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NORMALIZATION OF ROTATIONALLY SYMMETRIC SHAPES FOR PATTERN RECOGNITION

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Abstract—Image normalization is useful for image recognition by the matching method. Recently, a normalization algorithm was developed to normalize images under the distortion of translation, rotation, scaling and skew. However, the method fails when the image is rotationally symmetric. A method is introduced called the *modified Fourier descriptor* to normalize rotationally symmetric shapes. This method is simple and fast, and can be combined with the preceding normalization algorithm to make it more general and realizable.

Image normalization   Rotationally symmetric shapes   Modified Fourier descriptor

1. INTRODUCTION

Pattern recognition has been an important area in computer vision applications. In the case of a planar image, there are four basic forms of geometric distortion caused by the change of a camera's location. They are translation, rotation, scaling, and skew. So far, there have been a number of methods developed to solve these distortions, such as moment invariants,[1] Fourier descriptor,[2,3] Hough transformation,[4] shape matrix,[5] and principle axis method.[6] All of the above methods can be made invariant to translation, rotation, and scaling. However, they become useless when the pattern is skewed. When the direction of the camera is not vertical to the planar image or the sampling intervals in the x–y directions are not equal, the image is skewed.

Recently, we have developed a method of image normalization, which normalizes all the images before recognition. Thus we just compare the input normalized pattern with reference patterns by the matching method, which is very simple and fast.

The block diagram of pattern recognition by image normalization is shown in Scheme 1.

This method first extracts features from the input patterns, then normalizes the input pattern by the normalization algorithm. Here, we define normalization as a process which transforms the input pattern into a normal form which is invariant under translation, rotation, scaling, and skew. We call the transformed image a normalized image. Since the normalized image is invariant under translation, rotation, scaling, and skew, we may recognize patterns just by the simple matching method. This method has the following advantages:

(1) the method is suitable when patterns are large;
(2) the normalization algorithm is easy, does not need much computation;
(3) the similarity measure by matching is rapid;
(4) searching in the data base is efficient.

Unfortunately, the normalization algorithm becomes useless for rotationally symmetric shapes. This will be discussed later.

A shape is said to be n-fold rotationally symmetric if the shape, after being rotated through any multiple of $2\pi/n$, becomes identical to the original shape. This is frequently encountered in real applications.[7] Tsai and co-workers have developed a series of methods to solve the recognition problems about rotationally symmetric shapes.[7–9] Their methods do not seem efficient, since the fold number must be found before normalizing the rotationally symmetric shapes. In this paper, we introduce a method called the *modified Fourier descriptor* to normalize rotationally symmetric shapes which can detect the fold number and rotation angle simultaneously. Hence this method can normalize rotationally symmetric shapes efficiently.

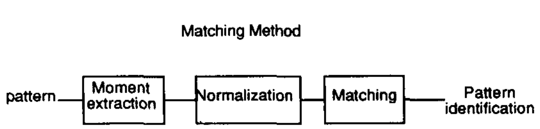
In the following, we review the normalization algorithm developed recently and see why it fails for rotationally symmetric shapes. A summary of the normalization algorithm is given below.

(1) Computing the mean vector **c** and the covariance matrix **M** of the original image

$$\mathbf{c} = [C_x C_y]^{\mathrm{T}} \tag{1}$$

and

$$\mathbf{M} = \begin{bmatrix} u_{20} & u_{11} \\ u_{11} & u_{02} \end{bmatrix}. \tag{2}$$



Scheme 1.

