that feedback cannot increase the classical capacity of entanglement-breaking channels. The question of whether or not feedback can increase the capacity of memoryless quantum channels when used across entangled input states remains open.

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Algebraic Identification for Optimal Nonorthogonality 4×4 Complex Space–Time Block Codes Using Tensor Product on Quaternions

Ming-Yang Chen, Hua-Chieh Li, and Soo-Chang Pei, *Fellow, IEEE*

***Abstract*—The design potential of using quaternionic numbers to identify a 4×4 real orthogonal space–time block code has been exploited in various communication articles. Although it has been shown that orthogonal codes in full-rate exist only for 2 Tx-antennas in complex constellations, a series of complex quasi-orthogonal codes for 4 Tx-antennas is still proposed to have good performance recently. This quasi-orthogonal scheme enables the codes to reach the optimal nonorthogonality, which can be measured by taking the expectation over all transmit signals of the ratios between the powers of the off-diagonal and diagonal components. This correspondence extends the quaternionic identification to the above encoding methods. Based upon tensor product for giving the quaternionic space a linear extension, a complete necessary and sufficient condition for identifying any given complex quasi-orthogonal code with the extended space is generalized by considering every possible two-dimensional $\mathbb{R}$-algebra.**

***Index Terms*—Algebraic codes, division algebras, quasi-orthogonal space–time block code (STBC), quaternions, $\mathbb{R}$-algebras, representations of finite groups, tensor product.**

I. INTRODUCTION

In addressing the issue of decoding complexity, space–time block codes (STBCs) have the advantages of supporting higher data rates, lower decoding complexity, and relatively simpler implementations and feasibility to combat detrimental effects in fading channels. Based upon the idea of orthogonizing transmission matrices, Alamouti [1] first defined the STBC from *orthogonal design* for 2 Tx-antennas as

$$\begin{pmatrix} s_1 & s_2 \\ -s_2^\dagger & s_1^\dagger \end{pmatrix}$$

where $s^\dagger$ stands for the complex conjugate of s. Later, [3], [8]–[11], [13] present more practical settings for orthogonal designs by defining a $T \times N$ unitary matrix with each of whose entries coming from the signal set

$$\{0\} \cup \left\{\pm s_k, \pm s_k^\dagger\right\}_{k=1}^{K}.$$

Particularly, a design is of full rate and minimal delay if $K = T$ and $T = N$, respectively. Due to mainly the unitary properties, the methodology of orthogonal designs ensures full diversity equal to the number of Tx-antennas, and how simple decoding can be accomplished by a linear maximum-likelihood algorithm. However, Tarokh, Jafarkhani, and Calderbank [11] applied *Clifford algebras*, an anticommutative matrix algebras for constellation signals, to prove that a full-rate orthogonal design with minimal delay exists only for 2 Tx-antennas in complex constellations. Similar conclusion was also developed through *division algebras* by Sethuraman and Sunder Rajan [9].

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M.-Y. Chen and S.-C. Pei are with the Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan 106, R. O. C. (e-mail: roc3.chen@msa.hinet.net; pei@cc.ee.ntu.edu.tw).

H.-C. Li is with the Department of Mathematics, National Taiwan Normal University, Taipei, Taiwan 116, R. O. C. (e-mail: li@math.ntnu.edu.tw).

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It boils down to the fact that a search for similar schemes in relaxing either the conditions of achieving full-rate, minimal delay, or the pairwise orthogonalities between all columns is motivated. In [11], the problem of building rate-1 full-diversity STBCs is completely solved by showing a generic algorithm which can find delay-optimal codes for any arbitrary number of Tx-antennas. Respectively, Tirkkonen and Hottinen, [3], [13] presented a maximal achievable rate of full-diversity STBCs when minimal delay is demanded. As shown independently in their framework, enabling finding an orthogonal design for certain nonoptimal rate or time-delay condition implies the necessity of efficiently computing corresponding Clifford algebras, so the key research topics in designing STBC with full diversity become the study of anticommutative division algebras. Recently, Sethuraman, Sundar Rajan, and Shashidhar successfully synthesized the conclusions of [11], [13] by embedding cyclic division algebras extended from *Hamilton's quaternions*

$$\mathbb{H} = \{x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k} \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$$

an anticommutative division algebra of order four over real numbers $\mathbb{R}$ with three imaginary units $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ such that

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$$

and

$$\boldsymbol{i}j = -\boldsymbol{j}i = \boldsymbol{k}, \quad \boldsymbol{j}k = -\boldsymbol{k}j = \boldsymbol{i}, \quad \boldsymbol{k}i = -\boldsymbol{i}k = \boldsymbol{j}$$

into transmit matrices, [8]–[10]. As hypercomplex numbers are known to be one of the most important abstract models in various scientific and engineering areas [2], [6], [8]–[10], *Hamilton's quaternions* have naturally emerged as a fast growing potential in designing STBCs.

