x}, E\boldsymbol{x}, 0) = \mathrm{col}(0, 0, 0)$ and is negative elsewhere.

Then, the family of controllers (14) with $\hat{E} = E$ solves the $\mathcal{H}_\infty$ output feedback control problem.

Proof: Set $U(\boldsymbol{x}_a) = V(\boldsymbol{x}) + M(\boldsymbol{x}_a)$. It follows that $(\partial U/\partial \boldsymbol{x}_a) = \tilde{U}^T(\boldsymbol{x}_a)E_a$ with $E_a^T \tilde{U}_{\boldsymbol{x}_a}(\boldsymbol{x}_a) = \tilde{U}_{\boldsymbol{x}_a}^T(\boldsymbol{x}_a)E_a$, where

$$\tilde{U}_{\boldsymbol{x}_a} = \begin{bmatrix} \tilde{V}_{\boldsymbol{x}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \tilde{M}_{\boldsymbol{x}_a}.$$

With the equations $Y_1(\boldsymbol{x})$ and $Y_3(\boldsymbol{x})$ in hand, we have the following Hamiltonian equation by using Taylor series expansion:

$$\begin{aligned}\frac{dU}{dt} &+ \|Z(\boldsymbol{x}, w, \alpha_2(\xi) + c(\eta))\|^2 - \gamma^2\|w\|^2 \\ =& Y_1(\boldsymbol{x}) + Y_3(\boldsymbol{x}_a) + \begin{bmatrix} w - \alpha_2(\boldsymbol{x}) \\ \alpha_2(\xi) + c(\eta) - \alpha_2(\boldsymbol{x}) \end{bmatrix}^T \\ &\cdot \begin{bmatrix} \epsilon_1 r_{11}(\boldsymbol{x}) & 0 \\ 0 & -\epsilon_2 r_{22}(\boldsymbol{x}) \end{bmatrix} \begin{bmatrix} w - \alpha_2(\boldsymbol{x}) \\ \alpha_2(\xi) + c(\eta) - \alpha_2(\boldsymbol{x}) \end{bmatrix} \\ &+ \|w - \tilde{w}(\boldsymbol{x}_a, \tilde{M}^T(\boldsymbol{x}_a))\|_{\tilde{R}(\boldsymbol{x}_a)}^2 \\ &+ o\left(\left\| \begin{matrix} w - \alpha_1(\boldsymbol{x}) \\ \alpha_2(\xi) + c(\eta) - \alpha_2(\boldsymbol{x}) \end{matrix} \right\|^3\right) \\ &+ o\left(\|w - \tilde{w}(\boldsymbol{x}_a, \tilde{M}^T(\boldsymbol{x}_a))\|^3\right)\end{aligned} \tag{16}$$

where

$$\tilde{R}(\boldsymbol{x}_a) \triangleq \frac{1}{2}\left(\frac{\partial^2 J(\boldsymbol{x}_a, \tilde{M}^T(\boldsymbol{x}_a), w)}{\partial w^2}\right)_{w = \tilde{w}(\boldsymbol{x}_a, \tilde{M}^T(\boldsymbol{x}_a))}$$

and the notation $\|v\|_{\tilde{R}}^2$ stands for $v^T\tilde{R}v$. It is easy to verify that $\tilde{R}(0) = (1 - \epsilon_1)(D_{11}^T D_{11} - \gamma^2 I)$. Since $Y_1(\boldsymbol{x})$ and $Y_3(\boldsymbol{x}_a)$ are nonpositive, (16) implies that

$$\frac{dU}{dt} + \|Z(\boldsymbol{x}, w, \alpha_2(\xi) + c(\eta))\|^2 - \gamma^2\|w\|^2 \leq 0 \tag{17}$$

which, in turn, implies that the closed-loop system has an $\mathcal{L}_2$-gain less than or equal to γ. Set $w = 0$, rearrange terms, and use (16) to get