(2) Aligning the coordinates with the eigenvectors of **M**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} e_{1x} & e_{1y} \\ -e_{1y} & e_{1x} \end{bmatrix} \begin{bmatrix} x - C_x \\ y - C_y \end{bmatrix}. \quad (3)$$

(3) Rescaling the coordinates according to the eigenvalues of **M**

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \dfrac{c}{\sqrt{\lambda_1}} & 0 \\ 0 & \dfrac{c}{\sqrt{\lambda_2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}. \quad (4)$$

(4) Computing the third-order central moments of the compact image, say

$$u_{30}, u_{21}, u_{12}, u_{03}.$$

(5) Calculating the tensors t^1 and t^2

$$t^1 = u_{12} + u_{30} \quad (5)$$

$$t^2 = u_{03} + u_{21}. \quad (6)$$

(6) Finding the angle α, which satisfies the following equation:

$$\tan\alpha = -\frac{t^1}{t^2}. \quad (7)$$

(7) Calculating the tensor $\bar{t}^2$

$$\bar{t}^2 = -t^1 \sin\alpha + t^2 \cos\alpha. \quad (8)$$

If $\bar{t}^2 < 0$, then $\alpha = \alpha + \pi$.

(8) Rotating the compact image clockwise by angle α

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x'' \\ y'' \end{bmatrix}. \quad (9)$$

Thus, we get the normalized image which is invariant to translation, rotation, scaling, and skew.

For the rotationally symmetric shape, the covariance matrix defined in equation (2) is already equal to the scaled identical matrix (the proof is shown in reference (7)) and the tensors defined in equations (5) and (6) are zero which makes equation (7) unsolvable. Therefore, the normalization algorithm proposed previously becomes useless for the normalization of the rotationally symmetric shape. In this paper, we introduce a method called "modified Fourier descriptor" to normalize the rotationally symmetric shapes.

The paper is arranged as follows. In Section 2, we introduce the theoretical deduction of the modified Fourier descriptor. In Section 3, we introduce how to realize the modified Fourier descriptor for the normalization of rotationally symmetric shapes. In Section 4, we show the experimental results and in Section 5, we make a conclusion.

2. THEORETICAL DEDUCTION FOR MODIFIED FOURIER DESCRIPTOR

Consider an N-fold rotationally symmetric shape. Let $p(r,\theta)$ denote the signature of the shape at the location (r,θ) in the polar coordinate system and throughout this section, we take the origin of the coordinate system to be the centroid of the shape

$$p(r,\theta) = \begin{cases} 1 & \text{if object exists} \\ 0 & \text{background} \end{cases}. \quad (10)$$

The characteristic of the N-fold rotationally symmetric shape is

$$p(r,\theta) = p(r, \theta + 2\pi k/N) \quad k \text{ is any integer.} \quad (11)$$

Thus $p(r,\theta)$ is a periodical signal.

Let $P(f_1, f_2)$ denote the Fourier transformation of the shape with respect to parameter (r,θ)

$$P(f_1,f_2) = \int_{\theta=-\infty}^{\infty} \int_{r=0}^{\infty} p(r,\theta) e^{-j2\pi(f_1\theta + f_2 r)} \, dr \, d\theta$$

$$= \int_{\theta=-\infty}^{\infty} G(f_2,\theta) e^{-j2\pi f_1 \theta} \, d\theta \quad (12)$$

where

$$G(f_2,\theta) = \int_{r=0}^{\infty} p(r,\theta) e^{-j2\pi f_2 r} \, dr. \quad (13)$$

Since $p(r,\theta)$ is a periodic function with period $2\pi/N$, the function $G(f_2,\theta)$ is also a periodic function with period $2\pi/N$, i.e.

$$G(f_2,\theta) = G(f_2, \theta + 2\pi k/N) \quad \text{where } k \text{ is any integer.} \quad (14)$$

Therefore, equation (12) may be expanded as a Fourier series

$$P(f_1,f_2) = \sum_{n=-\infty}^{\infty} C_n(f_2) e^{j2\pi n f_0 \theta} \quad (15)$$

where

$f_0 = N/2\pi$ is the fundamental frequency

$$C_n(f_2) = \frac{N}{2\pi} \int_0^{2\pi/N} G(f_2,\theta) e^{-j2\pi n f_0 \theta} \, d\theta.$$

As shown in Fig. 1, the Fourier transformation of the rotationally symmetric shape in polar coordinates is zero except at the lines $f_1 = nf_0$. Let $P(f)$ be the Fourier coefficients of $P(f_1, f_2)$ on the line $f_2 = 0$.