Nevertheless, Jafarkhani [5], Tirkkonen, Boariu, and Hottinen [12] proposed two different 4×4 complex STBCs from *quasi-orthogonal designs*, and showed the occurrences of good transmission rates and channel capacity. By a quasi-orthogonal design we mean a full-rate STBC with minimal delay such that the four transmission matrix columns are divided into two groups. While the columns within each group are not orthogonal to each other, different groups are orthogonal to each other. In a more recent article [4], Hou, Lee, and Park derived this character on 4×4 complex STBCs to complete the family of quasi-orthogonal designs and made a comparison on their performance. Moreover, a measure of nonorthogonality was proposed in [12] by taking the expectation value over all symbol constellations of the ratios between the signal powers of the off-diagonal and diagonal components. In fact, the above quasi-orthogonal scheme indeed reaches the optimal (minimal) nonorthogonality value. Generally speaking, the formulation of a quasi-orthogonal STBC is based on a real orthogonal design. For example, the four-antennas complex quasi-orthogonal code constructed in [5] has originated from the following orthogonal matrix:

$$\mathcal{C}(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{pmatrix}.$$

It was discovered in [8]–[11] that a 4×4 real orthogonal design (e.g., $\mathcal{C}(x_1, x_2, x_3, x_4)$) can be *identified* with $\mathbb{H}$ because the function f that maps $\mathcal{C}(x_1, x_2, x_3, x_4)$ to $x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k}$ is a linear transformation such that

$$f\left(\mathcal{C}(x_1, x_2, x_3, x_4) \cdot \mathcal{C}\left(x_1', x_2', x_3', x_4'\right)\right) = f(\mathcal{C}(x_1, x_2, x_3, x_4)) \cdot f\left(\mathcal{C}\left(x_1', x_2', x_3', x_4'\right)\right)$$

for all $x_j, x_j' \in \mathbb{R}$. In this correspondence, we generalize the above properties to all 4×4 complex STBCs from quasi-orthogonal designs, i.e., we are looking for all possible 4×4 complex STBCs $\mathcal{S}(s_1, s_2, s_3, s_4)$ that can be identified with an eight-dimensional $\mathbb{R}$-algebra $\mathcal{H}$, which is a linear extension of $\mathbb{H}$. In other words, if we let $s_1 = t_1 + t_2\boldsymbol{i}$, $s_2 = t_3 + t_4\boldsymbol{i}$, $s_3 = t_5 + t_6\boldsymbol{i}$, $s_4 = t_7 + t_8\boldsymbol{i}$, then the map defined by

$$g(\mathcal{S}(t_1 + t_2\boldsymbol{i}, t_3 + t_4\boldsymbol{i}, t_5 + t_6\boldsymbol{i}, t_7 + t_8\boldsymbol{i})) = \sum_{j=1}^{8} t_j(-1)^{\alpha_j}\sigma(j)$$

for all $t_j \in \mathbb{R}$ with positive integers α_j and a *bijective* map σ from $\{1, 2, \ldots, 8\}$ to a suitable basis of $\mathcal{H}$ satisfies the relation

$$g\left(\mathcal{S}(s_1, s_2, s_3, s_4) \cdot \mathcal{S}\left(s_1', s_2', s_3', s_4'\right)\right) = g(\mathcal{S}(s_1, s_2, s_3, s_4)) \cdot g\left(\mathcal{S}\left(s_1', s_2', s_3', s_4'\right)\right)$$

for all $s_j, s_j' \in \mathbb{C}$. By attempting the theory of *tensor product*, it is equivalent to consider every possible $\mathbb{R}$-algebra with dimension two in order to obtain a complete set of results. As we shall see later, the necessary and sufficient condition of whether a given 4×4 complex code can be identified with all possible tensored spaces is completed through methods in representations of finite groups. Dedicating from the graceful power of representation theory, every quasi-orthogonal which can be identified with *Hamilton's quaternions* shall be equal to one of our list (up to a reordering of the variables). Hopefully, our results will provide more insights into the original designing ideas of those widely developed schemes such as [4], [5], [12] today.

