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Short communication

A novel scaling scheme for fast Hartley transform

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Abstract. This paper presents some results on fixed-point error analysis of the fast Hartley transform algorithms. A novel scheme for preventing overflow is considered in the analysis. It is proved that error performance, defined based on the signal-to-noise ratio, can be improved for both the decimation-in-frequency and the decimation-in-time fast Hartley transform algorithms.

Zusammenfassung. Dieser Artikel stellt einige Ergebnisse zur Festkommafehleranalyse der schnellen Hartley Transformation vor. In der Analyse wird eine neue Methode betrachtet, um Überlauf zu verhindern. Es wird bewiesen daß der Fehlereinfluß, basierend auf der Definition des Signal zu Rauschverhältnisses, sowohl für frequenzdezimierende als auch für zeitdezimierende schnelle Algorithmen für die Hartley Transformation vermindert werden kann.

Résumé. Cet article présente quelques résultats sur l'analyse de l'erreur en précision finie des algorithmes de transformation de Hartley rapides. Nous considérons dans cette analyse une méthode nouvelle de prévention du dépassement. Nous prouvons que les performances d'erreur, définies par le rapport signal sur bruit, peuvent être améliorées à la fois pour les algorithmes de transformation de Hartley rapides basés sur la décimation en fréquence et pour ceux basés sur la décimation en temps.

Keywords. Discrete Hartley transform; fixed-point arithmetic; round-off error; error analysis.

1. Introduction

The discrete Hartley transform (DHT), defined by Bracewell [2], is an important tool for the processing of discrete signals. A finite N-point sequence $x[n]$ and its corresponding DHT $H[k]$ are related by

$$H[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]\operatorname{cas}\frac{2\pi nk}{N},\quad 0\leqslant k\leqslant N-1, \tag{1}$$

and

$$x[n]=\sum_{k=0}^{N-1}H[k]\operatorname{cas}\frac{2\pi nk}{N},\quad 0\leqslant n\leqslant N-1, \tag{2}$$

where

$$\operatorname{cas}\alpha=\cos\alpha+\sin\alpha.$$

The transform kernel of DHT is similar to that of DFT and, consequently, the DHT can be used to compute the DFT and vice versa. Some previous works [1, 4, 6] had proved that a variety of algorithms that traditionally utilized the DFT could be carried out with the DHT effectively.

Similar to the fast Fourier transform (FFT), the fast algorithms for computing DHT, the so-called fast Hartley transform (FHT), have been developed by many

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authors [3, 8]. In many practical situations, the FHT needs to be implemented using the fixed-point arithmetics. In this case the effect of the word length on the accuracy of the computation is of importance both with regard to the design of special-purpose machines and to the accuracy attainable from existing machines. An error analysis of the fixed-point DHT was considered by Prabhu and Narayanan [7] recently. They analyzed fixed-point errors for radix-2 decimation-in-frequency (DIF) and decimation-in-time (DIT) FHT algorithms, with a power of 4 input length; furthermore, a scaling scheme was also developed for preventing overflow in the computation.

In this paper, a novel step-by-step scaling scheme is proposed for computing the FHT with more general input lengths. Following the model given in [7], the averaged output signal-to-noise ratio (SNR) is derived for both the cases of DIT and DIF FHTs. The analyzed results show that better average output SNR can be obtained by using the newly proposed scaling scheme.

2. The novel scaling scheme for FHT

For fixed-point arithmetics, it is necessary to assure that the input data of a discrete transform is sufficiently small so that the numerical overflow is avoided. This can be accomplished by proper scaling of the inputs such that the values of the transform outputs are less than one under the conditions that the values of the input are also less than one. The step-by-step scaling scheme given in [9] can be used to prevent overflow in computing FHT. The FHT gain of the input magnitudes is bounded, from (1), by

$$|H[k]|_{\max} \leqslant |x[n]|_{\max}\left[\sum_{n=0}^{N-1} \operatorname{cas}\frac{2\pi nk}{N}\right] \leqslant N|x[n]|_{\max}. \tag{3}$$