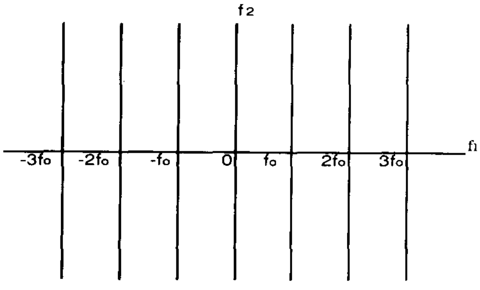


Fig. 1.

Then

$$P(f) = P(f_1, 0)$$
$$= \int_{\theta=-\infty}^{\infty} e^{-j2\pi f_1 \theta} \left\{ \int_{r=0}^{\infty} p(r,\theta)\,dr \right\} d\theta \quad (16)$$
$$= \int_{\theta=-\infty}^{\infty} G(\theta) e^{-j2\pi f_1 \theta} d\theta \quad (17)$$

where

$$G(\theta) = \int_{r=0}^{\infty} p(r,\theta)\,dr. \quad (18)$$

In fact, the function $G(\theta)$ may be viewed as a *modified Fourier descriptor*, since it is not a function of the boundary of the shape but of the radian angle. Therefore, $G(\theta)$ is also a period function with period $2\pi/N$

$$G(\theta) = G(\theta + 2\pi k/N) \quad \text{where } k \text{ is any integer.} \quad (19)$$

Similar to equation (12), equation (19) may be expanded as a Fourier series

$$P(f) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 \theta} \quad (20)$$

where

$f_0 = 2\pi/N$ is the fundamental frequency

$$C_n = \frac{N}{2\pi} \int_0^{2\pi/N} G(\theta) e^{-j2\pi n f_0 \theta} d\theta. \quad (21)$$

As shown in Fig. 2, the Fourier transformation of the Fourier descriptor $G(\theta)$ is zero except at the location $f = nf_0$. Observing equation (20), the first Fourier coefficient C_1 is located at the fundamental frequency f_0. By finding the fundamental frequency, we can derive the fold number N of the rotationally symmetric shape

$$N = 2\pi f_0. \quad (22)$$

In the following, we discuss how to use the Fourier descriptor to normalize the rotationally symmetric shape. Consider a shape as shown in Fig. 3 which is

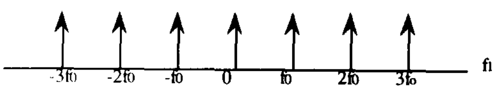


Fig. 2.

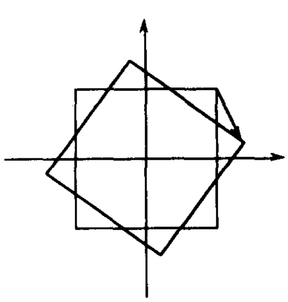


Fig. 3.

rotated clockwise by an angle Δ around the center of mass of the shape. Then the Fourier coefficients in equation (21) become

$$\tilde{C}_n = e^{-j2\pi n f_0 \Delta} C_n = e^{-jnN\Delta} C_n \quad (23)$$

where $\tilde{C}_n$ are the Fourier coefficients of the rotated shape and C_n the Fourier coefficients of the original shape. For $n = 1$, $\tilde{C}_1 = e^{-jN\Delta} C_1$. In other words, when the shape is rotated clockwise by an angle Δ, the phase of the first Fourier coefficient C_1 decreases by $N\Delta$. Thus, we may set the criterion of the normalization as follows.

Rotate the rotationally symmetric shape around the center of mass so that the phase of the first Fourier coefficient C_1 becomes zero. This may be done just by rotating the shape clockwise by an angle ϕ/N, where ϕ is the phase of C_1 and N may be found by equation (22).