$$\begin{aligned}\frac{dU}{dt} =& -\|Z(\boldsymbol{x}, 0, \alpha_2(\xi) + c(\eta))\|^2 + Y_2(\boldsymbol{x}) + Y_3(\boldsymbol{x}_a) \\ &+ \begin{bmatrix} -\alpha_2(\boldsymbol{x}) \\ \alpha_2(\xi) + c(\eta) - \alpha_2(\boldsymbol{x}) \end{bmatrix}^T \begin{bmatrix} \epsilon_1 r_{11}(\boldsymbol{x}) & 0 \\ 0 & -\epsilon_2 r_{22}(\boldsymbol{x}) \end{bmatrix} \\ &\cdot \begin{bmatrix} -\alpha_2(\boldsymbol{x}) \\ \alpha_2(\xi) + c(\eta) - \alpha_2(\boldsymbol{x}) \end{bmatrix} + \|\tilde{w}(\boldsymbol{x}_a, \tilde{M}^T(\boldsymbol{x}_a))\|_{\tilde{R}(\boldsymbol{x}_a)}^2 \\ &+ o\left(\left\| \begin{matrix} -\alpha_1(\boldsymbol{x}) \\ \alpha_2(\xi) + c(\eta) - \alpha_2(\boldsymbol{x}) \end{matrix} \right\|^3\right) \\ &+ o\left(\|\tilde{w}(\boldsymbol{x}_a, \tilde{M}^T(\boldsymbol{x}_a))\|^3\right)\end{aligned}$$

which is negative semidefinite near $\boldsymbol{x}_a = 0$ by hypothesis. This shows that the closed-loop system is stable locally around the equilibrium point. We claim that the DAE (14) has index one. To see this, observe that any trajectory satisfying $(dU/dt)(\boldsymbol{x}(t), \xi(t), \eta(t)) = 0$ for all $t \geq 0$ is necessarily a trajectory of

$$E\dot{\boldsymbol{x}}(t) = F(\boldsymbol{x}, 0, \alpha_2(\xi) + c(\eta)) \tag{18}$$

such that $\boldsymbol{x}(t)$ is bounded and $Z(\boldsymbol{x}, 0, \alpha_2(\xi) + c(\eta)) = 0$ for all $t \geq 0$. This shows that the previous DAE has index one. Moreover,

$$F_a \triangleq \begin{bmatrix} F(\boldsymbol{x}, w, \alpha_2(\xi) + c(\eta)) \\ \tilde{F}(\xi, \eta) + G(\xi)Y(\boldsymbol{x}, w, \alpha_2(\xi) + c(\eta)) + \hat{g}_1(\xi)c(\eta) + \hat{g}_2(\xi)d(\eta) \\ a(\eta, Y(\boldsymbol{x}, w, \alpha_2(\xi) + c(\eta)) - Y(\xi, \alpha_1(\xi), \alpha_2(\xi) + c(\eta))) \end{bmatrix}.$$

hypotheses H1 and H3 along with assumption A1) imply that the trajectory satisfying $(dU/dt)(\boldsymbol{x}(t), \xi(t), \eta(t)) = 0$ for all $t \geq 0$ is necessarily a trajectory such that $\boldsymbol{x}(t) = \xi(t)$ and $\eta(t) = 0$ for all $t \geq 0$. Setting $\boldsymbol{x}(t) = \xi(t) \equiv 0$ and $w(t) = 0$ in (15), we have $(\tilde{M}_\eta)^T a(\eta, Y(0, 0, c(\eta)) - Y(0, 0, c(\eta))) < 0$, for all $\eta \neq 0$, where $\tilde{M}^T(\boldsymbol{x}_a) = [(\tilde{M}_{\boldsymbol{x}})^T \ (\tilde{M}_\xi)^T \ (\tilde{M}_\eta)^T]$. This shows that the DAE

$$E_Q \dot{\eta} = a(\eta, Y(0, 0, c(\eta)) - Y(0, 0, c(\eta))) \tag{19}$$

has index one and is asymptotically stable. Hence, by hypothesis H2 and the fact that DAE (18) and (19) have index one, we can conclude that the closed-loop system (8)–(14) has index one. Asymptotical stability then easily follows by Theorem 4. Q.E.D.