The remainder of this correspondence is organized as follows. The theories behind tensor product and $\mathbb{R}$-algebras are introduced in Section II, whose developments help us solve the main problem for all possible $\mathbb{R}$-algebras with dimension two in Section III. Section IV concludes the correspondence with some remarks and future work.

II. Mathematical Principles

In this section, we present the linear extension of an algebraic space. in Section II-A, we show the definition of tensor product, combining a full search of two-dimensional $\mathbb{R}$-algebras to discuss all cases which are associated with $\mathbb{H}$.

A. Tensor Product

Functioning as the given name, a map $f : \mathbb{U} \times \mathbb{V} \rightarrow \mathbb{W}$, where $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{W}$ are vector spaces over $\mathbb{R}$, is called *bilinear* if and only if f is a linear transformation whenever one of the variables $\boldsymbol{u}, \boldsymbol{v}$ is fixed in $f(\boldsymbol{u}, \boldsymbol{v})$. The origin of the tensor product lies in classic differential geometry and physics, which had need of multiplying indexed geometric objects such as the first and second fundamental forms, the stress tensor, and so on. However, in mathematics, the subject of linear algebras is upgraded to bilinear algebras by introducing the tensor product of two vector spaces. First of all, let us see the definition of tensor product from its universal property given in the sense of bilinear maps.

Definition 2.1 (Tensor Product): Given two vector spaces $\mathbb{U}$ and $\mathbb{V}$ over $\mathbb{R}$. The tensor product $\mathbb{U} \otimes \mathbb{V}$ is a vector space $\mathbb{T}$ over $\mathbb{R}$, for which there exists a bilinear map $\beta : \mathbb{U} \times \mathbb{V} \rightarrow \mathbb{T}$ satisfying the *universal property*: for every bilinear map $\beta' : \mathbb{U} \times \mathbb{V} \rightarrow \mathbb{W}$, there is a unique linear transformation $T_{\beta'} : \mathbb{T} \rightarrow \mathbb{W}$ such that

$$T_{\beta'} \circ \beta = \beta'$$

where $\mathbb{W}$ is also a vector space over $\mathbb{R}$.

$$
\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & s_4^\dagger & -s_3^\dagger \\ s_3 & -s_4 & s_1 & -s_2 \\ -s_4^\dagger & -s_3^\dagger & s_2^\dagger & s_1^\dagger \end{pmatrix}
\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & -s_4^\dagger & s_3^\dagger \\ -s_3^\dagger & -s_4^\dagger & s_1^\dagger & s_2^\dagger \\ s_4 & -s_3 & -s_2 & s_1 \end{pmatrix}
\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_1 & -s_4 & -s_3 \\ -s_3^\dagger & s_4^\dagger & s_1^\dagger & -s_2^\dagger \\ -s_4^\dagger & s_3^\dagger & -s_2^\dagger & s_1^\dagger \end{pmatrix}
$$
$$
\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & -s_4^\dagger & s_3^\dagger \\ s_3 & s_4 & s_1 & s_2 \\ -s_4^\dagger & s_3^\dagger & -s_2^\dagger & s_1^\dagger \end{pmatrix}
\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & s_4^\dagger & -s_3^\dagger \\ -s_3^\dagger & s_4^\dagger & s_1^\dagger & -s_2^\dagger \\ s_4 & s_3 & s_2 & s_1 \end{pmatrix}
\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_1 & s_4 & s_3 \\ -s_3^\dagger & -s_4^\dagger & s_1^\dagger & s_2^\dagger \\ -s_4^\dagger & -s_3^\dagger & s_2^\dagger & s_1^\dagger \end{pmatrix}
$$

Fig. 1. Solution set corresponding to basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4\}$.