3. REAL IMPLEMENTATION IN DIGITAL FORM

In real implementation, we obtain the Fourier descriptor $G(\theta)$ in duration $[0 < \theta < 2\pi]$, denoted by $s(\theta)$

$$s(\theta) = \begin{cases} G(\theta) & 0 \le \theta \le 2\pi \\ 0 & \text{elsewhere} \end{cases}. \quad (24)$$

Let $S(f)$ be the spectrum of $s(\theta)$

$$S(f) = \left\{ \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_0) \right\} * 2\pi \,\mathrm{sinc}(2\pi f) \quad (25)$$

where $*$ denotes convolution and $f_0 = N/2\pi$. Obviously

$$S(f) = \begin{cases} 0 & \text{when } f = n/2\pi, \text{ but } n \ne kN \\ C_n & \text{when } f = nf_0 = nN/2\pi \\ \text{non-zero} & \text{elsewhere} \end{cases} \quad (26)$$

So, we can find the Fourier coefficients C_n by evaluating $S(f)$ at $f = nNf_0$.

In real implementation by computer, we must sample $s(\theta)$. Let $s(n)$ be the sampling sequence of $s(\theta)$ with sampling interval $T_s = 2\pi/M$ (where M is the length of $s(n)$)

$$s(n) = s(\theta)\,\delta(\theta - nT_s) \quad \text{for } n = 0, 1, 2, \ldots, M-1. \quad (27)$$

Let $S(k)$ be the M-point DFT of $s(n)$

$$S(k) = \sum_{n=0}^{M-1} s(n) e^{-j2\pi nk/M} \quad \text{for } k = 0, 1, 2, \ldots, M-1. \quad (28)$$

We may find that the kth coefficient $S(k)$ corresponds to the spectrum $S(f)$ at the frequency $f = k/2\pi$ neglecting the aliasing effect. Therefore, neglecting the aliasing effect

$$S(k) = \begin{cases} 0 & \text{if } k \ne nN \\ C_n & \text{if } k = nN \end{cases}. \quad (29)$$

If we consider the aliasing effect, $S(k)$ can be given as follows.

Case 1. M is not a multiple of N

$$S(k) = \begin{cases} \text{aliasing component} & \text{if } k \neq nN \\ C_n & \text{if } k = nN \end{cases}. \tag{30}$$

Case 2. M is a multiple of N

$$S(k) = \begin{cases} 0 & \text{if } k \neq nN \\ C_n + \text{aliasing component} & \text{if } k = nN \end{cases}. \tag{31}$$

By the conclusion in Section 2, the purpose is to detect the location of C_1 (where we can find the fold number N) and the phase of C_1 (by which we can normalize shape). For case 1, the detection of the location of C_1 is ambiguous, but the estimated phase of C_1 is more accurate. For case 2, the detection of location of C_1 is accurate, but there is little error in the estimation of the phase of C_1. Therefore, there is a trade-off between case 1 and case 2. In summary, if the sampling length M is not too small, the aliasing effect many be neglected. By experiment, we choose $M = 256$ when the fold number $N < 10$. In this case, we may set a threshold which is very small to detect the coefficient C_1 from the sequences $S(k)$.

Now, we summarize the normalization algorithm of the rotationally symmetric shapes by Fourier descriptors.

(1) Verifying whether the input pattern is rotationally symmetric or not. Since the covariance matrix of rotationally symmetric shapes is a scaled identical matrix,[7] we may apply this property to verify if patterns are rotationally symmetric:

if $\mathbf{M} = k\mathbf{I}$ the pattern is rotationally symmetric;
if $\mathbf{M} \neq k\mathbf{I}$ the pattern is not rotationally symmetric.

Note that the criterion is not strict but is suitable enough for real applications.

If the pattern is not a rotationally symmetric shape, normalize the shape by the normalization algorithm listed in Section 1.

