Robust H_∞ filter design with filter pole constraints via π-sharing theory

C.-M. Lee and I-K. Fong

Abstract: The robust H_∞ filtering problem subject to pole-placement constraints for continuous-time systems with the polytopic type uncertainties is considered. Different from those considered in the literature, the regional pole-placement constraints considered here focus on the filter dynamics. To solve the problem, the π-sharing theory is extended to offer a stability criterion that covers the bounded real lemma as a special case, and the linear matrix inequality approach is adopted to develop filter design methods based on the convex optimisation procedure. One numerical example is given to illustrate the proposed methods.

1 Introduction

The filter design problem for dynamic systems is important in many engineering applications. The filter design techniques therefore have received a large amount of attention in the literature, for both theoretical and practical aspects. In recent years, the convex optimisation based filter design methods under the linear matrix inequality (LMI) framework [1] are very popular because of the efficient computation algorithms that are available. Also, the consideration of system poles has become an important issue in the filter design problem [2–4], just as in the feedback control problems [5–7]. There are some studies trying to consider the poles in the robust filtering problem using the LMI approach [8–11]. However, in these works there is a common assumption. In order to conveniently impose the pole-placement constraints of the filter, the poles of the entire filtering error dynamics are required to lie inside some desired regions. Thus poles of the system for which signals are to be estimated must also be assumed to lie inside the desired regions. In general, such an assumption not only limits its applicable domain, but also often results in more conservative designs. Unlike in the feedback control problems, for which it is natural and feasible to require poles of the overall system be placed into a certain region, in the filtering problem the only poles that need to be placed are those of the filters. Thus it is more reasonable to merely assume that the system for which signals are to be estimated is stable, i.e., with all poles located in the open left half of the complex plane [12]. With this in mind, this paper considers the robust filtering problem subject to the $\mathcal{D}$-stability [5] constraint for the filter, which enables the filter designers to shape the filter characteristics in a flexible fashion [13, 14].

The concept of energy storage and dissipation in the circuit theory is well inherited in system theory [15], and is applied fruitfully to many control and engineering problems, such as the stability analysis problems [16–18], the design and synthesis problems for controllers and filters [19–21], and so on. Along this line, the π-sharing theory [22] is an extension of the concepts of passivity [23] and dissipativity [24]. It uses the so-called π-coefficients to describe the 'energy storage and dissipation' of systems, and the so-called π-stability [22] is able to deal with both the Lyapunov stability and input-output stability simultaneously. After proper development [25], the π-sharing theory is shown to be applicable to MIMO systems within a convenient LMI framework. Successful application of the π-sharing theory to the controller design problem can be found in [26]. However, the π-sharing theory in [25, 26] is limited to square systems. Here, an extended π-sharing theory will be developed under the LMI framework, and applied to the above filter design problem, which involves non-square systems. As can be seen subsequently, the π-stability from the extended π-sharing theory covers the bounded real lemma [1, 27] as a special case.

Some notations to be adopted are introduced first. The inequality $\boldsymbol{X} \geq \boldsymbol{0}$ means that the matrix $\boldsymbol{X}$ is symmetric and positive semi-definite, and $\boldsymbol{X} \geq \boldsymbol{Y}$ means $\boldsymbol{X} - \boldsymbol{Y} \geq \boldsymbol{0}$. Similar definitions apply to symmetric positive/negative definite matrices. Let $\boldsymbol{x}(t)$ and $\boldsymbol{\Phi}(t)$, respectively, be any real vector and symmetric matrix functions of the continuous-time time index t. Then $(\boldsymbol{\Phi})|\boldsymbol{x}(t)|^2 = \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{\Phi}(t)\boldsymbol{x}(t)$ and $(\boldsymbol{\Phi})\|\boldsymbol{x}\|_T^2 = \int_0^T \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{\Phi}(t)\boldsymbol{x}(t)dt$, where T is a nonnegative constant. If $\boldsymbol{\Phi} = \boldsymbol{I}$, the identity matrix, then it is omitted from the notations. Finally, for any matrix $\boldsymbol{Z}$, $\|\boldsymbol{Z}\|$ represents its induced two-norm, and for any symmetric matrix $\boldsymbol{X}$, $\lambda_{\max}(\boldsymbol{X})$ and $\lambda_{\min}(\boldsymbol{X})$ denote its maximal and minimal eigenvalues, respectively.

2 π-sharing theory and problem formulation

2.1 Multivariable π-sharing theory

First, the continuous-time multivariable π-sharing theory [25] is extended in this subsection, but restricted to the linear time-invariant (LTI) case. Consider the system Σ_{o} with the state-space model

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned} \tag{1}$$

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where $\boldsymbol{x}(t) \in \mathcal{R}^n$ is the state vector, $\boldsymbol{u}(t) \in \mathcal{R}^m$ is the input vector, and $\boldsymbol{y}(t) \in \mathcal{R}^p$ is the output vector. The system matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ are of appropriate dimensions. The system Σ_o is said [22, 25] to be π-sharing with respect to the constant π-coefficients $\{\boldsymbol{S}, \boldsymbol{\Gamma}, \boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{R}\}$, if for all $T \geq 0$

$$\int_0^T \boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{S}\boldsymbol{y}(t)dt \geq (\boldsymbol{\Gamma})|\boldsymbol{x}(T)|^2 - (\boldsymbol{\Gamma})|\boldsymbol{x}(0)|^2 + (\boldsymbol{Q})\|\boldsymbol{x}\|_T^2 + (\boldsymbol{P})\|\boldsymbol{y}\|_T^2 + (\boldsymbol{R})\|\boldsymbol{u}\|_T^2 \quad (2)$$

where $\boldsymbol{\Gamma}, \boldsymbol{Q} \in \mathcal{R}^{n\times n}$ are positive semi-definite symmetric, $\boldsymbol{P} \in \mathcal{R}^{p\times p}$ and $\boldsymbol{R} \in \mathcal{R}^{m\times m}$ are symmetric, and $\boldsymbol{S} \in \mathcal{R}^{m\times p}$ is subject to no special constraint. Define the dissipativity matrix [25] of the system Σ_o as

$$\boldsymbol{M}_{dis} = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{12}^{\mathrm{T}} & \boldsymbol{M}_{22} \end{bmatrix} \quad (3)$$

where

$$\boldsymbol{M}_{11} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Gamma} + \boldsymbol{\Gamma}\boldsymbol{A} + \boldsymbol{Q} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{C},$$

$$\boldsymbol{M}_{12} = \boldsymbol{\Gamma}\boldsymbol{B} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{D} - \frac{1}{2}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}},$$

$$\boldsymbol{M}_{22} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{D} + \boldsymbol{R} - \frac{1}{2}(\boldsymbol{S}\boldsymbol{D} + \boldsymbol{D}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}})$$

The dissipativity matrix may be used to qualify a set of π-coefficients in terms of its negative semi-definiteness in the next lemma.