$$
\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & -s_4^\dagger & s_3^\dagger \\ -s_3 & s_4 & s_1 & -s_2 \\ -s_4^\dagger & -s_3^\dagger & s_2^\dagger & s_1^\dagger \end{pmatrix}
\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & -s_4^\dagger & s_3^\dagger \\ -s_3^\dagger & s_4^\dagger & s_1^\dagger & -s_2^\dagger \\ -s_4 & -s_3 & s_2 & s_1 \end{pmatrix}
\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2 & s_1 & -s_4 & s_3 \\ -s_3^\dagger & s_4^\dagger & s_1^\dagger & -s_2^\dagger \\ -s_4^\dagger & -s_3^\dagger & s_2^\dagger & s_1^\dagger \end{pmatrix}
$$
$$
\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & s_4^\dagger & -s_3^\dagger \\ -s_3 & -s_4 & s_1 & s_2 \\ -s_4^\dagger & s_3^\dagger & -s_2^\dagger & s_1^\dagger \end{pmatrix}
\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & s_4^\dagger & -s_3^\dagger \\ -s_3^\dagger & -s_4^\dagger & s_1^\dagger & s_2^\dagger \\ -s_4 & s_3 & -s_2 & s_1 \end{pmatrix}
\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2 & s_1 & s_4 & -s_3 \\ -s_3^\dagger & -s_4^\dagger & s_1^\dagger & s_2^\dagger \\ -s_4^\dagger & s_3^\dagger & -s_2^\dagger & s_1^\dagger \end{pmatrix}
$$

Fig. 2. Solution set corresponding to basis $\{\boldsymbol{b}_1, \boldsymbol{b}_3, \boldsymbol{b}_5, \boldsymbol{b}_7\}$.

If $\mathbb{U}$ and $\mathbb{V}$ are two $\mathbb{R}$-algebras and we want to make a multiplication rule between $\mathbb{U}$ and $\mathbb{V}$, then the distribution rule enforces that this multiplication rule is a bilinear map on $\mathbb{U} \times \mathbb{V}$. Therefore, by Definition 2.1, we can make $\mathbb{U} \otimes \mathbb{V}$ an $\mathbb{R}$-algebra with multiplication defined by

$$(\boldsymbol{u}_1 \otimes \boldsymbol{v}_1) \cdot (\boldsymbol{u}_2 \otimes \boldsymbol{v}_2) = (\boldsymbol{u}_1 \cdot \boldsymbol{u}_2) \otimes (\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)$$

for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{U}$, $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{V}$, and extend linearly to $\mathbb{U} \otimes \mathbb{V}$. In other words, the tensor product is a formal bilinear multiplication of two vector spaces. Moreover, the tensor product is uniquely specified by the universal property (up to an isomorphism). We sort these properties in Lemma 2.2. See [7] for the proof.

Lemma 2.2: The tensor product is uniquely determined by the given vector spaces $\mathbb{U}$ and $\mathbb{V}$ (up to an isomorphism). Moreover, if $\mathbb{U}$ and $\mathbb{V}$ are finite dimensional with bases $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$, respectively, then

$$\{\boldsymbol{u}_i \otimes \boldsymbol{v}_j \mid 1 \le i \le m, 1 \le j \le n\}$$

is a basis of $\mathbb{U} \otimes \mathbb{V}$ and $\dim_{\mathbb{R}} \mathbb{T} = mn$.

Since we are looking for all linear extensions of $\mathbb{H}$ which are $\mathbb{R}$-algebras of dimension eight, by Lemma 2.2, we only have to consider those algebras of the form $\mathbb{H} \otimes \mathbb{L}$, where $\mathbb{L}$ is an $\mathbb{R}$-algebra of dimension two, as presented in Lemma 2.3.

B. $\mathbb{R}$-Algebras

Let $\mathbb{C}$ be the field of complex numbers. It is clear that $\mathbb{C}$ is an $\mathbb{R}$-algebra of dimension two with basis $\{1, \boldsymbol{i}\}$ and multiplication defined by $\boldsymbol{i}^2 = -1$. There are two other nonisomorphic $\mathbb{R}$-algebras. One is $\mathbb{E}$, which is an $\mathbb{R}$-algebra of dimension two with basis $\{1, \boldsymbol{e}\}$ and multiplication defined by $\boldsymbol{e}^2 = 1$. The other is $\mathbb{F}$, which is an $\mathbb{R}$-algebra of dimension two with basis $\{1, \boldsymbol{f}\}$ and multiplication defined by $\boldsymbol{f}^2 = 0$. In what follows, we show that these are the only $\mathbb{R}$-algebras of dimension two.

Lemma 2.3: Every two-dimensional $\mathbb{R}$-algebra is isomorphic to one of $\mathbb{C}$, $\mathbb{E}$, and $\mathbb{F}$.