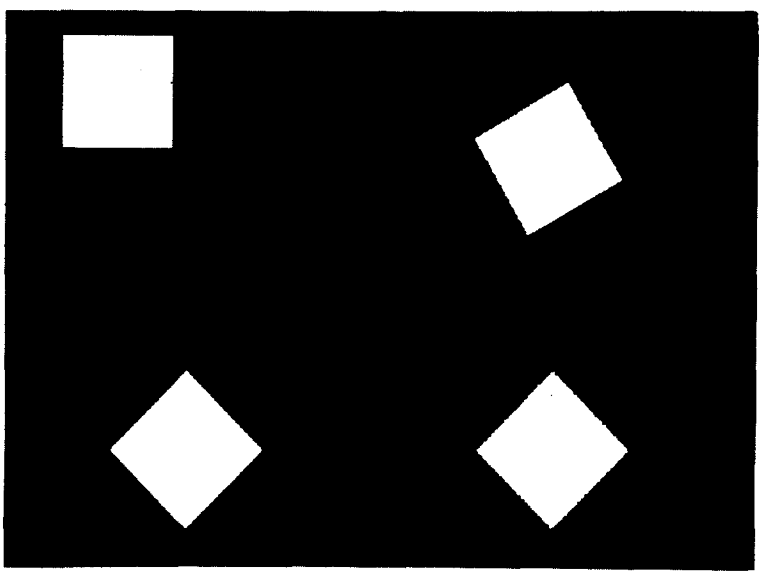
If the pattern is rotationally symmetric, carry on the procedure.

(2) Evaluating the Fourier descriptor $s(n)$

$$s(n) = \sum_{i=1}^{N'} p(i \cdot \Delta r, 2\pi n/M) \quad \text{for } n = 0, 1, 2, \ldots, M-1$$

Table 1

No. i	$\lvert S(i)/S(0) \rvert$
0	1
1	0
2	0
3	0
4	7.0280090E − 02
5	0
6	0
7	0
8	2.1902695E − 02
9	0
10	0
11	0
12	1.1369203E − 02
13	0
14	0
15	0
16	6.4240796E − 03
17	0
18	0
19	0
20	4.4142171E − 03



(a)	(c)
(b)	(d)

(a) Rotationally symmetric shape
(b) Nomalized image of fig.1(a)
(c) Rotation of fig.1(a)
(d) Normalized image of fig.1(c)

Fig. 4.

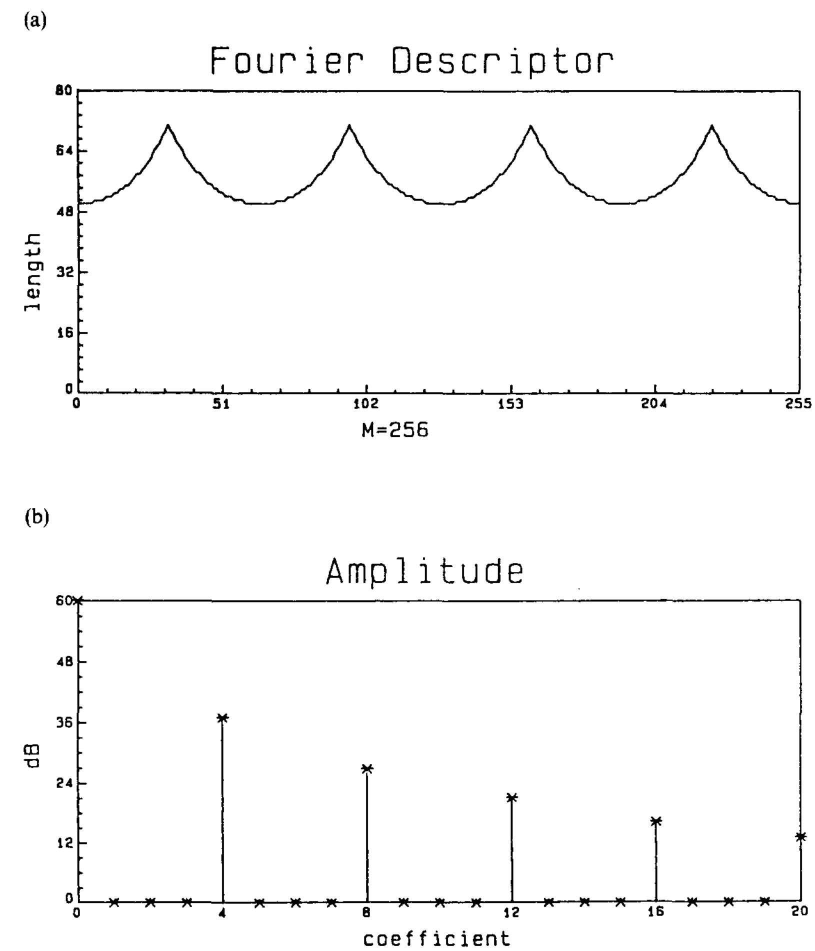


Fig. 5(a), (b).

where the choice of Δr, N' depends on application.