Lemma 1 [22, 25]: If $\boldsymbol{M}_{dis} \leq \boldsymbol{0}$, $\boldsymbol{\Gamma} \geq \boldsymbol{0}$, and $\boldsymbol{Q} \geq \boldsymbol{0}$, then system Σ_o in (1) is π-sharing with respect to $\{\boldsymbol{S}, \boldsymbol{\Gamma}, \boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{R}\}$.

Note that the conditions in the above lemma are LMIs with respect to $\{\boldsymbol{S}, \boldsymbol{\Gamma}, \boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{R}\}$ for the given Σ_o.

In the π-sharing theory, the π-stability is defined to include state and input-output stability at the same time. Below is the definition of the π-stability.

Definition 1 [22, 25]: The system (1) is π-stable, if there exist $\gamma_1, \ldots, \gamma_4 \in \mathcal{R}$ such that

$$\|\boldsymbol{y}\|_T \leq \gamma_1\|\boldsymbol{u}\|_T + \gamma_2|\boldsymbol{x}(0)|,$$

$$\sup_{0\leq t\leq T} |\boldsymbol{x}(t)| \leq \gamma_3\|\boldsymbol{u}\|_T + \gamma_4|\boldsymbol{x}(0)|$$

for all $\boldsymbol{u}(t) \in \mathcal{R}^m$, $\boldsymbol{x}(0) \in \mathcal{R}^n$, and $T \geq 0$.

Obviously, the first condition in Definition 1 is for the $\mathcal{L}_2$ stability, and the second one implies stability in the sense of Lyapunov when the external input $\boldsymbol{u} \equiv \boldsymbol{0}$. The next lemma, adapted from [22, 25] for the LTI case, gives a sufficient condition for the π-stability in terms of the π-coefficients.

Lemma 2: If the system Σ_o in (1) is π-sharing with respect to $\{\boldsymbol{S}, \boldsymbol{\Gamma}, \boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{R}\}$ with $\boldsymbol{\Gamma} \geq \gamma\boldsymbol{I} > \boldsymbol{0}$, $\boldsymbol{R} \geq \rho_0\boldsymbol{I}$, and $\boldsymbol{P} \geq p_0\boldsymbol{I} > \boldsymbol{0}$, then it is π-stable with $\gamma_1 = (s_0 + \sqrt{p_0\delta})/p_0$, $\gamma_2 = \sqrt{\gamma_0/p_0}$, and $\gamma_3 = \gamma_4 = \sqrt{\xi/\gamma}$, where $\delta = |\min\{0, \rho_0\}|$, $\gamma_0 = \lambda_{\max}(\boldsymbol{\Gamma})$, $\xi = \max\{\gamma_0, \delta + (s_0 + \sqrt{p_0\delta})/p_0, \sqrt{\gamma_0/p_0}\}$, and $s_0 = \|\boldsymbol{S}\|$.

Compared with the original π-sharing theory [22, 25], the extended π-sharing theory uses $\int_0^T \boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{S}\boldsymbol{y}(t)dt$ instead of $\int_0^T \boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{y}(t)dt$ in the left-hand side of (2). Thus the difference in form is not large, and the proofs for lemmas 1 and 2 can be extended from the corresponding proofs in [22, 25] in a straightforward fashion. For the sake of brevity, the proofs are omitted here. However, the introduction of $\boldsymbol{S}$ not only makes handling non-square systems (with $m \neq p$) possible, but also lets lemma 2 cover the bounded real lemma [27] as a special case. For example, the H_∞ performance specification $\|\boldsymbol{y}\|_T^2 \leq \eta^2\|\boldsymbol{u}\|_T^2$ for a given $\eta > 0$ may be formulated as the π-stability of system Σ_o with $\{\boldsymbol{S}, \boldsymbol{\Gamma}, \boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{R}\} = \{\boldsymbol{0}, \boldsymbol{\Gamma}, \boldsymbol{0}, \boldsymbol{I}, -\eta^2\boldsymbol{I}\}$.

2.2 A robust filtering problem

Consider the deconvolution filtering system in Fig. 1. The source signal $\boldsymbol{s}(t) \in \mathcal{R}^{p_s}$ is assumed to be generated by the signal model

$$\Sigma_S : \begin{cases} \dot{\boldsymbol{x}}_s(t) = \boldsymbol{A}_s\boldsymbol{x}_s(t) + \boldsymbol{B}_s\boldsymbol{w}(t) \\ \boldsymbol{s}(t) = \boldsymbol{C}_s\boldsymbol{x}_s(t) + \boldsymbol{D}_s\boldsymbol{w}(t) \end{cases} \quad (4)$$

where $\boldsymbol{x}_s(t) \in \mathcal{R}^{n_s}$ is the model state vector, $\boldsymbol{w}(t) \in \mathcal{R}^{m_s}$ is the driving signal vector with each element in $\mathcal{L}_2[0, \infty)$, and $\boldsymbol{A}_s$, $\boldsymbol{B}_s$, $\boldsymbol{C}_s$, $\boldsymbol{D}_s$ are known constant matrices of appropriate dimensions. The source signals are transmitted through a channel system with an uncertain characteristic modelled by

$$\Sigma_C : \begin{cases} \dot{\boldsymbol{x}}_c(t) = \boldsymbol{A}_c\boldsymbol{x}_c(t) + \boldsymbol{B}_c\boldsymbol{s}(t) \\ \boldsymbol{z}_c(t) = \boldsymbol{C}_c\boldsymbol{x}_c(t) + \boldsymbol{D}_c\boldsymbol{s}(t) \end{cases} \quad (5)$$

where $\boldsymbol{x}_c(t) \in \mathcal{R}^{n_c}$ and $\boldsymbol{z}_c(t) \in \mathcal{R}^{p_c}$ are the channel state and output signal vectors, respectively. The channel system matrices $\{\boldsymbol{A}_c, \boldsymbol{B}_c, \boldsymbol{C}_c, \boldsymbol{D}_c\}$ are only known to belong to a polytopic set

$$\mathcal{D}_c \equiv \left\{(\boldsymbol{A}_c, \boldsymbol{B}_c, \boldsymbol{C}_c, \boldsymbol{D}_c) = \sum_{i=1}^{l} \tau_i(\boldsymbol{A}_{ci}, \boldsymbol{B}_{ci}, \boldsymbol{C}_{ci}, \boldsymbol{D}_{ci})\right\} \quad (6)$$

where all uncertain parameters τ_i, $i = 1, 2, \ldots, l$, are non-negative and satisfy $\sum_{i=1}^{l} \tau_i = 1$, and all vertices $\{\boldsymbol{A}_{ci}, \boldsymbol{B}_{ci}, \boldsymbol{C}_{ci}, \boldsymbol{D}_{ci}\}$ of the polytope are known and designated by $i = 1, 2, \ldots, l$.