Thus, the overall increase of the magnitudes, due to FHT, will not exceed N. From [7], the maximum increase in each stage of the DIF or DIT FHT is $2\sqrt{2}$. Therefore, the conventional scaling scheme for FFT [10] which divides the signals by 2 in each stage cannot guarantee the numerical stability of the fixed-point FHT. As a solution to this problem the scaling factor, chosen to scale down the magnitude of the intermediate results, must be greater than $2\sqrt{2}$ and easy to be performed both in software and in hardware. A simple scaling scheme used by Prabhu and Narayanan is to scale down the intermediate stage by 4 in each of the first $\frac{1}{2}\log_2 N$ stages in DIF FHT and by a factor of 2 in the first two stages and a factor of 4 from stage 3 to stage $\frac{1}{2}\log_2 N+1$ DIT FHT, under the assumption that the transform length is a power of 4. In the following a novel scaling scheme is proposed and it will be proved that better SNR can be obtained by using the new approach. For convenience, it is assumed that the transform length is an integer power of two, i.e., $N=2^m$.

2.1. Scaling scheme for DIF FHT

Because the magnitudes of the signal could increase by a value of 8 in every two successive stages, the intermediate results are scaled down by a factor of 4 in the first stage and then by a factor of 2 in the second to prevent overflow. Such a scaling scheme is chosen because it can easily be implemented by right bit-shifts. Since the overall increase will not exceed N, such scalings are required only in the first $\frac{2}{3}\log_2 N$ stage. Three possible situations for different N are discussed in the following:

(i) $m=3k$. The scaling is performed by a factor of 4 and a factor of 2 in turns, and the process continues only for the first $\frac{2}{3}m$ stages. For notation convenience, this scaling process is denoted by $[(\frac{1}{4})(\frac{1}{2})]^k$.

(ii) $m=3k+1$. The scaling process of the first $[\frac{2}{3}m]$ stages is the same as that of (i). Additionally, one more scaling by a factor of 2 in the stage $2k+1$ is required and this process is denoted by $[(\frac{1}{4})(\frac{1}{2})]^k(\frac{1}{2})$.

(iii) $m=3k+2$. From stage 1 onwards till stage $2k$, the scaling process is the same as (i). But it requires one more scaling of $\frac{1}{4}$ in the $(2k+1)$-th stage, and is denoted by $[(\frac{1}{4})(\frac{1}{2})]^k(\frac{1}{4})$.

In the above three cases, the overall scaling of magnitude is equal to $1/N$, which would ensure that the magnitudes of the outputs for the last stage would be less than unity if the magnitudes of the inputs are in the

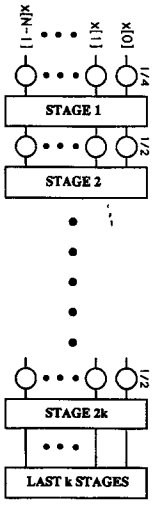

Fig. 1. Scaling scheme for the DIF algorithm.

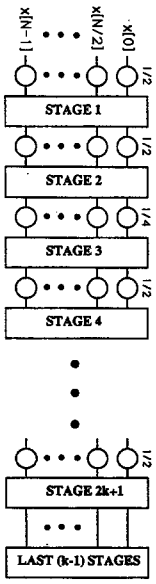

Fig. 2. Scaling scheme for the DIT algorithm.

interval $(-1, 1)$. The scaling scheme of case (i) is shown pictorially in Fig. 1.

2.2. Scaling scheme for DIT FHT

There are also three cases which are briefly described in the following by using the aforecited notations:

$$\begin{aligned} &\text{(i) } m=3k: && (\tfrac{1}{2})^2[(\tfrac{1}{4})(\tfrac{1}{2})]^{k-1}(\tfrac{1}{2}), \\ &\text{(ii) } m=3k+1: && (\tfrac{1}{2})^2[(\tfrac{1}{4})(\tfrac{1}{2})]^{k-1}(\tfrac{1}{4}), \\ &\text{(iii) } m=3k+2: && (\tfrac{1}{2})^2[(\tfrac{1}{4})(\tfrac{1}{2})]^{k}, \end{aligned} \tag{4}$$

where the heading factor $(\frac{1}{2})^2$ means that the scaling of a factor of 2 is performed in the first two stages of the DIT FHT. The DIT scaling scheme is illustrated in Fig. 2.

