DFT-Commuting Matrix With Arbitrary or Infinite Order Second Derivative Approximation

Soo-Chang Pei, Wen-Liang Hsue, and Jian-Jiun Ding

***Abstract*—Recently, Candan introduced higher order DFT-commuting matrices whose eigenvectors are better approximations to the continuous Hermite-Gaussian functions (HGFs). However, the highest order $2k$ of the $O(h^{2k})$ $N \times N$ DFT-commuting matrices proposed by Candan is restricted by $2k+1 \leq N$. In this paper, we remove this order upper bound restriction by developing two methods to construct arbitrary-order DFT-commuting matrices. Computer experimental results show that the Hermite-Gaussian-like (HGL) eigenvectors of the new proposed DFT-commuting matrices outperform those of Candan. In addition, the HGL eigenvectors of the infinite-order DFT-commuting matrix are shown to be the same as those of the n^2 DFT-commuting matrix recently discovered in the literature.**

***Index Terms*—Commuting matrix, discrete Fourier transform, discrete fractional Fourier transform, eigenvector, Hermite–Gaussian function.**

I. Introduction

The continuous fractional Fourier transform (FRT) is a generalized version of the conventional continuous Fourier transform [1]. The continuous HGFs are known as the eigenfunctions of the FRT [1]. The discrete fractional Fourier transform (DFRFT) is the fractional version of the discrete Fourier transform (DFT) [2]–[4]. To define the eigendecomposition-based DFRFT, we must first compute an orthonormal eigenvector basis of the DFT [3]. The DFT matrix has only four distinct multiple eigenvalues $\{1, -j, -1, j\}$. Consequently, it is ambiguous to determine an eigenvector basis of the DFT [3]. To resolve this eigenvector ambiguity problem, we can find the DFT eigenvectors from those of the DFT-commuting matrices [3]. Moreover, in order to define the DFRFT whose outputs are sample approximations of the continuous FRT, finding good DFT-commuting matrices with physical meaning Hermite–Gaussian-like (HGL) DFT-eigenvectors is important for this kind of DFRFT definitions [3]–[6].

In [3], [4], the DFRFT is defined based on HGL eigenvectors computed from the Dickinson–Steiglitz extended-tridiagonal commuting matrix of the DFT [7]. Inspired by the work of Grünbaum [8], Pei *et al.* [5] proposed another extended-tridiagonal DFT-commuting matrix whose eigenvectors are even closer to the continuous HGFs than those of the Dickinson–Steiglitz matrix. In fact, the Dickinson–Steiglitz matrix is the $O(h^2)$ approximation to the second-order differential equation generating the HGF [4], [9]. Recently, Candan [9] proposed the higher order DFT-commuting matrices whose eigenvectors approximate the continuous HGFs even more accurately than those of the Dickinson–Steiglitz matrix [7] and those of the DFT-commuting matrices introduced in [5]. However, the highest order $2k$ of the $O(h^{2k})$ $N \times N$ DFT-commuting matrices in [9] is restricted by $2\,k+1 \leq N$ [9].

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II. Preliminaries

A. Continuous Hermite–Gaussian Functions (HGFs)

The continuous HGFs are the eigenfunctions of the continuous Fourier transform and are the solutions of the following second-order differential equation [1]:

$$\frac{d^2}{dt^2} f(t) - 4\pi^2 t^2 f(t) = \lambda \cdot f(t). \tag{1}$$

It is shown in [4] that (1) can also be expressed as

$$\mathsf{S}\{f(t)\} = \lambda \cdot f(t) \tag{2}$$

where

$$\mathsf{S} \equiv \mathsf{D}^2 + \mathsf{F}\mathsf{D}^2\mathsf{F}^{-1} \tag{3}$$

with D and F being the differentiation operator and the continuous Fourier transform operator, respectively. Therefore, the HGFs are also the eigenfunctions of the operator S defined in (3).

B. General DFT-Commuting Matrices

The $N \times N$ DFT Matrix $\mathbf{F}$ is Defined as

$$[\mathbf{F}]_{kn} = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}kn}, \quad 0 \leq k, n \leq N-1. \tag{4}$$

It is shown in [9] that any $N \times N$ DFT-commuting matrix $\mathbf{A}$ can be expressed as

$$\mathbf{A} = \mathbf{L} + \mathbf{F}\mathbf{L}\mathbf{F}^{-1} + \mathbf{F}^2\mathbf{L}\mathbf{F}^{-2} + \mathbf{F}^3\mathbf{L}\mathbf{F}^{-3} \tag{5}$$

where $\mathbf{L}$ is an arbitrary $N \times N$ matrix. The general form of the DFT-commuting matrix $\mathbf{A}$ in (5) can be derived using the key fact $\mathbf{F}^4 = \mathbf{I}$ [9]. In fact, (5) can be further simplified as follows.