At the receiving end, the measured signal vector $\boldsymbol{y}(t) \in \mathcal{R}^{p_c}$ is equal to $\boldsymbol{z}_c(t) + \boldsymbol{v}(t)$, where $\boldsymbol{v}(t)$ is an energy-bounded channel noise vector. To integrate, one can combine the signal and channel models as

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}_e(t) \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}_e(t) \\ \boldsymbol{s}(t) = \boldsymbol{L}\boldsymbol{x}(t) + \boldsymbol{J}\boldsymbol{u}_e(t) \end{cases} \quad (7)$$

where $\boldsymbol{x}^{\mathrm{T}} = [\boldsymbol{x}_s^{\mathrm{T}}\ \boldsymbol{x}_c^{\mathrm{T}}]$, $\boldsymbol{u}_e^{\mathrm{T}} = [\boldsymbol{w}^{\mathrm{T}}\ \boldsymbol{v}^{\mathrm{T}}]$, $\boldsymbol{L} = [\boldsymbol{C}_s\ \boldsymbol{0}]$, and $\boldsymbol{J} = [\boldsymbol{D}_s\ \boldsymbol{0}]$, and define the polytopic set

$$\mathcal{D}_z = \left\{(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}) = \sum_{i=1}^{l} \tau_i(\boldsymbol{A}_i, \boldsymbol{B}_i, \boldsymbol{C}_i, \boldsymbol{D}_i)\right\} \quad (8)$$

where

$$\boldsymbol{A}_i = \begin{bmatrix} \boldsymbol{A}_s & \boldsymbol{0} \\ \boldsymbol{B}_{ci}\boldsymbol{C}_s & \boldsymbol{A}_{ci} \end{bmatrix}, \quad \boldsymbol{B}_i = \begin{bmatrix} \boldsymbol{B}_s & \boldsymbol{0} \\ \boldsymbol{B}_{ci}\boldsymbol{D}_s & \boldsymbol{0} \end{bmatrix},$$
$$\boldsymbol{C}_i = [\boldsymbol{D}_{ci}\boldsymbol{C}_s\ \ \boldsymbol{C}_{ci}], \quad \boldsymbol{D}_i = [\boldsymbol{D}_{ci}\boldsymbol{D}_s\ \ \boldsymbol{I}] \quad (9)$$

To optimally recover the source signals $\boldsymbol{s}(t)$, the signal vector $\boldsymbol{y}(t)$ is deconvoluted by a filter with order

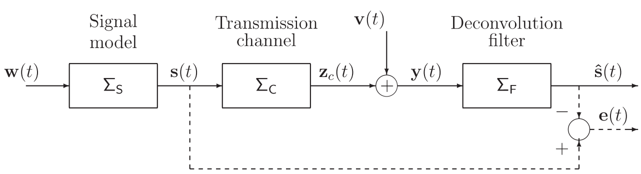

Fig. 1 *Deconvolution filtering system model*

$n_f = n_s + n_c$:

$$\Sigma_F : \begin{cases} \dot{\boldsymbol{x}}_f(t) = \boldsymbol{A}_f \boldsymbol{x}_f(t) + \boldsymbol{B}_f \boldsymbol{y}(t) \\ \boldsymbol{s}_f(t) = \boldsymbol{C}_f \boldsymbol{x}_f(t) + \boldsymbol{D}_f \boldsymbol{y}(t) \end{cases} \tag{10}$$

where $\boldsymbol{x}_f(t) \in \mathcal{R}^{n_f}$ is the filter state vector, $\boldsymbol{s}_f(t) \in \mathcal{R}^{p_s}$ is the filter output vector, and $\boldsymbol{A}_f, \boldsymbol{B}_f, \boldsymbol{C}_f, \boldsymbol{D}_f$ are filter system matrices to be designed. Although choosing a fixed filter under the uncertain channel environment may cause some conservativeness than choosing parameter-dependent filters [28, 29], there are also advantages. A fixed filter does not require the knowledge of exact parameter values of $\tau_i, i = 1, 2, \ldots, l$, and is easier to implement.

Define the filtering error as $\boldsymbol{e}(t) = \boldsymbol{s}(t) - \boldsymbol{s}_f(t)$, which satisfies

$$\Sigma_e : \begin{cases} \dot{\boldsymbol{x}}_e(t) = \boldsymbol{A}_e \boldsymbol{x}_e(t) + \boldsymbol{B}_e \boldsymbol{u}_e(t) \\ \boldsymbol{e}(t) = \boldsymbol{C}_e \boldsymbol{x}_e(t) + \boldsymbol{D}_e \boldsymbol{u}_e(t) \end{cases} \tag{11}$$

with $\boldsymbol{x}_e^{\mathrm{T}}(t) = [\boldsymbol{x}^{\mathrm{T}}(t) \ \boldsymbol{x}_f^{\mathrm{T}}(t)]$. The polytopic set $\mathcal{D}_e$ is defined as

$$\mathcal{D}_e = \left\{ (\boldsymbol{A}_e, \boldsymbol{B}_e, \boldsymbol{C}_e, \boldsymbol{D}_e) = \sum_{i=1}^{l} \tau_i (\boldsymbol{A}_{ei}, \boldsymbol{B}_{ei}, \boldsymbol{C}_{ei}, \boldsymbol{D}_{ei}) \right\} \tag{12}$$

where

$$\boldsymbol{A}_{ei} = \begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{0} \\ \boldsymbol{B}_f \boldsymbol{C}_i & \boldsymbol{A}_f \end{bmatrix}, \quad \boldsymbol{B}_{ei} = \begin{bmatrix} \boldsymbol{B}_i \\ \boldsymbol{B}_f \boldsymbol{D}_i \end{bmatrix}$$
$$\boldsymbol{C}_{ei} = \begin{bmatrix} \boldsymbol{L} - \boldsymbol{D}_f \boldsymbol{C}_i & -\boldsymbol{C}_f \end{bmatrix}, \quad \boldsymbol{D}_{ei} = \boldsymbol{J} - \boldsymbol{D}_f \boldsymbol{D}_i \tag{13}$$

The purpose of this paper is to design a filter Σ_F with the three desired properties below.

DP-1: The filtering error dynamics Σ_e is bounded stable [30], i.e., there exists a constant $\beta \geq 0$ such that $|\boldsymbol{x}_e(t)| \leq \beta$ for all $t \geq 0$, no matter what initial condition $\boldsymbol{x}_e(0)$ and input $\boldsymbol{u}_e(\cdot) \in \mathcal{L}_2[0, \infty)$ are.