3. FHT error analysis with fixed-point arithmetic

Under the same conditions given in [7], the fixed-point error analysis of both DIF and DIT FHT algorithms are performed. Throughout the analysis, we assume fixed-point arithmetics with $(b+1)$-bit word length and sign magnitude representation of the binary numbers is used. It is assumed that all the N real random variables comprising the input sequence $x[n]$ are uncorrelated. Also, they are distributed uniformly in the interval $(-1, 1)$ with zero-mean and a variance of $\frac{1}{3}$. The input $x[n]$ and the multiplying coefficients are assumed to be represented with infinite precision. Finally, the averaged output signal variance is given by

$$\sigma_{\mathrm{H}}^2 = 1/3N. \tag{5}$$

In the fixed-point error analysis, two kinds of errors are involved which are round-off errors and truncation errors, respectively. It is easy to show [11] that the variance of the b-bit round-off error is

$$\sigma_r^2 = 2^{-2b}/12. \tag{6}$$

Scaling by a factor of $\frac{1}{2}$ involves shifting the input to the right by one bit and truncation of the last bit. It can be shown that the variance of this truncation error is given by

$$\sigma_{t1}^2 = 2^{-2b}/8. \tag{7}$$

Similarly, it also can be shown that the variance of the truncation error for scaling by $\frac{1}{4}$ is

$$\sigma_{t2}^2 = \frac{7}{32}\,2^{-2b}. \tag{8}$$

In the following, only the situation $N=2^{3k}$ for DIF

FHT is discussed in detail. The deviations, however, are applicable with minor modifications to other cases.

3.1. Fixed-point error analysis for DIF FHT

Since errors of each stage are uncorrelated, the total error variance can be computed as sums of all errors involving the output. The truncation errors occurring in odd stages are scaled down by a factor of 4 and the error variance σ_{t2}^2 is resulted. For the same reason, the truncation errors occurring in the even stages are scaled down by a factor of 2 and the error variance σ_{t1}^2 results. At any odd stage $2l-1$, $1 \leqslant l \leqslant k$, there are N truncation errors and each error propagates through $k-l$ odd stages and $k-l+1$ even stages and $N/4^{l-1}$ output points are involved. The error variance from an odd stage $2l-1$ in the first $2k$ stages is scaled $(k-l)$ times by the scaling factor $\frac{1}{4}$ and is scaled $(k-l+1)$ times by the scaling factor $\frac{1}{2}$. When the error is multiplied by $\frac{1}{4}$ and $\frac{1}{2}$, the variance gets reduced by $\frac{1}{16}$ and $\frac{1}{4}$, respectively. The overall truncation errors which are introduced in even stages can be computed following the same way and the total averaged truncation error variance of the output is given by

$$
\begin{aligned}
[\sigma_{\mathrm{T}}^2]_{\mathrm{av}} = & \frac{\sigma_{t2}^2}{N}\left[NN\left(\frac{1}{16}\right)^{k-1}\left(\frac{1}{4}\right)^{k} + N\frac{N}{4}\left(\frac{1}{16}\right)^{k-2}\left(\frac{1}{4}\right)^{k-1} + \cdots + N\frac{N}{4^{k-1}}\left(\frac{1}{16}\right)^{0}\left(\frac{1}{4}\right)^{1}\right] \\
& + \frac{\sigma_{t1}^2}{N}\left[N\frac{N}{2}\left(\frac{1}{16}\right)^{k-1}\left(\frac{1}{4}\right)^{k-1} + N\frac{N}{2}\frac{1}{4}\left(\frac{1}{16}\right)^{k-2}\left(\frac{1}{4}\right)^{k-2} + \cdots + N\frac{N}{2}\frac{1}{4^{k-1}}\left(\frac{1}{16}\right)^{0}\left(\frac{1}{4}\right)^{0}\right] \\
= & \frac{16}{15}(2\sigma_{t1}^2 + \sigma_{t2}^2)(2^k - 2^{-3k}). \qquad (9)
\end{aligned}
$$

At any stage l, $1 \leqslant l \leqslant 3k-2$, there are $N-2^{l+1}$ round errors and they will propagate to $N/2^l$ output points. There are no round-off errors in the $(3k-l)$-th stage. The round-off errors can be computed as the sums of the following three terms:

(1) round-off errors introduced in the first k odd stages.
(2) round-off errors introduced in the first k even stages.
(3) round-off errors introduced in the last $k-2$ stages.