Definition: An $N \times N$ matrix $\mathbf{B}$ is defined to be $\mathbf{K}$-symmetric [10] if

$$\mathbf{K}\mathbf{B}\mathbf{K} = \mathbf{B} \tag{6}$$

where $\mathbf{K}$ is the circular reversal matrix given by

$$\mathbf{K} \equiv \mathbf{F}^2 = \mathbf{F}^{-2} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix}_{N \times N} \tag{7}$$

with $\mathbf{J}$ being the $(N-1) \times (N-1)$ reversal matrix whose nonzero entries are ones on the antidiagonal. An $N \times 1$ vector $\mathbf{x}$ is defined to be $\mathbf{K}$-symmetric if $\mathbf{K}\mathbf{x} = \mathbf{x}$.

Property 1: If $\mathbf{B}_1$ and $\mathbf{B}_2$ are both $N \times N$ $\mathbf{K}$-symmetric matrices, then i) $\mathbf{B}_1 \cdot \mathbf{B}_2$ is $\mathbf{K}$-symmetric and ii) $\beta_1 \cdot \mathbf{B}_1 + \beta_2 \cdot \mathbf{B}_2$ is $\mathbf{K}$-symmetric, where β_1 and β_2 are arbitrary scalars.

Proof: i) $\mathbf{K}(\mathbf{B}_1\mathbf{B}_2)\mathbf{K} = (\mathbf{K}\mathbf{B}_1\mathbf{K})(\mathbf{K}\mathbf{B}_2\mathbf{K}) = \mathbf{B}_1\mathbf{B}_2$. Proof of Part ii) is straightforward. ■

Property 2: Let us define an $N \times N$ matrix $\mathbf{S}$ as

$$\mathbf{S} = \mathbf{M} + \mathbf{F}\mathbf{M}\mathbf{F}^{-1} \tag{8}$$

where $\mathbf{M}$ is an $N \times N$ $\mathbf{K}$-symmetric matrix. Then $\mathbf{S}$ commutes with $\mathbf{F}$.

Proof:

$$\begin{aligned}\mathbf{FS} &= \mathbf{FM} + \mathbf{F}^2\mathbf{M}\mathbf{F}^{-1} = \mathbf{FM} + \mathbf{F}^2\mathbf{M}\mathbf{F}^{-2}\mathbf{F} \\ &= \mathbf{FM} + \mathbf{KMKF} = \mathbf{FM} + \mathbf{MF} = \mathbf{SF}.\end{aligned}$$

■

From Property 2, a DFT-commuting matrix $\mathbf{S}$ can be constructed by first choosing a $\mathbf{K}$-symmetric generating matrix $\mathbf{M}$ and then substituting it into (8).

C. Candan's Higher Order DFT-Commuting Matrices

The $(2k)^{\text{th}}$ -order approximation to second derivative is derived by Candan [9], [11] and is:

$$f''(x_k) = \left(\sum_{m=1}^{k} (-1)^{m-1} \frac{2[(m-1)!]^2}{(2m)!} \delta^{2m}\right) \frac{f_k}{h^2} + O(h^{2k}), \tag{9}$$

where $\delta^2 f_k \equiv f_{k+1} - 2f_k + f_{k-1}$ is the second central differencing. Equation (9) is used by Candan [9] to derive the $O(h^{2k})$ DFT-commuting matrix. For example, from (9), $O(h^4)$ approximation of second derivative is

$$f''(x_k) = \frac{1}{h^2}\left(-\frac{1}{12}f_{k+2} + \frac{4}{3}f_{k+1} - \frac{5}{2}f_k + \frac{4}{3}f_{k-1} - \frac{1}{12}f_{k-2}\right). \tag{10}$$