DP-2: When $\boldsymbol{x}_e(0) = \boldsymbol{0}$, the filter has the H_∞ performance

$$\|\boldsymbol{e}(t)\|_T^2 \leq \mu^2 \|\boldsymbol{u}_e(t)\|_T^2 \tag{14}$$

for some scalar $\mu > 0$ and all $\boldsymbol{u}_e \neq 0$, as $T \to \infty$.

DP-3: Poles of the filter must be constrained in the following region(s) [5, 31] of the x–y plane:

- PC-1: Disc region $\mathcal{P}_D(c_0, r_0)$ centred at the point $(c_0, 0)$ with radius r_0.
- PC-2: Vertical strip $\mathcal{P}_V(r_1, r_2)$ lying between the lines $x = r_1$ and $x = r_2$ on the x–y plane, where $r_1 < r_2 \in \mathcal{R}$.
- PC-3: Left conic sector $\mathcal{P}_L(c_l, \theta_l)$ with the apex at the point $(c_l, 0)$ and inner angle θ_l, where $0 \leq \theta_l \leq \pi$.
- PC-4: Right conic sector $\mathcal{P}_R(c_r, \theta_r)$ with the apex at the point $(c_r, 0)$ and the inner angle θ_r, where $c_r \leq 0$ and $0 \leq \theta_r \leq \pi$.

From the above subsection, it is easy to see that both DP-1 and DP-2 may be implied by the π-stability. More specifically, if Σ_e is π-sharing with respect to the π-coefficients $\{\boldsymbol{S}_e, \boldsymbol{\Gamma}_e, \boldsymbol{Q}_e, \boldsymbol{P}_e, \boldsymbol{R}_e\}$ satisfying the matrix inequalities

$$\begin{bmatrix} \boldsymbol{A}_{ei}^{\mathrm{T}} \boldsymbol{\Gamma}_e + \boldsymbol{\Gamma}_e \boldsymbol{A}_{ei} + \boldsymbol{Q}_e & \boldsymbol{\Gamma}_e \boldsymbol{B}_{ei} - \frac{1}{2} \boldsymbol{C}_{ei}^{\mathrm{T}} \boldsymbol{S}_e^{\mathrm{T}} & \boldsymbol{C}_{ei}^{\mathrm{T}} \\ \boldsymbol{B}_{ei}^{\mathrm{T}} \boldsymbol{\Gamma}_e - \frac{1}{2} \boldsymbol{S}_e \boldsymbol{C}_{ei} & \boldsymbol{R}_e - \frac{1}{2} (\boldsymbol{S}_e \boldsymbol{D}_{ei} + \boldsymbol{D}_{ei}^{\mathrm{T}} \boldsymbol{S}_e^{\mathrm{T}}) & \boldsymbol{D}_{ei}^{\mathrm{T}} \\ \boldsymbol{C}_{ei} & \boldsymbol{D}_{ei} & -\boldsymbol{P}_e^{-1} \end{bmatrix} \leq \boldsymbol{0} \tag{15}$$

$$\boldsymbol{\Gamma}_e > \boldsymbol{0}, \quad \boldsymbol{Q}_e \geq \boldsymbol{0} \tag{16}$$

$$\begin{bmatrix} \boldsymbol{P}_e & -\frac{1}{2} \boldsymbol{S}_e^{\mathrm{T}} \\ -\frac{1}{2} \boldsymbol{S}_e & \boldsymbol{R}_e \end{bmatrix} \geq \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\mu^2 \boldsymbol{I} \end{bmatrix} \tag{17}$$

for all $i = 1, 2, \ldots, l$, then not only the H_∞ performance specification (14), but also the bounded stability of Σ_e is ensured, since from lemma 2

$$\|\boldsymbol{e}(t)\|_T \leq \mu \|\boldsymbol{u}_e(t)\|_T + \gamma_2 |\boldsymbol{x}_e(0)| \tag{18}$$

for some $\gamma_2 > 0$.

The desired property DP-3 is useful for shaping the filter characteristics [13]. The corresponding conditions [5, 31] are described below, which can be applied individually or together.

- PC-1: Existence of a matrix $\boldsymbol{\Gamma}_f > \boldsymbol{0}$ such that

$$(\boldsymbol{A}_f - c_0 \boldsymbol{I})^{\mathrm{T}} \boldsymbol{\Gamma}_f (\boldsymbol{A}_f - c_0 \boldsymbol{I}) - r_0^2 \boldsymbol{\Gamma}_f < \boldsymbol{0} \tag{19}$$

- PC-2: Existence of a matrix $\boldsymbol{\Gamma}_f > \boldsymbol{0}$ such that

$$\begin{aligned} (\boldsymbol{A}_f - r_2 \boldsymbol{I})^{\mathrm{T}} \boldsymbol{\Gamma}_f + \boldsymbol{\Gamma}_f (\boldsymbol{A}_f - r_2 \boldsymbol{I}) < \boldsymbol{0} \\ (r_1 \boldsymbol{I} - \boldsymbol{A}_f)^{\mathrm{T}} \boldsymbol{\Gamma}_f + \boldsymbol{\Gamma}_f (r_1 \boldsymbol{I} - \boldsymbol{A}_f) < \boldsymbol{0} \end{aligned} \tag{20}$$

- PC-3: Existence of a matrix $\boldsymbol{\Gamma}_f > \boldsymbol{0}$ such that

$$\begin{bmatrix} \sin\frac{\theta_l}{2}[(\boldsymbol{A}_f - c_l \boldsymbol{I})^{\mathrm{T}} \boldsymbol{\Gamma}_f + \boldsymbol{\Gamma}_f (\boldsymbol{A}_f - c_l \boldsymbol{I})] & \cos\frac{\theta_l}{2}[(\boldsymbol{A}_f - c_l \boldsymbol{I})^{\mathrm{T}} \boldsymbol{\Gamma}_f - \boldsymbol{\Gamma}_f (\boldsymbol{A}_f - c_l \boldsymbol{I})] \\ \cos\frac{\theta_l}{2}[\boldsymbol{\Gamma}_f (\boldsymbol{A}_f - c_l \boldsymbol{I}) - (\boldsymbol{A}_f - c_l \boldsymbol{I})^{\mathrm{T}} \boldsymbol{\Gamma}_f] & \sin\frac{\theta_l}{2}[(\boldsymbol{A}_f - c_l \boldsymbol{I})^{\mathrm{T}} \boldsymbol{\Gamma}_f + \boldsymbol{\Gamma}_f (\boldsymbol{A}_f - c_l \boldsymbol{I})] \end{bmatrix} < \boldsymbol{0} \tag{21}$$