The total averaged round-off error variance of the output points can be computed as

$$
\begin{aligned}
[\sigma_{\mathrm{R}}^2]_{\mathrm{av}} = & \frac{\sigma_r^2}{N}\left[(N-2^2)\frac{N}{2}\left(\frac{1}{16}\right)^{k-1}\left(\frac{1}{4}\right)^{k} + (N-2^4)\frac{N}{2^3}\left(\frac{1}{16}\right)^{k-2}\left(\frac{1}{14}\right)^{k-1} + \cdots \right. \\
& + \frac{N}{2^{2k-1}}(N-2^{2k})\left(\frac{1}{16}\right)^{0}\left(\frac{1}{4}\right)^{1} \\
& + (N-2^3)\frac{N}{2^2}\left(\frac{1}{16}\right)^{k-1}\left(\frac{1}{4}\right)^{k-1} \\
& + (N-2^5)\frac{N}{2^4}\left(\frac{1}{16}\right)^{k-2}\left(\frac{1}{4}\right)^{k-2} + \cdots \\
& + \frac{N}{2^{2k}}(N-2^{2k+1})\left(\frac{1}{16}\right)^{0}\left(\frac{1}{4}\right)^{0} \\
& + (N-2^{2k+2})\frac{N}{2^{2k+1}} \\
& + (N-2^{2k+3})\frac{N}{2^{2k+2}} + \cdots \\
& \left. + (N-2^{3k-1})\frac{N}{2^{3k-2}}\right] \\
= & \sigma_r^2\left[\frac{13}{5}2^k - 2k - \frac{160}{63} - \frac{8}{5}2^{-3k} + \frac{160}{63}2^{-6k}\right]. \qquad (10)
\end{aligned}
$$

The averaged variance of the output noise is given by

$$
\begin{aligned}
\sigma_{\mathrm{A}}^2 &= [\sigma_{\mathrm{T}}^2]_{\mathrm{av}} + [\sigma_{\mathrm{R}}^2]_{\mathrm{av}} \\
&= 2^{-2b}[\tfrac{43}{60}2^k - \tfrac{1}{6}k - \tfrac{40}{189} - \tfrac{19}{30}2^{-3k} + \tfrac{40}{189}2^{-6k}], \qquad (11)
\end{aligned}
$$

where the variances of σ_r^2, σ_{t1}^2 and σ_{t2}^2, defined in (6), (7) and (8), respectively, have been substituted. Since

the signal variance at the output is $1/(3N)$, the average signal-to-noise ratio at the output is

$$\left[\frac{\sigma_{\mathrm{H}}^2}{\sigma_{\mathrm{A}}^2}\right]_{\mathrm{av}} = \frac{2^{2b}}{N[\frac{43}{20}2^k - \frac{1}{2}k - \frac{40}{63} - \frac{19}{10}2^{-3k} + \frac{40}{63}2^{-6k}]}. \tag{12}$$

Following the above analysis, the averaged signal-to-noise ratio at the output for an $N=2^{3k+1}$ DIF FHT can be obtained as

$$\left[\frac{\sigma_{\mathrm{H}}^2}{\sigma_{\mathrm{A}}^2}\right]_{\mathrm{av}} = \frac{2^{2b}}{N[\frac{22}{10}2^k - \frac{1}{2}k - \frac{83}{126} - \frac{19}{20}2^{-3k} + \frac{10}{63}2^{-6k}]}, \tag{13}$$

and for the case of $N=2^{3k+2}$ the result is

$$\left[\frac{\sigma_{\mathrm{H}}^2}{\sigma_{\mathrm{A}}^2}\right]_{\mathrm{av}} = \frac{2^{2b}}{N[\frac{41}{10}2^k - \frac{1}{2}k - \frac{131}{126} - \frac{19}{40}2^{-3k} + \frac{5}{126}2^{-6k}]}. \tag{14}$$