According to the work of Candan [9], the $O(h^4)$ 7×7 generating matrix $\mathbf{M}_4$ which approximates the second derivative can then be constructed by circularly shifting the $O(h^4)$ coefficient series $\{-1/12, 4/3, -2/5, 4/3, -1/12\}$ in (10) as follows. First, define the $O(h^4)$ 1×7 generating vector $\mathbf{m}_4$ as

$$\mathbf{m}_4 = [-2/5, 4/3, -1/12, 0, 0, -1/12, 4/3]. \tag{11}$$

Then the $O(h^4)$ 7×7 generating matrix $\mathbf{M}_4$ is [9] [see (12), shown at the bottom of the page], where $\mathbf{m}_4(p)$ is obtained from $\mathbf{m}_4$ by circularly shifting it p positions to the right. Substituting $\mathbf{M}_4$ of (12) into (8), the $O(h^4)$ DFT-commuting matrix $\mathbf{S}_4$ can be obtained. From (8) and (3), the resulting DFT-commuting matrix $\mathbf{S}_4$ is a discrete approximation of the continuous HGF generating operator $\mathbf{S}$ in (3) and thus $\mathbf{S}_4$ has HGL eigenvectors. In [9], Candan showed that the eigenvectors of higher order DFT-commuting matrices are closer to HGFs than those of the lower order ones. From (9), the length of the coefficient series for $O(h^{2k})$ approximation to second derivative is $2k+1$. Therefore, it is important to notice that in Candan's work the highest order $2k$ of the $N \times N$ $O(h^{2k})$ DFT-commuting matrix is restricted by $2k+1 \le N$, such that the length-$(2k+1)$ coefficient series for (9) can be accommodated into the rows of the $N \times N$ generating matrix $\mathbf{M}$. That is $k \le \lfloor (N-1)/(2) \rfloor$.

III. Improved High-Order DFT-Commuting Matrices

A. Arbitrary-Order DFT-Commuting Matrices

From Section II, the $N \times N$ $O(h^2)$ generating matrix $\mathbf{M}_2$ can be easily constructed and the result is the $N \times N$ second difference matrix which is denoted as $\mathbf{D}$

$$\mathbf{M}_2 = \mathbf{D} \equiv \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \ddots & \ddots & 0 \\ 0 & 1 & -2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}_{N \times N}. \tag{13}$$

Note that $\mathbf{D}$ is $\mathbf{K}$-symmetric from the definition (6). We will then show that any $N \times N$ $O(h^{2k})$ generating matrix $\mathbf{M}_{2k}$, which is $O(h^{2k})$ approximation to second derivative, can be constructed from the $O(h^2)$ generating matrix $\mathbf{D}$. For example, different from the approach introduced in Subsection II.C, it is interesting to see that the $O(h^4)$ 7×7 generating matrix $\mathbf{M}_4$ in (12) can also be computed from $\mathbf{D}^2$ and $\mathbf{D}^1$, where $\mathbf{D}$ is the 7×7 second difference matrix, using the same coefficients in (9) as follows:

$$\mathbf{M}_4 = -\frac{1}{12}\mathbf{D}^2 + \mathbf{D}. \tag{14}$$

From (9), the above observation can be generalized to formulate the $N \times N$ $O(h^{2k})$ generating matrix $\mathbf{M}_{2k}$ as a matrix polynomial of the $N \times N$ second difference matrix $\mathbf{D}$ in (13). The result is

$$\mathbf{M}_{2k} = \sum_{m=1}^{k} (-1)^{m-1} \frac{2[(m-1)!]^2}{(2m)!} \mathbf{D}^m. \tag{15}$$

It is then easy to verify that $\mathbf{M}_{2k}$ is $\mathbf{K}$-symmetric for all k from Property 1 and the fact that $\mathbf{D}$ is $\mathbf{K}$-symmetric. Therefore, $\mathbf{M}_{2k}$ is a valid generating matrix from Property 2. Unlike Candan's approach in [9], when k exceeds $\lfloor (N-1)/(2) \rfloor$, $\mathbf{M}_{2k}$ can still be computed from $\mathbf{D}$ using (15). Substituting $\mathbf{M}_{2k}$ of (15) into (8), we can obtain the $N \times N$ $O(h^{2k})$ DFT-commuting matrix $\mathbf{S}_{2k}$. The order $2k$ of $\mathbf{S}_{2k}$ computed using the above method can be extended to an arbitrary large number, even to infinite order.

$$\begin{aligned}\mathbf{M}_4 &= \begin{bmatrix} \mathbf{m}_4(0) \\ \mathbf{m}_4(1) \\ \vdots \\ \mathbf{m}_4(6) \end{bmatrix} \\ &= \begin{bmatrix} -5/2 & 4/3 & -1/12 & 0 & 0 & -1/12 & 4/3 \\ 4/3 & -5/2 & 4/3 & -1/12 & 0 & 0 & -1/12 \\ -1/12 & 4/3 & -5/2 & 4/3 & -1/12 & 0 & 0 \\ 0 & -1/12 & 4/3 & -5/2 & 4/3 & -1/12 & 0 \\ 0 & 0 & -1/12 & 4/3 & -5/2 & 4/3 & -1/12 \\ -1/12 & 0 & 0 & -1/12 & 4/3 & -5/2 & 4/3 \\ 4/3 & -1/12 & 0 & 0 & -1/12 & 4/3 & -5/2 \end{bmatrix},\end{aligned} \tag{12}$$