The previous lemma gives a general form of the output feedback controllers. However, it does not explicitly specify how we can choose the free system parameters E_Q, $a(\bullet, \bullet)$ and $c(\bullet)$ in order to meet the hypothesis in Lemma 7. In the sequel, we give a way to meet the condition in Lemma 7, and in the mean time, to reduce the number of independent variables. Consider the following DAE:

$$E_Q \dot{\eta} = a(\eta, \bullet). \tag{20}$$

If there exists a smooth function $L(\eta)$, locally defined on a neighborhood of $\eta = 0$, which vanishes at the points where $E_Q \eta = 0$, is positive elsewhere, satisfies $(\partial L(\eta)/\partial \eta) = \tilde{L}(\eta)^T E_Q$ with $E_Q^T \tilde{L}_\eta(\eta) = \tilde{L}_\eta(\eta)^T E_Q$, and is such that $\tilde{L}^T(\eta) a(\eta, \bullet) < 0$, then we can conclude from Theorem 2 that DAE (20) is locally asymptotically stable and has index one. Henceforth, if some further hypotheses are imposed in the above inequality, the condition in Lemma 7 can be met. This is summarized in the following theorem.

Theorem 8: Consider (8) and (14). Suppose assumptions A1)–A4) are satisfied. Suppose hypotheses H1 of Lemma 5 and H2 of Theorem 6 hold. Suppose also that the following hypothesis holds.

H4: There exists a smooth function $L(\eta)$, defined as previously shown, such that the function

$$Y_4(\eta, w) = \tilde{L}^T(\eta) a(\eta, Y(0, w, 0)) + \begin{bmatrix} w \\ c(\eta) \end{bmatrix}^T R(0) \begin{bmatrix} w \\ c(\eta) \end{bmatrix}$$

at $w = w^+(\eta)$, viewed as a function of η, is negative definite near $\eta = 0$, and its Hessian matrix is nonsingular at $\eta = 0$. The function $w^+(\eta)$ is defined on a neighborhood of $\eta = 0$, which satisfies $(\partial Y_4(\eta, w)/\partial w)_{w=w^+(\eta)} = 0$ with $w^+(0) = 0$ (This function exists, for $R(0)$ is nonsingular).

Then, if $\hat{g}_1(\bullet)$ and $\hat{g}_2(\bullet)$ satisfy

$$\tilde{Q}(\boldsymbol{x})\hat{g}_1(\boldsymbol{x}) = 2\beta^T(\boldsymbol{x}, 0, 0) r_{12}(\boldsymbol{x}) - 2(1 + \epsilon_2)\alpha_2^T(\boldsymbol{x}) r_{22}(\boldsymbol{x})$$

and

$$\tilde{Q}(\boldsymbol{x})\hat{g}_2(\boldsymbol{x}) = a^T(0, Y(\boldsymbol{x}, \alpha_1(\boldsymbol{x}) + \beta(\boldsymbol{x}, 0, 0), 0))$$

respectively, where $\beta(\boldsymbol{x}, \xi, \eta) = \tilde{w}(\boldsymbol{x}_a, [\tilde{Q}(\boldsymbol{x} - \xi) - \tilde{Q}(\boldsymbol{x} - \xi)\tilde{L}(\eta)])$, the family of controllers (14) with $d(\eta) \triangleq \tilde{L}(\eta)$ solves the $\mathcal{H}_\infty$ output feedback control problem. ◇◇◇

Proof: It is straightforward to verify that $M(\boldsymbol{x}_a) \triangleq Q(\boldsymbol{x} - \xi) + L(\eta)$ satisfies the hypothesis of Lemma 7.