- PC-4: Existence of a matrix $\boldsymbol{\Gamma}_f > \boldsymbol{0}$ such that

$$\begin{bmatrix} \sin\frac{\theta_r}{2}[(c_r \boldsymbol{I} - \boldsymbol{A}_f)^{\mathrm{T}} \boldsymbol{\Gamma}_f + \boldsymbol{\Gamma}_f (c_r \boldsymbol{I} - \boldsymbol{A}_f)] & \cos\frac{\theta_r}{2}[(c_r \boldsymbol{I} - \boldsymbol{A}_f)^{\mathrm{T}} \boldsymbol{\Gamma}_f - \boldsymbol{\Gamma}_f (c_r \boldsymbol{I} - \boldsymbol{A}_f)] \\ \cos\frac{\theta_r}{2}[\boldsymbol{\Gamma}_f (c_r \boldsymbol{I} - \boldsymbol{A}_f) - (c_r \boldsymbol{I} - \boldsymbol{A}_f)^{\mathrm{T}} \boldsymbol{\Gamma}_f] & \sin\frac{\theta_r}{2}[(c_r \boldsymbol{I} - \boldsymbol{A}_f)^{\mathrm{T}} \boldsymbol{\Gamma}_f + \boldsymbol{\Gamma}_f (c_r \boldsymbol{I} - \boldsymbol{A}_f)] \end{bmatrix} < \boldsymbol{0} \tag{22}$$

It is noted that there are other regions and conditions for the pole placement constraint, such as those presented in [5, 32]. Basically it is also possible to accommodate these conditions in this paper, but the details are omitted for the sake of brevity.

3 Main results

3.1 Robust H_∞ filter design

To start developing the filter design method, the π-stability conditions for the filtering error dynamics Σ_e are utilised in the following theorem.

Theorem 1: If there exist feasible solutions μ^2, $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$, $\boldsymbol{P}_e^{-1}$, $\boldsymbol{R}_e$, $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{M}$, $\boldsymbol{N}$, $\boldsymbol{Z}$, and $\boldsymbol{D}_f$ to the following matrix inequalities

$$\begin{bmatrix} \boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{\Phi}+\boldsymbol{\Phi}\boldsymbol{A}_i+\boldsymbol{\Psi} & (\boldsymbol{X}\boldsymbol{A}_i+\boldsymbol{Z}\boldsymbol{C}_i+\boldsymbol{M})^{\mathrm{T}}+\boldsymbol{\Phi}\boldsymbol{A}_i+\boldsymbol{\Psi} & \boldsymbol{\Phi}\boldsymbol{B}_i-\frac{1}{2}(\boldsymbol{L}-\boldsymbol{D}_f\boldsymbol{C}_i-\boldsymbol{N})^{\mathrm{T}}\boldsymbol{S}_e^{\mathrm{T}} & (\boldsymbol{L}-\boldsymbol{D}_f\boldsymbol{C}-\boldsymbol{N})^{\mathrm{T}} \\ \boldsymbol{X}\boldsymbol{A}_i+\boldsymbol{Z}\boldsymbol{C}_i+\boldsymbol{M}+\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{\Phi}+\boldsymbol{\Psi} & \boldsymbol{X}\boldsymbol{A}_i+\boldsymbol{Z}\boldsymbol{C}_i+(\boldsymbol{X}\boldsymbol{A}_i+\boldsymbol{Z}\boldsymbol{C}_i)^{\mathrm{T}}+\boldsymbol{Y} & \boldsymbol{X}\boldsymbol{B}_i+\boldsymbol{Z}\boldsymbol{D}_i-\frac{1}{2}(\boldsymbol{L}-\boldsymbol{D}_f\boldsymbol{C})^{\mathrm{T}}\boldsymbol{S}_e^{\mathrm{T}} & (\boldsymbol{L}-\boldsymbol{D}_f\boldsymbol{C}_i)^{\mathrm{T}} \\ \boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{\Phi}-\frac{1}{2}\boldsymbol{S}_e(\boldsymbol{L}-\boldsymbol{D}_f\boldsymbol{C}_i-\boldsymbol{N}) & (\boldsymbol{X}\boldsymbol{B}_i+\boldsymbol{Z}\boldsymbol{D}_i)^{\mathrm{T}}-\frac{1}{2}\boldsymbol{S}_e(\boldsymbol{L}-\boldsymbol{D}_f\boldsymbol{C}_i) & \boldsymbol{R}_e-\frac{1}{2}[\boldsymbol{S}_e(\boldsymbol{J}-\boldsymbol{D}_f\boldsymbol{D}_i)+(\boldsymbol{J}-\boldsymbol{D}_f\boldsymbol{D}_i)^{\mathrm{T}}\boldsymbol{S}_e^{\mathrm{T}}] & (\boldsymbol{J}-\boldsymbol{D}_f\boldsymbol{D}_i)^{\mathrm{T}} \\ \boldsymbol{L}-\boldsymbol{D}_f\boldsymbol{C}_i-\boldsymbol{N} & \boldsymbol{L}-\boldsymbol{D}_f\boldsymbol{C}_i & \boldsymbol{J}-\boldsymbol{D}_f\boldsymbol{D}_i & -\boldsymbol{P}_e^{-1} \end{bmatrix} \le \mathbf{0} \tag{23}$$

$$\begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{\Phi} \\ \boldsymbol{\Phi} & \boldsymbol{X} \end{bmatrix} > \mathbf{0}, \quad \boldsymbol{Y} \ge \boldsymbol{\Psi} \ge \mathbf{0}, \quad \mu^2 > 0 \tag{24}$$

$$\begin{bmatrix} -\boldsymbol{P}_e^{-1} & \frac{1}{2}\boldsymbol{P}_e^{-1}\boldsymbol{S}_e^{\mathrm{T}} & \boldsymbol{P}_e^{-1} \\ \frac{1}{2}\boldsymbol{S}_e\boldsymbol{P}_e^{-1} & -\mu^2\boldsymbol{I}-\boldsymbol{R}_e & \mathbf{0} \\ \boldsymbol{P}_e^{-1} & \mathbf{0} & -\boldsymbol{I} \end{bmatrix} \le \mathbf{0} \tag{25}$$

for all $i = 1, \ldots, l$, then the filter Σ_{F} with the gain matrices

$$\begin{aligned} \boldsymbol{A}_f &= -\boldsymbol{U}^{-1}\boldsymbol{M}\boldsymbol{U}^{-\mathrm{T}}, \quad \boldsymbol{C}_f = -\boldsymbol{N}\boldsymbol{U}^{-\mathrm{T}}, \\ \boldsymbol{B}_f &= \boldsymbol{U}^{-1}\boldsymbol{Z}, \quad \boldsymbol{D}_f = \boldsymbol{D}_f \end{aligned} \tag{26}$$

satisfies the desired properties DP-1 and DP-2 of the filtering problem, where $\boldsymbol{U}$ is any non-singular matrix satisfying $\boldsymbol{U}\boldsymbol{U}^{\mathrm{T}} = \boldsymbol{X} - \boldsymbol{\Phi}$.