From (15), when $2k+1>N$, the coefficients outside the central N positions in the coefficient series approximating the second derivative of (9) are in effect wrapped around (modulous N) and added to the central N coefficients to form the $N\times N$ $O(h^{2k})$ generating matrix $\mathbf{M}_{2k}$. In [12, p. 15], it is shown that the infinite-order $O(h^{\infty})$ coefficient series for (9) is

$$t_n=\begin{cases}-\frac{\pi^2}{3}, & n=0\\ 2\frac{(-1)^{n+1}}{n^2}, & n\neq 0.\end{cases} \tag{16}$$

Consequently, the $O(h^{\infty})$ $1\times N$ generating vector for the $O(h^{\infty})$ $N\times N$ generating matrix $\mathbf{M}_{\infty}$ are given by (17), shown at the bottom of the page, where

$$s_n=\sum_{k=-\infty}^{\infty}t_{n+kN},\quad 0\le n\le\left\lfloor\frac{N}{2}\right\rfloor. \tag{18}$$

With $\mathbf{m}_{\infty}$, the $O(h^{\infty})$ $N\times N$ generating matrix $\mathbf{M}_{\infty}$ can be constructed similar to (12).

B. Coefficient-Truncated Arbitrary-Order DFT-Commuting Matrices

Before introducing the coefficient truncation technique to construct arbitrary-order DFT-commuting matrices [14], we first observe the distribution of coefficient series for $O(h^{2k})$ approximation to second derivative in (9). For example, from (10), the $O(h^4)$ coefficient series are $\{-1/12,4/3,-2/5,4/3,-1/12\}$. From the coefficient series of the above 4th -order and other higher orders, we find that absolute values of the $O(h^{2k})$ coefficients near the central positions are larger and dominant for all k.

According to the above observation, we propose the $O(h^{2k})$ $N\times N$ DFT-commuting matrices using a coefficient truncation technique as follows. For the following discussions in this subsection, we assume that $2k+1>N$. Because $2k+1>N$, Candan's method in [9] cannot be directly applied for these cases. However, because the central coefficients dominate in the series for $O(h^{2k})$ approximation in (9), we can use only the central N dominant coefficients to construct the generating matrix and set all of the remaining minor coefficients as zeros. Therefore, assume that the $O(h^{2k})$ coefficient series for (9) are $\{a_k,a_{k-1},\ldots,a_1,a_0,a_1,\ldots,a_k\}$. Then the $1\times N$ $O(h^{2k})$ coefficient-truncated generating vector $\mathbf{m}_{2k,N}$ can be constructed as

$$\mathbf{m}_{2k,N}=\begin{cases}[a_0,a_1,\ldots,a_v,a_v,a_{v-1},\ldots,a_1], & \text{if } N \text{ is odd}\\ [a_0,a_1,\ldots,a_v,a_{v-1},\ldots,a_1], & \text{if } N \text{ is even}\end{cases} \tag{19}$$

where $v=\lfloor(N)/(2)\rfloor$. In (19), the first subscript $2k$ indicates that the generating vector $\mathbf{m}_{2k,N}$ is constructed from the $O(h^{2k})$ approximation to second derivative in (9) and the second subscript N indicates that the length-$(2k+1)$ coefficients series for (9) are truncated to reserve only the central N dominant coefficients. With $\mathbf{m}_{2k,N}$ in (19), the corresponding $O(h^{2k})$ $N\times N$ generating matrix $\mathbf{M}_{2k,N}$ can be easily constructed as

$$\mathbf{M}_{2k,N}=\begin{bmatrix}\mathbf{m}_{2k,N}(0)\\ \mathbf{m}_{2k,N}(1)\\ \vdots\\ \mathbf{m}_{2k,N}(N-1)\end{bmatrix} \tag{20}$$