IV. Converse Result—A Necessary Condition

Suppose that the $\mathcal{H}_\infty$ control problem is solved by the output feedback controller Γ which has the following representation:

$$\begin{aligned} \hat{E}\dot{\xi} &= \Phi(\xi, y) \\ u &= \Theta(\xi) \end{aligned} \tag{21}$$

and let U be a smooth function satisfying

$$W(\boldsymbol{x}, \xi, w) = \begin{bmatrix} \tilde{U}_{\boldsymbol{x}} & \tilde{U}_\xi \end{bmatrix} \begin{bmatrix} F(\boldsymbol{x}, w, \Theta(\xi)) \\ \Phi(\xi, Y(\boldsymbol{x}, w, \Theta(\xi)) \end{bmatrix} + \|Z(\boldsymbol{x}, w, \Theta(\xi))\|^2 - \gamma^2\|w\|^2 \leq 0 \tag{22}$$

for all $(\boldsymbol{x}, \xi, w)$ in a neighborhood of (0, 0, 0). Consider the case that $\xi \neq 0$ and $\tilde{U}_\xi(\boldsymbol{x}, \xi) \neq 0$. Since $\xi \neq 0$, we have $\Phi \neq 0$. Hence, from (22), we have $\inf_{\Phi, \Theta} \max_w W(\boldsymbol{x}, \xi, w) = -\infty$, because inequality (22) contains a term linearly in Φ. Next, consider the case that $\xi \neq 0$ but $\tilde{U}_\xi(\boldsymbol{x}, \xi) = 0$. Suppose that $\tilde{U}_{\xi\xi}$ is nonsingular for every $(\boldsymbol{x}, \xi)$ satisfying $\tilde{U}_\xi(\boldsymbol{x}, \xi) = 0$. Then, by the implicit function theorem, the previous identity has a differentiable solution $\xi = \ell(\boldsymbol{x})$ with $\ell(0) = 0$. The previous statement is needed in the subsequent proof. We take it as a standing assumption.

A5) $\tilde{U}_\xi(\boldsymbol{x}, \xi) = 0$ if and only if $\xi = \ell(\boldsymbol{x})$ for some smooth function ℓ with $\ell(0) = 0$. Furthermore, $\tilde{U}_{\xi\xi}(\boldsymbol{x}, \xi)|_{\xi=\ell(\boldsymbol{x})}$ is nonsingular.

Setting $\xi = \ell(\boldsymbol{x})$ in (12) yields

$$\tilde{V}^T F(\boldsymbol{x}, w, \Theta(\ell(\boldsymbol{x})) + \|Z(\boldsymbol{x}, w, \Theta(\ell(\boldsymbol{x}))\|^2 - \gamma^2\|w\|^2 \leq 0, \quad \forall w. \tag{23}$$

This shows that $\inf_{0, \Theta(\xi)} \max_w W(\boldsymbol{x}, \xi, w) = Y_1(\boldsymbol{x})$, where $\xi = \ell(\boldsymbol{x})$. Hence, the state feedback law $u = \Theta(\ell(\boldsymbol{x}))$ solves the state feedback $\mathcal{H}_\infty$ control problem for Σ. This shows that V is a solution of Y_1. A further necessary condition is obtained by restricting to the class of controller Γ which produces zero control input u. Consider the Hamiltonian function $K_\gamma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^p \to \mathbb{R}$ defined as

$$K_\gamma(\boldsymbol{x}, p, w, y) = p^T F(\boldsymbol{x}, w, 0) - y^T Y(\boldsymbol{x}, w, 0) + \|Z(\boldsymbol{x}, w, 0)\|^2 - \gamma^2\|w\|^2. \tag{24}$$

It is easy to verify that

$$\left(\frac{\partial^2 K_\gamma(\boldsymbol{x}, p, w, y)}{\partial w^2}\right)_{(\boldsymbol{x}, p, w, y) = (0,0,0,0)} = 2(D_{11}^T D_{11} - \gamma^2 I).$$

This shows that there exists a smooth function $\hat{w}(\boldsymbol{x}, p, y)$ defined in a neighborhood of (0,0,0) such that

$$\left(\frac{\partial K_\gamma(\boldsymbol{x}, p, w, y)}{\partial w}\right)_{w = \hat{w}(\boldsymbol{x}, p, y)} = 0 \quad \hat{w}(0, 0, 0) = 0.$$