(3) Making an M-point DFT of $s(n)$

$$S(k) = \sum_{n=0}^{M-1} s(n)\,\mathrm{e}^{-j2\pi nk/M} \quad \text{for } k = 0,1,2,\ldots,M-1.$$

(4) Searching the sequence $\{S(i)/S(0)\}$ with the beginning from $i = 1$ if $S(i)/S(0) >$ threshold, then $N = i$ and end the searching.

(5) Find the phase of C_1 by evaluating the phase of $S(N)$, say ϕ.

(6) Rotating the shape clockwise by the angle ϕ/N. Thus, we get the normalized shape.

4. EXPERIMENTAL RESULTS

(1) Figure 4(a) shows a square shape which is made by computer. It is a rotationally symmetric shape. The Fourier descriptor of Fig. 4(a) is evaluated with $M = 256$ and shown in Fig. 5(a). Then we make a 256-point DFT of the Fourier descriptor, the resulting first 20 Fourier coefficients are listed in Table 1, and plotted in Fig. 5(b) by dB. Observing these coefficients, they satisfy case 2 in equation (31).

Table 2

No. i	$\lvert S(i)/S(0)\rvert$
0	1.000000
1	4.0429183E − 03
2	2.2288881E − 02
3	9.1571994E − 03
4	0.1692139
5	1.4055817E − 02
6	2.0911200E − 02
7	4.8770332E − 03
8	4.3639529E − 02
9	8.5320082E − 03
10	1.9210672E − 02
11	5.4477747E − 03
12	5.8793712E − 02
13	1.2609311E − 02
14	1.9429194E − 02
15	2.2740040E − 03
16	1.9058874E − 02
17	7.1299216E − 03
18	1.7912557E − 02
19	4.4469251E − 03
20	2.9510433E − 02

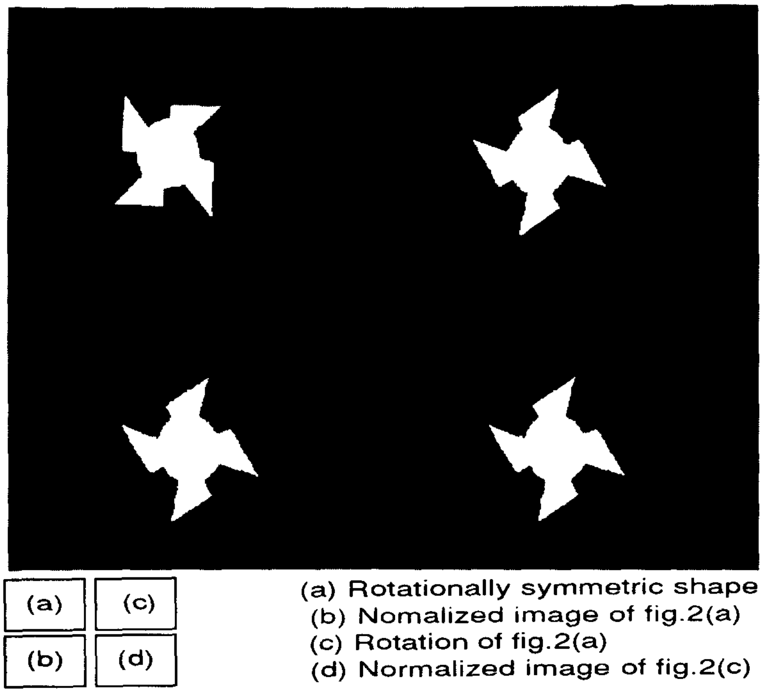
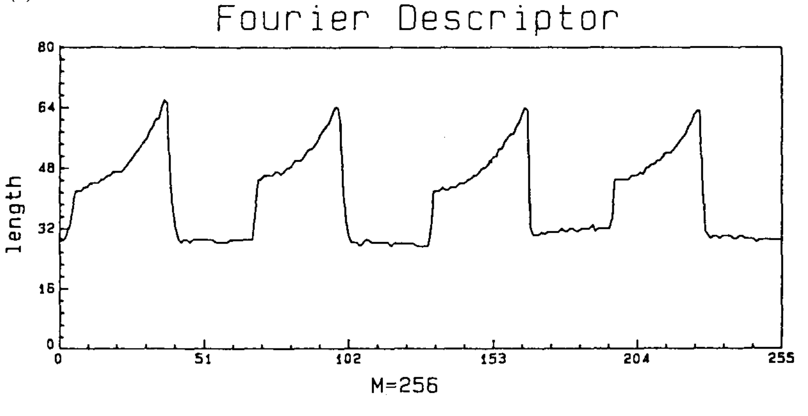