where $\mathbf{m}_{2k,N}(p)$ is obtained from $\mathbf{m}_{2k,N}$ by circularly shifting it p positions to the right. $\mathbf{M}_{2k,N}$ in (20) is $\mathbf{K}$-symmetric because $(\mathbf{m}_{2k,N})^T$ is $\mathbf{K}$-symmetric, with T being the transpose operation. Consequently, $\mathbf{M}_{2k,N}$ is a valid generating matrix. Then the corresponding coefficient-truncated $O(h^{2k})$ $N\times N$ DFT-commuting matrix $\mathbf{S}_{2k,N}$ can be computed by substituting $\mathbf{M}_{2k,N}$ of (20) into (8), i.e.,

$$\mathbf{S}_{2k,N}=\mathbf{M}_{2k,N}+\mathbf{F}\mathbf{M}_{2k,N}\mathbf{F}^{-1}. \tag{21}$$

Property 3: The explicit expression of the coefficient-truncated $O(h^{2k})$ $N\times N$ DFT-commuting matrix $\mathbf{S}_{2k,N}$ in (21) is

$$\mathbf{S}_{2k,N}=\mathbf{M}_{2k,N}+\mathrm{diag}(d_0,d_1,\ldots,d_{N-1}) \tag{22}$$

where $\mathbf{M}_{2k,N}$ is given by (20), and $d_0,d_1,\ldots,d_{N-1}$ are given by

$$d_\mu=\begin{cases}a_0+\sum_{\alpha=1}^{s}2a_\alpha\cos\left(\alpha\cdot\mu\cdot\frac{2\pi}{N}\right), & \text{if } N \text{ is odd}\\ a_0+(-1)^{\mu}\cdot a_{\frac{N}{2}}+\sum_{\alpha=1}^{s}2a_\alpha\cos\left(\alpha\cdot\mu\cdot\frac{2\pi}{N}\right), & \text{if } N \text{ is even}\end{cases} \tag{23}$$

with $s=\lfloor(N-1)/(2)\rfloor$.

Proof: See the Appendix. ■

For example, the $O(h^{10})$ 5×5 generating matrix $\mathbf{M}_{10,5}$ can be constructed as follows. Assume that the coefficient series for the $O(h^{10})$ approximation to second derivative in (9) is $\{b_5,b_4,b_3,b_2,b_1,b_0,b_1,b_2,b_3,b_4,b_5\}$. It should be noted that, from (9), the $O(h^{10})$ coefficient series b_i is different from the $O(h^{2k})$ coefficient series a_i defined in the previous discussion when $k\neq 5$. The length-11 coefficient series b_i can then be truncated to a length-5 coefficient series as $\{b_2,b_1,b_0,b_1,b_2\}$. From (19), the $O(h^{10})$ 1×5 generating vector $\mathbf{m}_{10,5}$ is

$$\mathbf{m}_{10,5}=[b_0,b_1,b_2,b_2,b_1]. \tag{24}$$

Therefore, from (20), the $O(h^{10})$ 5×5 generating matrix $\mathbf{M}_{10,5}$ is

$$\mathbf{M}_{10,5}=\begin{bmatrix}\mathbf{m}_{10,5}(0)\\ \mathbf{m}_{10,5}(1)\\ \vdots\\ \mathbf{m}_{10,5}(4)\end{bmatrix}=\begin{bmatrix}b_0&b_1&b_2&b_2&b_1\\ b_1&b_0&b_1&b_2&b_2\\ b_2&b_1&b_0&b_1&b_2\\ b_2&b_2&b_1&b_0&b_1\\ b_1&b_2&b_2&b_1&b_0\end{bmatrix}. \tag{25}$$

$$\mathbf{m}_{\infty}=\begin{cases}[s_0,s_1,\ldots,s_v,s_v,s_{v-1},\ldots,s_1], & \text{if } N \text{ is odd,}\\ [s_0,s_1,\ldots,s_v,s_{v-1},\ldots,s_1], & \text{if } N \text{ is even}\end{cases}\quad \text{with } v=\left\lfloor\frac{N}{2}\right\rfloor \tag{17}$$

It is easy to verify that $\mathbf{M}_{10,5}$ is $\mathbf{K}$-symmetric from the definition (6). Besides, from *Property 3* and (25), the explicit expression of the coefficient-truncated $O(h^{10})$ 5×5 DFT-commuting matrix $\mathbf{S}_{10,5}$ is [see (26), shown at the bottom of the page].