Furthermore, it is also easy to check that

$$\left(\frac{\partial^2 K_\gamma(\boldsymbol{x}, p, \hat{w}(\boldsymbol{x}, p, y), y)}{\partial y^2}\right)_{(\boldsymbol{x}, p, y) = (0,0,0)} = \frac{1}{2}(\gamma^2 I - D_{11}^T D_{11})^{-1} D_{21} D_{21}^T. \tag{25}$$

Thus, there exists a smooth function $y^*(x, p)$ defined in a neighborhood of (0,0) such that

$$\left(\frac{\partial K_\gamma(x, p, \hat{w}(\boldsymbol{x}, p, y), y)}{\partial y}\right)_{y = y^*(\boldsymbol{x}, p)} = 0 \quad y^*(0, 0) = 0.$$

Set $w^*(\boldsymbol{x}, p) = \hat{w}(\boldsymbol{x}, p, y^*(\boldsymbol{x}, p))$. Then, we have

$$K_\gamma(\boldsymbol{x}, p, w, y) \leq K_\gamma(\boldsymbol{x}, p, \hat{w}(\boldsymbol{x}, p, y), y) \tag{26}$$

for all $(\boldsymbol{x}, p, w, y)$ in a neighborhood of the origin and

$$K_\gamma(\boldsymbol{x}, p, \hat{w}(\boldsymbol{x}, p, y), y) \geq K_\gamma(\mathrm{x}, p, w^*(\boldsymbol{x}, p), y^*(\boldsymbol{x}, p)), y^*(\boldsymbol{x}, p)) \tag{27}$$

for all $(\boldsymbol{x}, p, y)$ in a neighborhood of the origin. We will show that it is necessary

$$K_\gamma(\boldsymbol{x}, \tilde{P}(\boldsymbol{x}), w^*(x, \tilde{P}(\boldsymbol{x})), y^*(\boldsymbol{x}, \tilde{P}(\boldsymbol{x}))) \leq 0 \tag{28}$$

for some storage function $P(\boldsymbol{x})$ with $(\partial P/\partial \boldsymbol{x}) = \tilde{P}^T E$. This is summarized in the following statement.

Theorem 9: Consider system (8) and suppose assumptions A1)–A5) hold. Suppose that the $\mathcal{H}_\infty$ control problem is solved by the output feedback controller (21). Suppose that there exists a smooth real-valued function $U(\boldsymbol{x}, \xi)$, which vanishes at the points where $E^e \boldsymbol{x}^e = 0$ and is positive elsewhere with $E^{eT} \tilde{U}_{\boldsymbol{x}^e} = \tilde{U}_{\boldsymbol{x}^e}^T E^e$, and satisfies (21) for all $(\boldsymbol{x}, \xi, w)$ in a neighborhood of (0, 0, 0). Then, the Hamilton–Jacobi inequalities

$$Y_1(\boldsymbol{x}) \le 0 \text{ and } K_\gamma(\boldsymbol{x}, \tilde{P}(\boldsymbol{x}), w^*(x, \tilde{P}(\boldsymbol{x})), y^*(\boldsymbol{x}, \tilde{P}(\boldsymbol{x}))) \le 0$$

have solutions $V(\boldsymbol{x})$ and, respectively, $P(\boldsymbol{x})$ (with $(\partial P/\partial \boldsymbol{x}) = \tilde{P}^T E$) *given by* $V(\boldsymbol{x}) = U(\boldsymbol{x}, \ell(\boldsymbol{x})) \ge 0$ and, respectively, $P(\boldsymbol{x}) = U(\boldsymbol{x}, 0) \ge 0$. Furthermore, $Q(\boldsymbol{x}) \triangleq P(\boldsymbol{x}) - V(\boldsymbol{x}) \ge 0$. $\diamond\diamond\diamond$