Proof: By the Schur complement [1] and the first inequality in (24), $\boldsymbol{\Phi} > \mathbf{0}$ and $\boldsymbol{X} - \boldsymbol{\Phi} > \mathbf{0}$. Thus, $\boldsymbol{I} - \boldsymbol{X}\boldsymbol{\Phi}^{-1}$ is non-singular and there exist non-singular matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ such that $\boldsymbol{I} - \boldsymbol{X}\boldsymbol{\Phi}^{-1} = \boldsymbol{U}\boldsymbol{V}^{\mathrm{T}}$. Let

$$\hat{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{\Phi}^{-1} & \boldsymbol{I} \\ \boldsymbol{V}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \text{ and } \breve{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{X} \\ \mathbf{0} & \boldsymbol{U}^{\mathrm{T}} \end{bmatrix} \tag{27}$$

where $\hat{\boldsymbol{T}}$ is non-singular since

$$\hat{\boldsymbol{T}}^{-1} = \begin{bmatrix} \mathbf{0} & \boldsymbol{V}^{-\mathrm{T}} \\ \boldsymbol{I} & -\boldsymbol{\Phi}^{-1}\boldsymbol{V}^{-\mathrm{T}} \end{bmatrix}$$

Define

$$\boldsymbol{\Gamma}_e = \breve{\boldsymbol{T}}\hat{\boldsymbol{T}}^{-1} = \begin{bmatrix} \boldsymbol{\Gamma}_{e1} & \boldsymbol{\Gamma}_{e0} \\ \boldsymbol{\Gamma}_{e0}^{\mathrm{T}} & \boldsymbol{\Gamma}_{e2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{U} \\ \boldsymbol{U}^{\mathrm{T}} & \boldsymbol{I} \end{bmatrix}$$

by letting $\boldsymbol{U} = -\boldsymbol{\Phi}\boldsymbol{V}$. Under this arrangement $\boldsymbol{\Gamma}_e > \mathbf{0}$ because $\boldsymbol{X} - \boldsymbol{U}\boldsymbol{U}^{\mathrm{T}} = \boldsymbol{X} + \boldsymbol{U}\boldsymbol{V}^{\mathrm{T}}\boldsymbol{\Phi} = \boldsymbol{\Phi} > \mathbf{0}$.

Next, pre- and post-multiply (23) by $diag(\boldsymbol{\Phi}^{-1}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I})$ at the same time. Then with (13), (26), (27), $\boldsymbol{U} = -\boldsymbol{\Phi}\boldsymbol{V}$, $\boldsymbol{\Gamma}_e = \breve{\boldsymbol{T}}\hat{\boldsymbol{T}}^{-1}$, and

$$\boldsymbol{Q}_e = \begin{bmatrix} \boldsymbol{Y} & (\boldsymbol{Y}-\boldsymbol{\Psi})\boldsymbol{U}^{-\mathrm{T}} \\ \boldsymbol{U}^{-1}(\boldsymbol{Y}-\boldsymbol{\Psi}) & \boldsymbol{U}^{-1}(\boldsymbol{Y}-\boldsymbol{\Psi})\boldsymbol{U}^{-\mathrm{T}} \end{bmatrix} \tag{28}$$

the result can be re-written as

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \hat{\boldsymbol{T}}^{\mathrm{T}}\boldsymbol{C}_{ei}^{\mathrm{T}} \\ \Upsilon_{12}^{\mathrm{T}} & \Upsilon_{22} & \boldsymbol{D}_{ei}^{\mathrm{T}} \\ \boldsymbol{C}_{ei}\hat{\boldsymbol{T}} & \boldsymbol{D}_{ei} & -\boldsymbol{P}_e^{-1} \end{bmatrix} \le \mathbf{0} \tag{29}$$

where

$$\begin{aligned} \Upsilon_{11} &= \hat{\boldsymbol{T}}^{\mathrm{T}}\boldsymbol{A}_{ei}^{\mathrm{T}}\boldsymbol{\Gamma}_e\hat{\boldsymbol{T}} + \hat{\boldsymbol{T}}^{\mathrm{T}}\boldsymbol{\Gamma}_e\boldsymbol{A}_{ei}\hat{\boldsymbol{T}} + \hat{\boldsymbol{T}}^{\mathrm{T}}\boldsymbol{Q}_e\hat{\boldsymbol{T}} \\ \Upsilon_{12} &= \hat{\boldsymbol{T}}^{\mathrm{T}}\boldsymbol{\Gamma}_e\boldsymbol{B}_{ei} - \frac{1}{2}\hat{\boldsymbol{T}}^{\mathrm{T}}\boldsymbol{C}_{ei}^{\mathrm{T}}\boldsymbol{S}_e^{\mathrm{T}} \\ \Upsilon_{22} &= \boldsymbol{R}_e - \frac{1}{2}(\boldsymbol{S}_e\boldsymbol{D}_{ei} + \boldsymbol{D}_{ei}^{\mathrm{T}}\boldsymbol{S}_e^{\mathrm{T}}) \end{aligned}$$

Note that $\boldsymbol{Q}_e \ge \mathbf{0}$ by (24). Now, pre- and post-multiplying (29), respectively, by $diag(\hat{\boldsymbol{T}}^{-\mathrm{T}}, \boldsymbol{I}, \boldsymbol{I})$ and $diag(\hat{\boldsymbol{T}}^{-1}, \boldsymbol{I}, \boldsymbol{I})$ results in (15). Therefore from lemma 2 the error dynamics Σ_e in (11) is π-stable with respect to the π-coefficients $\{\boldsymbol{S}_e, \boldsymbol{\Gamma}_e, \boldsymbol{Q}_e, \boldsymbol{P}_e, \boldsymbol{R}_e\}$ found in the proof.

Finally, with $\mu^2 > 0$, (25) may be re-written as (17) using the Schur complement. Hence the H_∞ performance constraint in (14) is satisfied.

It is worth noting that, the result in [10] may be regarded as a special case of what is obtained here with $\{\boldsymbol{S}_e, \boldsymbol{\Gamma}_e, \boldsymbol{Q}_e, \boldsymbol{P}_e, \boldsymbol{R}_e\} = \{\mathbf{0}, \boldsymbol{\Gamma}_e, \mathbf{0}, \boldsymbol{I}, -\mu^2\boldsymbol{I}\}$.

Apparently, the matrix inequalities in theorem 1 are not linear, but bilinear with respect to the variables. However, closer examination reveals that these inequalities are linear with respect to μ^2 and $\mathcal{V} = \{\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{P}_e^{-1}, \boldsymbol{R}_e, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{M}, \boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{D}_f\}$ if $\boldsymbol{S}_e$ is held constant. Also, these inequalities are linear with respect to μ^2 and $\boldsymbol{S}_e$ if $\mathcal{V}$ is held constant. Thus a coordinate-by-coordinate minimisation [33] procedure is proposed in the following for finding a filter Σ_{F} that satisfies the desired properties DP-1 and DP-2, and gives the minimal value of μ. Note that convergence of the procedure is guaranteed at least to a local optimum, because after each convex optimisation step μ is monotonically non-increasing.