IV. Computer Experiments

First, we compute the HGL eigenvectors of the 32×32 DFT-commuting matrices of 2nd -order ($\mathbf{S}_2$), 30th -order ($\mathbf{S}_{30}$), 1000th -order ($\mathbf{S}_{1000}$), and coefficient-truncated 1000th -order ($\mathbf{S}_{1000,32}$). Fig. 1(a) plots the error norms for the HGL eigenvectors of those DFT-commuting matrices. The error norm of the HGL eigenvector is defined as the norm of the difference vector between the HGL eigenvector and the samples of its corresponding HGF. Log scales of the same results in Fig. 1(a) are plotted in Fig. 1(b). In fact, $\mathbf{S}_{30}$ is the highest-order 32×32 DFT-commuting matrix in Candan's work [9], and $\mathbf{S}_2$ is the Dickinson-Steiglitz matrix. From Fig. 1, the eigenvectors of higher order DFT-commuting matrices are closer to samples of HGFs than those of the lower order ones. The total error norms of $\mathbf{S}_2$, $\mathbf{S}_{30}$, $\mathbf{S}_{1000}$, and $\mathbf{S}_{1000,32}$ in Fig. 1 are 17.4411, 7.2127, 5.6276, and 5.6503, respectively. Besides, from Fig. 1, the error norms of $\mathbf{S}_{1000}$ and $\mathbf{S}_{1000,32}$ are almost the same. Compared with $\mathbf{S}_{1000}$, coefficient truncation for $\mathbf{S}_{1000,32}$ in Fig. 1(b) results in negligible degradation of HGL eigenvectors with smaller numbers of zero-crossings.

The following $N \times N$ diagonal generating matrix was proposed in [6] and [13]:

$$[\mathbf{E}]_{n,n} = \begin{cases} n^2, & 0 \le n \le \lfloor \frac{N}{2} \rfloor \\ (N-n)^2, & \lceil \frac{N}{2} \rceil \le n \le N-1. \end{cases} \tag{27}$$

The diagonal entries of the generating matrix $\mathbf{E}$ in (27) are basically n^2. The reasons why the second half diagonal entries of $\mathbf{E}$ are given in the form of $(N\text{-}n)^2$ in (27) are that the resulting diagonal matrix $\mathbf{E}$ is $\mathbf{K}$-symmetric and thus $\mathbf{E}$ is a valid generating matrix (from *Property 2*). Substituting the above n^2 generating matrix $\mathbf{E}$ in (8), it is pointed out in [13] that the resulting n^2 DFT-commuting matrix

$$\mathbf{S}_{\mathbf{E}} = \mathbf{E} + \mathbf{F}\mathbf{E}\mathbf{F}^{-1} \tag{28}$$

possesses excellent HGL eigenvectors of DFT. For comparison, error norms for HGL eigenvectors of the 32×32 $\mathbf{S}_{\mathbf{E}}$ are also plotted in Fig. 1. The total error norm of $\mathbf{S}_{\mathbf{E}}$ in Fig. 1 is 5.5191. The reason that the error norms of HGL eigenvectors of $\mathbf{S}_{1000}$ and $\mathbf{S}_{\mathbf{E}}$ in Fig. 1 are almost the same can be explained as follows. First, since the $O(h^{2k})$ generating matrix $\mathbf{M}_{2k}$ in (15) is Toeplitz (in fact also circulant [12]) and symmetric, $\mathbf{F}\mathbf{M}_{2k}\mathbf{F}^{-1}$ is a diagonal matrix. From computer experiment, we find that when k is large

$$\mathbf{F}\mathbf{M}_{2k}\mathbf{F}^{-1} \cong c \cdot \mathbf{E} \tag{29}$$

where c is a scalar constant. Equation (29) indicates that the diagonalized $O(h^{2k})$ generating matrix $\mathbf{F}\mathbf{M}_{2k}\mathbf{F}^{-1}$ is a scalar multiple of the n^2 generating matrix $\mathbf{E}$ defined in (27) when k is large. For example, when $2\,k \ge 400$

$$\left\|\operatorname{diag}\left((1/c) \cdot \mathbf{F}\mathbf{M}_{2k}\mathbf{F}^{-1}\right) - \operatorname{diag}(\mathbf{E})\right\| \le 0.02 \times \|\operatorname{diag}(\mathbf{E})\| \tag{30}$$