Proof: It is obvious that $V(\boldsymbol{x})$ is a solution satisfying $Y_1(\boldsymbol{x}) \le 0$ from our previous observation. It is claimed that $P(\boldsymbol{x})$ is a solution of inequality (28). To see this, setting $\xi = 0$ in (12) yields

$$\tilde{P}^T F(\boldsymbol{x}, w, 0) + \tilde{U}^\xi(\boldsymbol{x}, 0)\Phi(0, Y(\mathbf{x}, w, 0)) + \|Z(\boldsymbol{x}, w, 0)\|^2 - \gamma^2\|w\|^2 \le 0. \quad (29)$$

Let $\tilde{U}^\xi(\boldsymbol{x}, 0)\Phi(\boldsymbol{x}, y) = \Pi^T(\boldsymbol{x}, y)y$, where $\Pi(\boldsymbol{x}, y)$ is a vector of smooth functions. This can always be done because the function $\Phi(0, y)$ vanishes at $y = 0$. Use $\Pi^T(\boldsymbol{x}, y)y$ and choose $w = \hat{w}(\boldsymbol{x}, \tilde{P}, y)$ in (12) to obtain

$$K_\gamma(\boldsymbol{x}, \tilde{P}, \hat{w}(\boldsymbol{x}, \tilde{P}, y), \Pi(\boldsymbol{x}, Y(\boldsymbol{x}, \hat{w}(\boldsymbol{x}, \tilde{P}, y)))) \le 0. \quad (30)$$

Observe that the Hessian matrix of $y - \Pi(\boldsymbol{x}, Y(\boldsymbol{x}, \hat{w}(\boldsymbol{x}, \tilde{P}, y)))$ is nonsingular [from assumption A4) and (25)]. Hence, by the implicit function theorem, there exists a unique solution, denoted by $\hat{y}(\boldsymbol{x})$, satisfying $\hat{y}(\boldsymbol{x}) - \Pi(\boldsymbol{x}, Y(\boldsymbol{x}, \hat{w}(\boldsymbol{x}, \tilde{P}, \hat{y}(\boldsymbol{x})))) = 0$, $\hat{y}(0) = 0$. Set $y = \hat{y}(\boldsymbol{x})$ in (30) to obtain

$$K_\gamma(\boldsymbol{x}, \tilde{P}(\boldsymbol{x}), \hat{w}(\boldsymbol{x}, \tilde{P}(\boldsymbol{x}), \hat{y}(\boldsymbol{x})), \hat{y}(\boldsymbol{x})) \le 0.$$

This shows that

$$K(\boldsymbol{x}, \tilde{P}(\boldsymbol{x}), w^*(\boldsymbol{x}, \tilde{P}(\boldsymbol{x})), y^*(\boldsymbol{x}, \tilde{P}(\boldsymbol{x}))) \le 0$$

from (27). In order to complete the proof, we have to show that $Q(\boldsymbol{x}) \triangleq P(\boldsymbol{x}) - V(\boldsymbol{x}) \ge 0$. Note that the function $U(\boldsymbol{x}, \xi)$ has the following Taylor series expansion:

$$\begin{aligned} U(\boldsymbol{x}, \xi) =& U(0,0) + \frac{\partial U}{\partial \boldsymbol{x}}(0,0)\boldsymbol{x} + \frac{\partial U}{\partial \xi}(0,0)\xi \\ &+ \begin{bmatrix} \boldsymbol{x} \\ \xi \end{bmatrix}^T \begin{bmatrix} \frac{1}{2}\frac{\partial^2 U}{\partial \boldsymbol{x}^2}(0,0) & \frac{1}{2}\frac{\partial^2 U}{\partial \boldsymbol{x}\partial \xi}(0,0) \\ \left(\frac{1}{2}\frac{\partial^2 U}{\partial \boldsymbol{x}\partial \xi}\right)^T(0,0) & \frac{1}{2}\frac{\partial^2 U}{\partial \xi^2}(0,0) \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \xi \end{bmatrix} \\ &+ \text{h.o.t.} \end{aligned} \quad (31)$$

where "h.o.t." means high order terms. Let $\xi = \ell(\boldsymbol{x})$, then we have the following two Taylor series expansions:

$$\begin{aligned} \frac{\partial^2 U}{\partial \boldsymbol{x}^2}(\boldsymbol{x}, \ell(\boldsymbol{x})) =& \frac{\partial^2 U}{\partial \boldsymbol{x}^2}(0,0) + \frac{\partial^2 U}{\partial \boldsymbol{x}\partial \xi}(0,0)\ell_{\boldsymbol{x}}(\boldsymbol{x}) \\ &+ (h.o.t.)_{\boldsymbol{xx}}|_{\xi=\ell(\boldsymbol{x})} \end{aligned} \quad (32)$$

$$\begin{aligned} \frac{\partial^2 U}{\partial \xi\partial \boldsymbol{x}}(\boldsymbol{x}, \ell(\boldsymbol{x})) =& \frac{\partial^2 U^T}{\partial \boldsymbol{x}\partial \xi}(0,0) + \frac{\partial^2 U}{\partial \xi^2}(0,0)\ell_{\boldsymbol{x}}(\boldsymbol{x}) \\ &+ (h.o.t.)_{\xi\boldsymbol{x}}|_{\xi=\ell(\boldsymbol{x})}. \end{aligned} \quad (33)$$

Note that $(\partial^2 U/\partial \boldsymbol{x}^2)(\boldsymbol{x}, \ell(\boldsymbol{x})) = (\partial^2 V/\partial \boldsymbol{x}^2)(\boldsymbol{x})$, and $(\partial^2 U/\partial \xi\partial \boldsymbol{x})(\boldsymbol{x}, \ell(\boldsymbol{x})) = 0$. Set $\boldsymbol{x} = 0$ in (32) and (33), respectively, to get

$$\frac{\partial^2 U}{\partial \boldsymbol{x}^2}(0, \ell(0)) = \frac{\partial^2 V}{\partial \boldsymbol{x}^2}(0) = \frac{\partial^2 U}{\partial \boldsymbol{x}^2}(0,0) + \frac{\partial^2 U}{\partial \boldsymbol{x}\partial \xi}(0,0)\ell_{\boldsymbol{x}}(0)$$

and

$$0 = \frac{\partial^2 U}{\partial \xi\partial \boldsymbol{x}}(0,0) = \frac{\partial^2 U^T}{\partial \boldsymbol{x}\partial \xi}(0,0) + \frac{\partial^2 U}{\partial \xi^2}(0,0)\ell_{\boldsymbol{x}}(0)$$

respectively. Next, observe that

$$\begin{aligned} \frac{\partial^2 Q}{\partial \boldsymbol{x}^2}(0) =& \frac{\partial^2 P}{\partial \boldsymbol{x}^2}(0) - \frac{\partial^2 V}{\partial \boldsymbol{x}^2}(0) \\ =& \frac{\partial^2 U}{\partial \boldsymbol{x}^2}(0,0) - \left(\frac{\partial^2 U}{\partial \boldsymbol{x}^2}(0,0) + \frac{\partial^2 U}{\partial \boldsymbol{x}\partial \xi}(0,0)\ell_{\boldsymbol{x}}(0)\right) \\ =& -\frac{\partial^2 U}{\partial \boldsymbol{x}\partial \xi}(0,0)\ell_{\boldsymbol{x}}(0) = \ell_{\boldsymbol{x}}^T(0)\frac{\partial^2 U}{\partial \xi^2}(0,0)\ell_{\boldsymbol{x}}(0) \ge 0. \end{aligned} \quad (34)$$

The last inequality holds by assumption A5). This concludes that $Q(\boldsymbol{x}) \ge 0$ by noting that $Q(\boldsymbol{x})$ has the following Taylor series expansion:

$$Q(\boldsymbol{x}) = Q(0) + \frac{\partial Q}{\partial \boldsymbol{x}}(0)\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T\frac{\partial^2 Q}{\partial \boldsymbol{x}^2}(0)\boldsymbol{x} + \text{h.o.t.} \ge 0.$$

It is nonnegative around the origin because it vanishes at the origin together with its first-order derivative, and its second-order derivative is positive by (34). This completes the proof. Q.E.D.

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