Procedure A

Step 0. Initialise $\boldsymbol{S}_e$ with a pre-selected matrix.

Step 1. Fix $\boldsymbol{S}_e$ to the value obtained in the previous step, and solve the convex optimisation problem

$$\min_{\mu^2, \mathcal{V}} \mu^2 \quad \text{subject to (23)–(25)}$$

Step 2. Fix $\mathcal{V}$ to the set obtained in the previous step, and solve the convex optimisation problem

$$\min_{\mu^2, \boldsymbol{S}_e} \mu^2 \quad \text{subject to (23), (25), and } \mu^2 > 0$$

Step 3. Repeat steps 1 and 2 until μ converges to a local optimum. At convergence, compute the filter gain matrices.

3.2 Regional pole-placement constraints

To deal with the third desired property DP-3, i.e. the regional pole-placement constraints, it is shown first that another set of filter gain matrices

$$\begin{aligned} \boldsymbol{A}_f &= (\boldsymbol{\Phi} - \boldsymbol{X})^{-1}\boldsymbol{M}, \quad \boldsymbol{C}_f = \boldsymbol{N}, \\ \boldsymbol{B}_f &= (\boldsymbol{\Phi} - \boldsymbol{X})^{-1}\boldsymbol{Z}, \quad \boldsymbol{D}_f = \boldsymbol{D}_f \end{aligned} \tag{30}$$

may also be used. Consider the transfer function matrix $\boldsymbol{G}_f(s)$ of the filter Σ_F with the gain matrices (26) and recall the relationship $\boldsymbol{U}\boldsymbol{U}^T = \boldsymbol{X} - \boldsymbol{\Phi}$. The following derivation

$$\begin{aligned}\boldsymbol{G}_f(s) &= \boldsymbol{C}_f(s\boldsymbol{I} - \boldsymbol{A}_f)^{-1}\boldsymbol{B}_f + \boldsymbol{D}_f \\ &= -\boldsymbol{N}\boldsymbol{U}^{-T}[s\boldsymbol{I} - (-\boldsymbol{U}^{-1}\boldsymbol{M}\boldsymbol{U}^{-T})]^{-1}\boldsymbol{U}^{-1}\boldsymbol{Z} + \boldsymbol{D}_f \\ &= \boldsymbol{N}[\boldsymbol{U}(s\boldsymbol{I} - (-\boldsymbol{U}^{-1}\boldsymbol{M}\boldsymbol{U}^{-T}))(-\boldsymbol{U}^T)]^{-1}\boldsymbol{Z} + \boldsymbol{D}_f \\ &= \boldsymbol{N}[(-\boldsymbol{U}\boldsymbol{U}^T)\cdot s\boldsymbol{I} - \boldsymbol{M}]^{-1}\boldsymbol{Z} + \boldsymbol{D}_f \\ &= \boldsymbol{N}[s\boldsymbol{I} - (\boldsymbol{\Phi} - \boldsymbol{X})^{-1}\boldsymbol{M}]^{-1}(\boldsymbol{\Phi} - \boldsymbol{X})^{-1}\boldsymbol{Z} + \boldsymbol{D}_f\end{aligned} \tag{31}$$

shows the equivalence of Σ_F with (30) to Σ_F with (26) in the sense that the filter state vector is transformed from $\boldsymbol{x}_f$ to $\boldsymbol{U}^{-T}\boldsymbol{x}_f$. Hence the input-output characteristic and stability of the filter are not affected.

With the new formulas (31) for the filter gain matrices, and with the choice $\boldsymbol{\Gamma}_f = \boldsymbol{X} - \boldsymbol{\Phi}$ in (19)–(22), the conditions for regional pole-placement constraints may be easily converted to LMIs with respect to the variables $\boldsymbol{\Phi}$, $\boldsymbol{X}$, and $\boldsymbol{M}$:

- PC-1: Existence of $\boldsymbol{\Phi}$, $\boldsymbol{X}$, and $\boldsymbol{M}$ such that

$$\begin{bmatrix} r_0^2(\boldsymbol{\Phi} - \boldsymbol{X}) & c_0(\boldsymbol{\Phi} - \boldsymbol{X}) - \boldsymbol{M}^T \\ c_0(\boldsymbol{\Phi} - \boldsymbol{X}) - \boldsymbol{M} & \boldsymbol{\Phi} - \boldsymbol{X} \end{bmatrix} < \boldsymbol{0} \tag{32}$$

- PC-2: Existence of $\boldsymbol{\Phi}$, $\boldsymbol{X}$, and $\boldsymbol{M}$ such that

$$\begin{aligned} -\boldsymbol{M} - \boldsymbol{M}^T + 2r_2\boldsymbol{\Phi} - 2r_2\boldsymbol{X} < \boldsymbol{0} \\ \boldsymbol{M} + \boldsymbol{M}^T - 2r_1\boldsymbol{\Phi} + 2r_1\boldsymbol{X} < \boldsymbol{0} \end{aligned} \tag{33}$$

- PC-3: Existence of $\boldsymbol{\Phi}$, $\boldsymbol{X}$, and $\boldsymbol{M}$ such that

$$\begin{bmatrix} \boldsymbol{F}(\theta_l, c_l) & \cos\frac{\theta_l}{2}(\boldsymbol{M} - \boldsymbol{M}^T) \\ \cos\frac{\theta_l}{2}(\boldsymbol{M}^T - \boldsymbol{M}) & \boldsymbol{F}(\theta_l, c_l) \end{bmatrix} < \boldsymbol{0} \tag{34}$$

where

$$\boldsymbol{F}(\theta, c) = \sin\frac{\theta}{2}[2c(\boldsymbol{\Phi} - \boldsymbol{X}) - \boldsymbol{M} - \boldsymbol{M}^T]$$

- PC-4: Existence of $\boldsymbol{\Phi}$, $\boldsymbol{X}$, and $\boldsymbol{M}$ such that

$$\begin{bmatrix} -\boldsymbol{F}(\theta_r, c_r) & \cos\frac{\theta_r}{2}(\boldsymbol{M}^T - \boldsymbol{M}) \\ \cos\frac{\theta_r}{2}(\boldsymbol{M} - \boldsymbol{M}^T) & -\boldsymbol{F}(\theta_r, c_r) \end{bmatrix} < \boldsymbol{0} \tag{35}$$

The constraints in (32)–(35) are much simpler than the ones developed for the entire filtering error dynamics [9, 10], and can be augmented into procedure A to produce a procedure in the following for finding a filter Σ_F satisfying all three desired properties, and with the minimal value of μ.