Proof: In fact, suppose that $\mathbb{L}$ is a two-dimensional $\mathbb{R}$-algebra with basis $\{1, \boldsymbol{l}\}$. Since $\mathbb{L}$ is an $\mathbb{R}$-algebra, we have $\boldsymbol{l}^2 = c + d\boldsymbol{l}$ with $c, d \in \mathbb{R}$. Our goal is to find an element $a + b\boldsymbol{l}$ in $\mathbb{L}$ which is linearly independent with 1 (i.e., $b \neq 0$) such that $(a + b\boldsymbol{l})^2$ is either $-1, 1,$ or 0. Considering

$$(a + b\boldsymbol{l})^2 = a^2 + 2ab\boldsymbol{l} + b^2(c + d\boldsymbol{l}) = (a^2 + b^2 c) + (2ab + b^2 d)\boldsymbol{l}.$$

Let $a = -\frac{bd}{2}$ then the imaginary part is eliminated and the real part becomes

$$b^2 \left(\frac{d^2}{4} + c \right).$$

Therefore, if $d^2 + 4c > 0$ then we can find $b \neq 0$ in $\mathbb{R}$ such that $(a + b\boldsymbol{l})^2 = 1$. Similarly, we can find b such that $(a + b\boldsymbol{l})^2 = -1$ or $(a + b\boldsymbol{l})^2 = 0$ if $d^2 + 4c < 0$ or $d^2 + 4c = 0$, respectively. □

III. The 4×4 Complex STBCs From Quasi-Orthogonal Designs and Quaternions

In this section, we consider the main existence problem. In order to let our results be complete, we separate each possible case of tensor products between $\mathbb{H}$- and the $\mathbb{R}$-algebras obtained in Lemma 2.3 for individual analysis. Due to the properties of representation theory, every quasi-orthogonal STBC which can be identified with $\mathbb{H}$ shall be equal to one of our final list (up to a reordering of the variables).

Lemma 3.1: A 4×4 complex STBC from quasi-orthogonal design can be identified with $\mathbb{H}$ only if it is equal to one of the matrices listed in Figs. 1 and 2 (up to a reordering of the variables).

Proof: By Lemma 2.3, it suffices for us to look for all possible 4×4 complex STBCs which can be identified with $\mathbb{H} \otimes \mathbb{C}$, $\mathbb{H} \otimes \mathbb{E}$, or $\mathbb{H} \otimes \mathbb{F}$.

- *Part A:* $\mathbb{H} \otimes \mathbb{E}$

Lemma 2.2 implies that

$$\boldsymbol{b}_1 = 1 \otimes 1, \boldsymbol{b}_2 = 1 \otimes \boldsymbol{e}, \boldsymbol{b}_3 = \boldsymbol{i} \otimes 1, \boldsymbol{b}_4 = \boldsymbol{i} \otimes \boldsymbol{e}$$
$$\boldsymbol{b}_5 = \boldsymbol{j} \otimes 1, \boldsymbol{b}_6 = \boldsymbol{j} \otimes \boldsymbol{e}, \boldsymbol{b}_7 = \boldsymbol{k} \otimes 1, \boldsymbol{b}_8 = \boldsymbol{k} \otimes \boldsymbol{e}$$

is a basis of $\mathbb{H} \otimes \mathbb{E}$. The product on $\mathbb{H} \otimes \mathbb{E}$ is defined as

$$(\boldsymbol{h}_1 \otimes \boldsymbol{e}_1)(\boldsymbol{h}_2 \otimes \boldsymbol{e}_2) = (\boldsymbol{h}_1 \boldsymbol{h}_2) \otimes (\boldsymbol{e}_1 \boldsymbol{e}_2)$$

for all $\boldsymbol{h}_1, \boldsymbol{h}_2 \in \mathbb{H}$ and $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{E}$. The general definition [11] of a rate-1 complex STBC $\mathcal{S}(s_1, s_2, \ldots, s_n)$ is an $n \times n$ matrix with entries the indeterminates $\pm s_1, \pm s_2, \ldots, \pm s_n$ and their conjugates