3.2. Fixed-point error analysis for DIT FHT

Following the same derivation given in Section 3.1, the average round-off error variance of the output for an N-point DIT FHT with $N=2^{3k}$ can be obtained as

$$[\sigma_{\mathrm{R}}^2]_{\mathrm{av}} = \sigma_r^2\left[\frac{14}{5}2^k - 8\cdot 2^{-k} - 2 + \frac{56}{5}2^{-3k}\right], \tag{15}$$

and the corresponding average truncation error variance of the output is

$$[\sigma_{\mathrm{T}}^2]_{\mathrm{av}} = \sigma_{t1}^2[\tfrac{23}{15}2^k + \tfrac{52}{15}2^{-3k}] + \sigma_{t2}^2[\tfrac{4}{15}2^k - \tfrac{64}{15}2^{-3k}]. \tag{16}$$

And the average signal-to-noise ratio at the output of the DIT FHT algorithm is

$$\left[\frac{\sigma_{\mathrm{H}}^2}{\sigma_{\mathrm{A}}^2}\right]_{\mathrm{av}} = \frac{2^{2b}}{N[\frac{29}{30}2^k - \frac{1}{2} - 2\cdot 2^{-k} + \frac{13}{10}2^{-3k}]}. \tag{17}$$

Following the above analysis, the corresponding expressions for the averaged signal-to-noise ratio with $N=2^{3k+1}$ and $N=2^{3k+2}$ for DIT FHT can be shown to be

$$\left[\frac{\sigma_{\mathrm{H}}^2}{\sigma_{\mathrm{A}}^2}\right]_{\mathrm{av}} = \frac{2^{2b}}{N[\frac{13}{5}2^k - \frac{1}{2} - 2\cdot 2^{-k} + \frac{13}{20}2^{-3k}]}$$

$$\text{for } N=2^{3k+1} \tag{18}$$

and

$$\left[\frac{\sigma_{\mathrm{H}}^2}{\sigma_{\mathrm{A}}^2}\right]_{\mathrm{av}} = \frac{2^{2b}}{N[\frac{14}{5}2^k - \frac{1}{2} - \frac{3}{2}2^{-k} + \frac{13}{40}2^{-3k}]},$$

$$\text{for } N=2^{3k+2}. \tag{19}$$

4. Discussions and conclusions

The average signal-to-noise ratio expressions for the fixed-point FHTs derived by Prabhu and Narayanan are [7]

$$\left[\frac{\sigma_{\mathrm{H}}^2}{\sigma_{\mathrm{A}}^2}\right]_{\mathrm{av}} = \frac{2^{2b}}{N[\frac{57}{28}2^k - \frac{1}{2}k - \frac{8}{15} - \frac{25}{14}2^{-2k} + \frac{8}{15}2^{-4k}]}$$

$$\text{for DIF FHT} \tag{20}$$

and

$$\left[\frac{\sigma_{\mathrm{H}}^2}{\sigma_{\mathrm{A}}^2}\right]_{\mathrm{av}} = \frac{2^{2b}}{N[\frac{9}{7}2^k - \frac{4}{3} - \frac{19}{42}2^{-2k}]}$$

$$\text{for DIT FHT,} \tag{21}$$

where $N=2^{2k}$.

The average signal-to-noise ratio expressions for the fixed-point FHTs using the same scaling scheme for the transform length $N=2^{2k+1}$, are [5]

$$\left[\frac{\sigma_{\mathrm{H}}^2}{\sigma_{\mathrm{A}}^2}\right]_{\mathrm{av}} = \frac{2^{2b}}{N[\frac{15}{7}2^k - \frac{1}{2}k - \frac{19}{30} - \frac{25}{28}2^{-2k} + \frac{2}{15}2^{-4k}]}$$

$$\text{for DIF FHT} \tag{22}$$

and

$$\left[\frac{\sigma_{\mathrm{H}}^2}{\sigma_{\mathrm{A}}^2}\right]_{\mathrm{av}} = \frac{2^{2b}}{N[\frac{39}{28}2^k - \frac{7}{6} - \frac{19}{84}2^{-2k}]}$$

$$\text{for DIT FHT,} \tag{23}$$

where $N=2^{2k+1}$.