Procedure B

Step 0. Initialise $\boldsymbol{S}_e$ with a pre-selected matrix.

Step 1. Fix $\boldsymbol{S}_e$ to the value obtained in the previous step, and solve the convex optimisation problem

$$\min_{\mu^2, \mathcal{V}} \mu^2 \quad \text{subject to (23)–(25) and any combination of (32)–(35)}$$

Step 2. Fix $\mathcal{V}$ to the set obtained in the previous step, and solve the convex optimisation problem

$$\min_{\mu^2, \boldsymbol{S}_e} \mu^2 \quad \text{subject to (23), (25) and } \mu^2 > 0$$

Step 3. Repeat steps 1 and 2 until μ converges to a local optimum. At convergence, compute the filter gain matrices.

4 A numerical example

In this Section, an example is worked out to illustrate the proposed filter design procedures. Suppose that the system shown in Fig. 1 has the signal model Σ_S with the following system matrices

$$\boldsymbol{A}_s = \begin{bmatrix} -1.7270 & -0.3405 \\ 1 & 0 \end{bmatrix}, \ \boldsymbol{B}_s = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\boldsymbol{C}_s = [0 \quad 1], \ \boldsymbol{D}_s = 0$$

and the channel model Σ_C with the following system matrices

$$\boldsymbol{A}_c = \begin{bmatrix} -2.7731 + 0.3\alpha_2 & -2.1222 & -1.2302 - 0.23\alpha_1 \\ 1 & -0.4\alpha_2 & 0 \\ -0.35\alpha_2 & 1 & -0.5\alpha_1 \end{bmatrix}$$

$$\boldsymbol{B}_c^T = [1 + \alpha_1 \quad \alpha_2 \quad 0], \ \boldsymbol{C}_c = [0 \quad 1 \quad \alpha_1], \ \boldsymbol{D}_c = 1 + 2\alpha_2$$

Assume the uncertain parameters $\alpha_1 \in [0, 0.5]$ and $\alpha_2 \in [-0.4, 0.5]$. An optimal H_∞ filter Σ_F is designed by using procedure A implemented with the LMI Toolbox [31]. The initial value of $\boldsymbol{S}_e^T$ is set to [0 0], and the corresponding optimal μ^2 is 8.0664 after step 1 is executed once. Then the procedure is executed until convergence, and the final optimal μ^2 obtained is 8.0653, corresponding to $\boldsymbol{S}_e^T = [0.0010 \ \ -0.0012]$. These results are put in the first three rows under the column title 'Procedure A' in Table 1, which show the advantage of having the extra matrix variable $\boldsymbol{S}_e^T$.

Next, by applying procedure B, four regional pole-placement constraints, PC-1 with $\mathcal{P}_D(-2, 0.5)$, PC-2 with $\mathcal{P}_V(-2, -0.5)$, PC-3 with $\mathcal{P}_L(1.5, \pi/6)$, and PC-4 with $\mathcal{P}_R(-1.5, \pi/6)$ are separately considered. As in the above, the initial value of $\boldsymbol{S}_e^T$ is set to [0 0], and step 1 is executed once first. Then the procedure is carried out until it converges. The results are put in the first three rows of Table 1 under the column title 'Procedure B'. Corresponding to these constraints, the poles of the filters and the system, including the signal and channel models at

Table 1: Filter design results from the example

	Procedure A	Procedure B			
		$\mathcal{P}_D$	$\mathcal{P}_V$	$\mathcal{P}_L$	$\mathcal{P}_R$
μ^2	8.0664	8.3682	8.0923	8.0896	8.1446
μ^2	8.0653	8.3660	8.0914	8.0886	8.1425
$\boldsymbol{S}_e$	$\begin{bmatrix} 0.0010 \\ -0.0012 \end{bmatrix}$	$\begin{bmatrix} 0.0032 \\ -0.0003 \end{bmatrix}$	$\begin{bmatrix} 0.0007 \\ -0.0013 \end{bmatrix}$	$\begin{bmatrix} 0.0022 \\ -0.0011 \end{bmatrix}$	$\begin{bmatrix} 0.0022 \\ -0.0020 \end{bmatrix}$
μ^2	8.0686	8.3744	8.0934	8.0956	8.1463

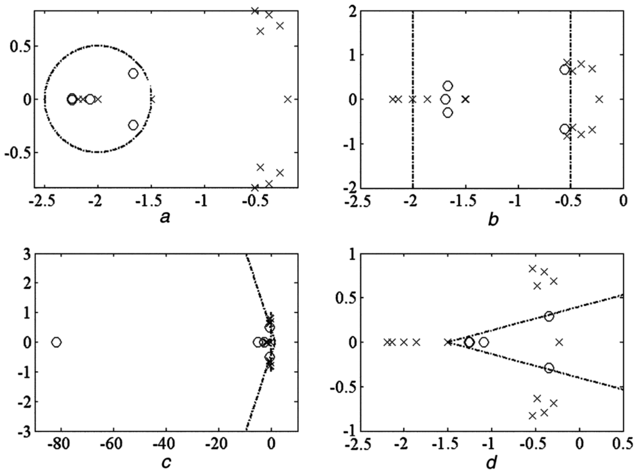

Fig. 2 *Poles of the robust filter, marked by ○, and poles of the system with signal and channel models at the vertices of $\mathcal{D}_c$, marked by ×, obtained from procedure B with respect to four regional pole-placement constraints*

the vertices of $\mathcal{D}_c$, are shown in Fig. 2. Clearly, the poles of the filters indeed lie inside the desired regions.

Finaly, the proposed method is applied with a special constraint, i.e., with the π-coefficients $\{\boldsymbol{S}_e, \boldsymbol{\Gamma}_e, \boldsymbol{Q}_e, \boldsymbol{P}_e, \boldsymbol{R}_e\}$ set to $\{\boldsymbol{0}, \boldsymbol{\Gamma}_e, \boldsymbol{0}, \boldsymbol{I}, -\mu^2\boldsymbol{I}\}$ in (23)–(25), where $\boldsymbol{Q}_e = \boldsymbol{0}$ is accomplished by setting $\boldsymbol{Y} = \boldsymbol{\Psi} = \boldsymbol{0}$. This simulates the solution of the current filtering problem by applying the bounded real lemma [27] originating from the concept of positive realness. The last row in Table 1 shows the more conservative results.

5 Conclusion

The robust H_∞ filtering problem for linear signal models and uncertain channels are solved by using the extended π-sharing theory. Regional pole-placement constraints on the filter, rather than the entire filtering error dynamics, are also considered and formulated as LMIs that can be augmented to the robust H_∞ filter design procedure. An example is worked out to illustrate the effectiveness of the proposed method, in addition to the improvement in reducing the conservativeness from the use of the bounded real lemma.

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