Fig. 6.

(a)



(b)

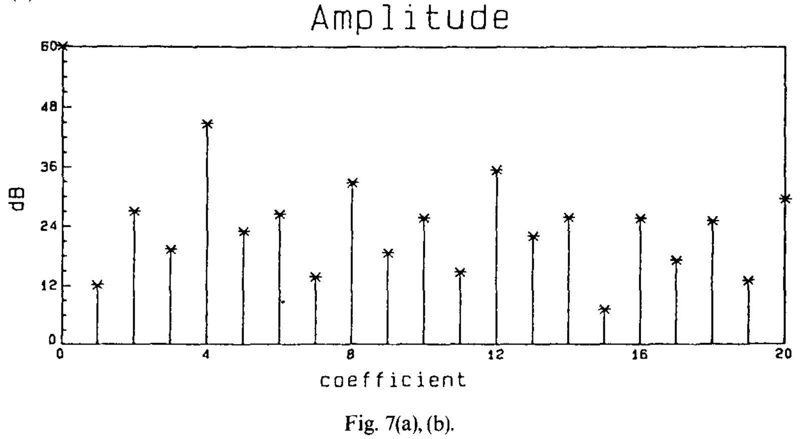


Fig. 7(a), (b).

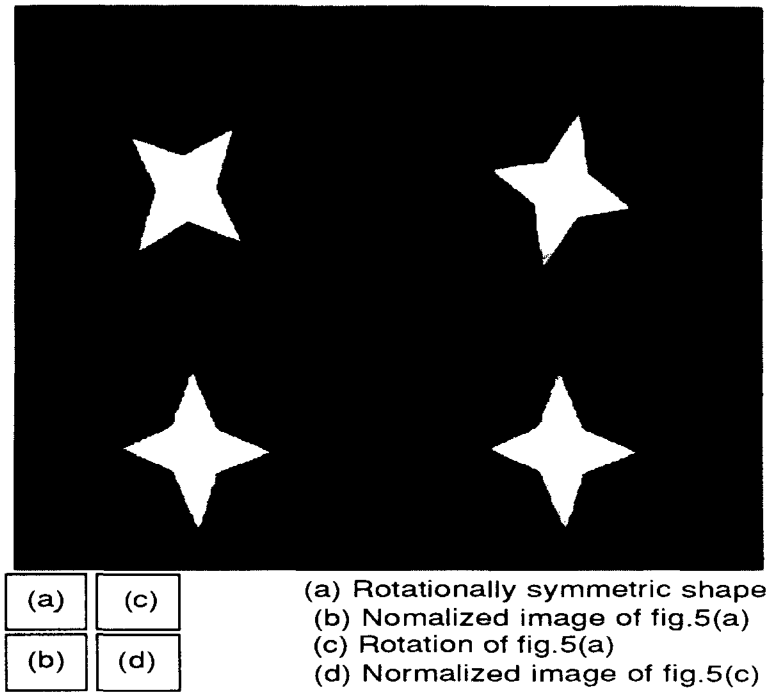

(a) (c)
(b) (d)
(a) Rotationally symmetric shape
(b) Nomalized image of fig.5(a)
(c) Rotation of fig.5(a)
(d) Normalized image of fig.5(c)

Fig. 8.