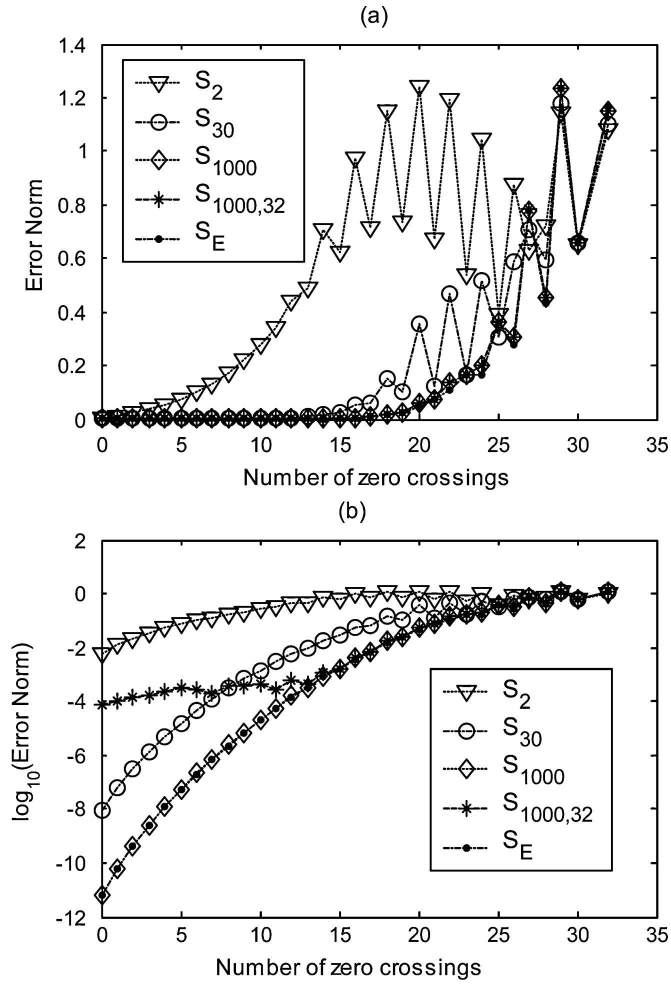

Fig. 1. Error norms for HGL eigenvectors of the 32×32 $\mathbf{S}_{\mathbf{E}}$ matrix defined in (28) as well as the 32×32 DFT-commuting matrices of 2nd -order ($\mathbf{S}_2$), 30th -order ($\mathbf{S}_{30}$), 1000th -order ($\mathbf{S}_{1000}$), and coefficient-truncated 1000th -order ($\mathbf{S}_{1000,32}$). (a) Normal scale. (b) Log scale.

if $\mathbf{M}_{2k}$ and $\mathbf{E}$ are 32×32, where $\operatorname{diag}(\mathbf{E})$ represents the vector formed from the diagonal entries of the matrix $\mathbf{E}$ and $\|\cdot\|$ is the vector norm. Therefore, since $\mathbf{M}_{1000}$ is $\mathbf{K}$-symmetric

$$\begin{aligned} \mathbf{S}_{1000} &= \mathbf{M}_{1000} + \mathbf{F}\mathbf{M}_{1000}\mathbf{F}^{-1} \\ &= \mathbf{F}^2\mathbf{M}_{1000}\mathbf{F}^{-2} + \mathbf{F}\mathbf{M}_{1000}\mathbf{F}^{-1} \\ &\cong c \cdot (\mathbf{E} + \mathbf{F}\mathbf{E}\mathbf{F}^{-1}) = c \cdot \mathbf{S}_{\mathbf{E}}. \end{aligned} \tag{31}$$

From (31), $\mathbf{S}_{1000}$ accurately approximates a scalar multiple of $\mathbf{S}_{\mathbf{E}}$. Consequently, $\mathbf{S}_{1000}$ and $\mathbf{S}_{\mathbf{E}}$ almost share the same eigenvector set. We conjecture that the n^2 DFT-commuting matrix $\mathbf{S}_{\mathbf{E}}$ in [6] and [13] is optimal in the sense that its HGL eigenvectors are the same as those of the infinite-order DFT-commuting matrix S_∞.

$$\mathbf{S}_{10,5} = \begin{bmatrix} \sum_{\alpha=0}^{2} 2b_\alpha & b_1 & b_2 & b_2 & b_1 \\ b_1 & \sum_{\alpha=0}^{2} 2b_\alpha \cos\left(\alpha \cdot \frac{2\pi}{5}\right) & b_1 & b_2 & b_2 \\ b_2 & b_1 & \sum_{\alpha=0}^{2} 2b_\alpha \cos(2\alpha \cdot \frac{2\pi}{5}) & b_1 & b_2 \\ b_2 & b_2 & b_1 & \sum_{\alpha=0}^{2} 2b_\alpha \cos(3\alpha \cdot \frac{2\pi}{5}) & b_1 \\ b_1 & b_2 & b_2 & b_1 & \sum_{\alpha=0}^{2} 2b_\alpha \cos(4\alpha \cdot \frac{2\pi}{5}) \end{bmatrix} \tag{26}$$