The two SNRs, defined in (12)–(14) and (20), (22) and defined in (17)–(19) and (21), (23) are

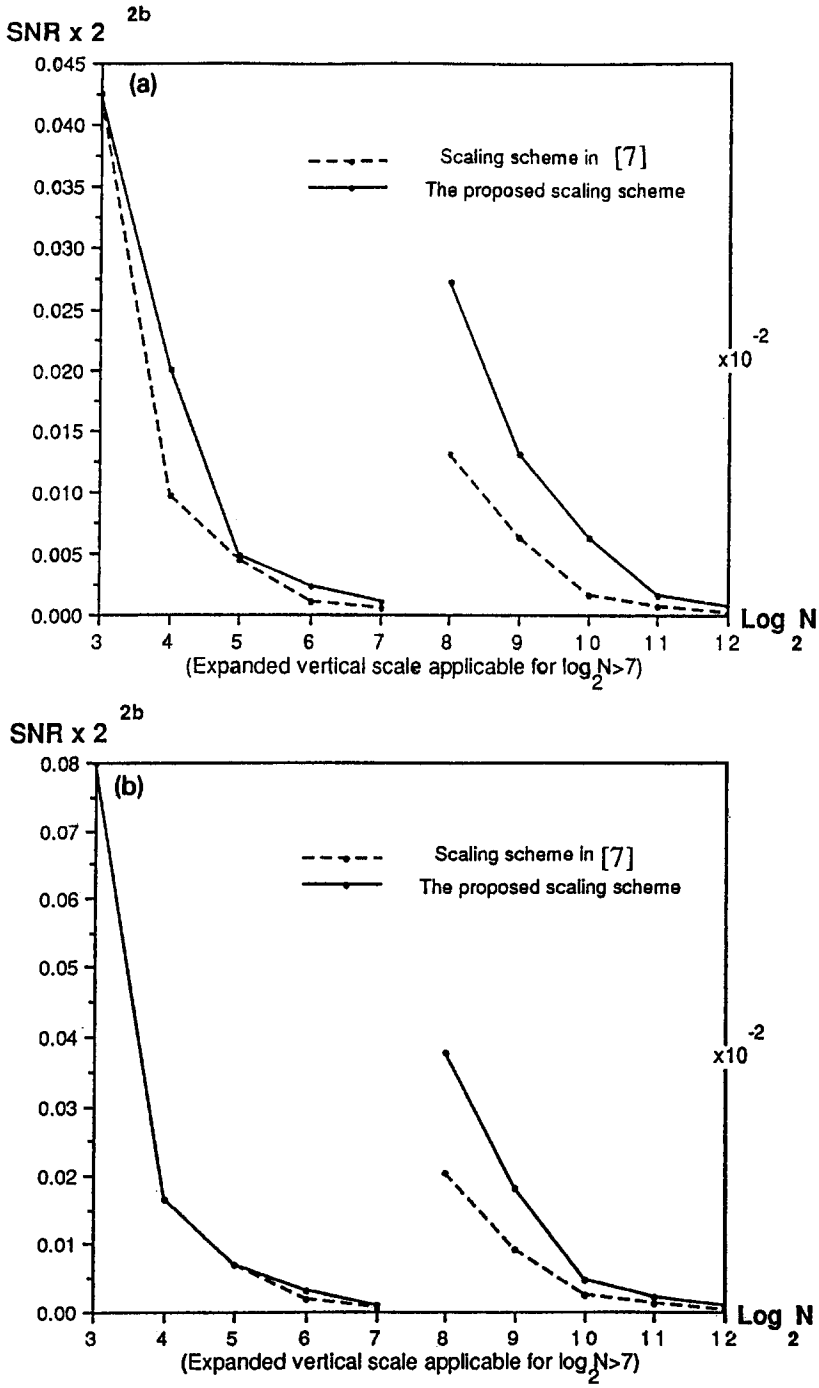

Fig. 3. Average signal-to-noise ratio for (a) DIF FHT, (b) DIT FHT.

plotted for various N in Figs. 3(a) and 3(b), respectively. It is clear, from Fig. 3, that the proposed step-by-step scaling scheme produces better SNR values than those of the ones proposed in [7]. As indicated in Fig. 3, the newly proposed scaling scheme yields a maximum improvement of about 6 dB in both DIF and DIT cases. However, the signal-to-noise ratio of the proposed scaling scheme is still less than that of the standard FFT algorithms [9].

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