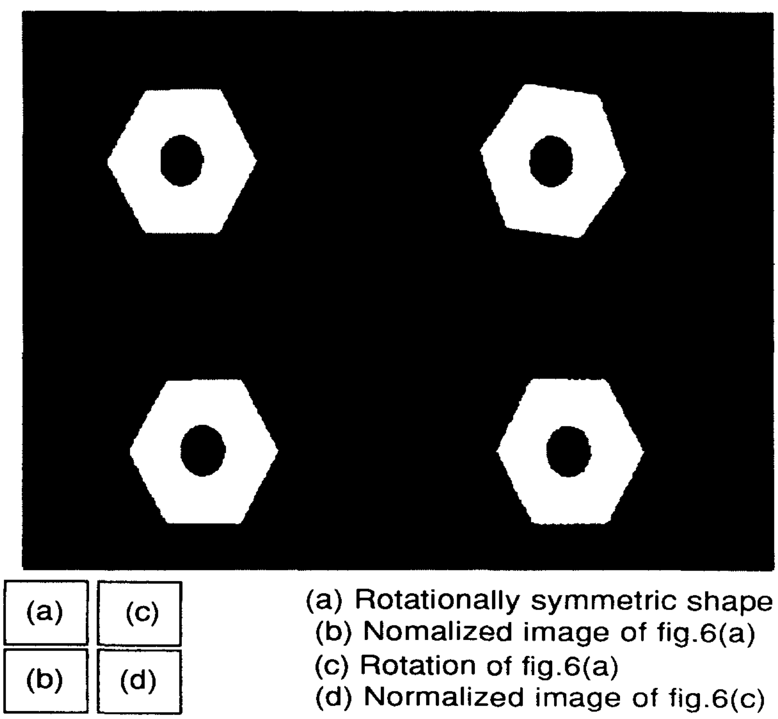

(a) (c)
(b) (d)
(a) Rotationally symmetric shape
(b) Nomalized image of fig.6(a)
(c) Rotation of fig.6(a)
(d) Normalized image of fig.6(c)

Fig. 9.

Figure 4(c) is the rotation of Fig. 4(a).

Figures 4(b) and (d) are the normalized patterns of Figs 4(a) and (c), respectively. Observing the two normalized patterns, they are the same.

(2) Figure 6(a) is a rotationally symmetric shape which is not made by computer but input by scanner. Thus, the pattern is not ideal for rotationally symmetric shapes but there is some noise.

The Fourier descriptor of Fig. 6(a) is evaluated with $M = 256$ and shown in Fig. 7(a). Then we make a 256-point DFT of the Fourier descriptor, the resulting first 20 Fourier coefficients are listed in Table 2, and plotted in Fig. 7(b) by dB. Observing Fig. 7(a), the Fourier descriptor is periodic but there is some noise. Thus, the Fourier coefficients listed in Table 2 are not the ideal case of equation (29). The noise is little compared with C_1. Thus, we can also detect the fold number and then normalize the pattern.

Figure 6(c) is the rotation of Fig. 6(a) and also is input by scanner.

Figures 6(b) and (d) are the normalized patterns of Figs 6(a) and (c), respectively. Observing the two normalized patterns, they are the same.

(3) Figures 8 and 9 are other examples.

5. CONCLUSION

Over the years, a large number of recognition methods have been developed to solve the distortion of translation, rotation and scaling. The matching method is very easy and suitable for a large pattern, but the method must normalize an image before matching. In this paper, we support the normalization algorithm listed in Section 1 to a more perfect, general and realizable state which can normalize the image under the distortion of translation, rotation, scaling, and skew. Advantages of this method are given as follows:

(1) the method is suitable when patterns are large;

(2) the normalization algorithm is easy and does not need much computation;

(3) the similarity measure by matching is rapid;

(4) the searching in the data base is efficient.

Summarizing the above analysis, image normalization is very useful in image understanding systems.

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