V. Conclusion

In this paper, we have extended Candan's work in [9] by developing two methods to construct arbitrary-order DFT-commuting matrices. In the first method, for any arbitrary k the $O(h^{2k})$ generating matrix was formulated as a matrix polynomial of the $O(h^2)$ generating matrix. In the second method, the coefficient series with length $2k+1$ for the $O(h^{2k})$ approximation to second derivative were truncated to the length- N dominant coefficient series, based on which the coefficient-truncated $N \times N$ $O(h^{2k})$ generating matrix was then constructed. Computer experiments were performed to demonstrate the superiority of the HGL eigenvectors computed from the arbitrary-order DFT-commuting matrices proposed in this paper.

Appendix

Proof of Property 3: We only give the proof of this property when N is odd. When N is even, this property can be proved similarly. Assume that N is odd. From (20), $\mathbf{M}_{2k,N}$ is Toeplitz and symmetric. Therefore, from (19) and (20), $\mathbf{M}_{2k,N}$ can be rewritten as

$$\mathbf{M}_{2k,N} = a_0\mathbf{I} + \sum_{\alpha=1}^{s} a_\alpha \mathbf{I}_\alpha \tag{A1}$$

where $\mathbf{I}_\alpha$ is given by

$$\mathbf{I}_\alpha = \begin{bmatrix} \mathbf{e}_\alpha(0) \\ \mathbf{e}_\alpha(1) \\ \vdots \\ \mathbf{e}_\alpha(N-1) \end{bmatrix} \tag{A2}$$

with

$$\mathbf{e}_\alpha \equiv [0, \underbrace{0,\cdots,0}_{(\alpha-1)\text{zeros}}, 1, 0, \cdots, 0, 1, \underbrace{0,\cdots,0}_{(\alpha-1)\text{zeros}}]_{1\times N}.$$

In (A2), $\mathbf{e}_\alpha(p)$ is obtained from $\mathbf{e}_\alpha$ by circularly shifting it p positions to the right. From direct matrix multiplications, we have

$$\mathbf{F}\mathbf{I}_\alpha\mathbf{F}^{-1} = \mathrm{diag}\left(2\cos\left(\alpha\cdot 0\cdot\frac{2\pi}{N}\right), 2\cos\left(\alpha\cdot 1\cdot\frac{2\pi}{N}\right), \ldots, 2\cos\left(\alpha\cdot(N-1)\cdot\frac{2\pi}{N}\right)\right) \tag{A3}$$

From (A1) and (A3)

$$\begin{aligned}\mathbf{F}\mathbf{M}_{2k,N}\mathbf{F}^{-1} &= a_0\mathbf{I} + \sum_{\alpha=1}^{s} a_\alpha \mathbf{F}\mathbf{I}_\alpha\mathbf{F}^{-1} \\ &= \mathrm{diag}(d_0, \quad d_1, \quad \cdots, \quad d_{N-1})\end{aligned} \tag{A4}$$

where

$$d_\mu = a_0 + \sum_{\alpha=1}^{s} 2a_\alpha \cos\left(\alpha\cdot\mu\cdot\frac{2\pi}{N}\right). \tag{A5}$$

Therefore, from (21), *Property 3* is proved when N is odd. ■

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On the Relationship Between MIMO and SIMO Radars

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***Abstract*—This correspondence explores the close relationship between multiple-input multiple-output (MIMO), single-input multiple-output (SIMO), and phased-array radars. It is shown that under certain conditions, the signals received by a MIMO radar and by a properly defined SIMO radar are identical. Several equivalent variations of MIMO and SIMO radars are presented. These relationships make it possible to study MIMO radars in terms of their equivalent SIMO radars, using the extensive body of algorithms and performance analysis which have been developed for SIMO radars.**

***Index Terms*—Antennas, array manifold, beamforming, matched filter, multi-input multi-output, phased-array, radar.**

I. Introduction

Recently, there has been considerable interest in a novel class of radar systems called "MIMO radar," where the term multiple-input multiple-output (MIMO) refers to the use of multiple-transmit as well as multiple-receive antennas. The use of multiple antennas both on transmit and receive is, of course, not new. Phased arrays which form beams both on transmit and receive have been around for a long time. See [1] and [2] and the references therein for a historical overview of

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