$\pm s_1^\dagger, \pm s_2^\dagger, \ldots, \pm s_n^\dagger$. Without loss of generality, we may assume that the first row of $\mathcal{S}$ is $s_1, s_2, \ldots, s_n$. Suppose that

$$\mathcal{S}\,(s_1, s_2, s_3, s_4) = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

is a complex STBC which can be identified with $\mathbb{H} \otimes \mathbb{E}$, where the $*$'s present these unknown indeterminates. Recalling that we are looking for all 4×4 complex STBCs $\mathcal{S}(s_1, s_2, s_3, s_4)$ such that the map defined by

$$g(\mathcal{S}(t_1 + t_2\boldsymbol{i}, t_3 + t_4\boldsymbol{i}, t_5 + t_6\boldsymbol{i}, t_7 + t_8\boldsymbol{i})) = \sum_{j=1}^{8} t_j(-1)^{\alpha_j}\sigma(j)$$

for all $t_j \in \mathbb{R}$ with positive integers α_j and a bijective map σ from $\{1, 2, \ldots, 8\}$ to $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_8\}$ satisfies the relation

$$\begin{aligned} &g(\mathcal{S}(s_1, s_2, s_3, s_4) \cdot \mathcal{S}(s_1', s_2', s_3', s_4')) \\ &\qquad = g(\mathcal{S}(s_1, s_2, s_3, s_4)) \cdot g(\mathcal{S}(s_1', s_2', s_3', s_4')) \end{aligned}$$

for all $s_j, s_j' \in \mathbb{C}$ where $s_1 = t_1 + t_2\boldsymbol{i}$, $s_2 = t_3 + t_4\boldsymbol{i}$, $s_3 = t_5 + t_6\boldsymbol{i}$, $s_4 = t_7 + t_8\boldsymbol{i}$. Since the product of two matrices with real entries is also a matrix with real entries, the image of all $\mathcal{S}(s_1, s_2, s_3, s_4)$ with $s_1, s_2, s_3, s_4 \in \mathbb{R}$ in $\mathbb{H} \otimes \mathbb{E}$ is a subspace $\mathbb{A}$ of dimension four over $\mathbb{R}$ whose elements are closed under multiplication. At this step, we may find all possible bases for $\mathbb{A}$. For example, because identity must be in $\mathbb{A}$, we have $\boldsymbol{b}_1 \in \mathbb{A}$. Suppose that $\boldsymbol{b}_2$ and $\boldsymbol{b}_3$ are in $\mathbb{A}$. Then by

$$\boldsymbol{b}_2 \cdot \boldsymbol{b}_3 = (1 \otimes \boldsymbol{e}) \cdot (\boldsymbol{i} \otimes 1) = \boldsymbol{i} \otimes \boldsymbol{e} = \boldsymbol{b}_4$$

we know that $\boldsymbol{b}_4$ is also in $\mathbb{A}$. Therefore, $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4\}$ is a possible basis of $\mathbb{A}$. Using similar methods, we obtain that the basis of $\mathbb{A}$ has only two possibilities among $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_8\}$ (omitting symmetrical combinations)

$$\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4\}, \{\boldsymbol{b}_1, \boldsymbol{b}_3, \boldsymbol{b}_5, \boldsymbol{b}_7\}.$$

∘ *Case i):* the basis of $\mathbb{A}$ is $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4\}$.

Let M_1 be the identity of $\{\mathcal{S}\,(s_1, s_2, s_3, s_4) \mid s_1, s_2, s_3, s_4 \in \mathbb{R}\}$ with the conventional matrix multiplication. We must have

$$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \cdot M_1 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

for all $s_1, s_2, s_3, s_4 \in \mathbb{R}$. It is easy to check that M_1 must be the identity matrix. By $M_1 \in \{\mathcal{S}\,(s_1, s_2, s_3, s_4) \mid s_1, s_2, s_3, s_4 \in \mathbb{R}\}$, we get that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M_1 = \mathcal{S}(1, 0, 0, 0).$$

Therefore, $\mathcal{S}\,(s_1, s_2, s_3, s_4)$ must be of the form

$$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ * & s_1 & * & * \\ * & * & s_1 & * \\ * & * & * & s_1 \end{pmatrix}$$

where now $*$'s do not include any indeterminates concerned about s_1. Moreover, if $\mathcal{S}\,(s_1, s_2, s_3, s_4)$ is identified with the representation of $\mathbb{A}$

$$s_1(-1)^{\beta_1}\sigma(1) + s_2(-1)^{\beta_2}\sigma(2) + s_3(-1)^{\beta_3}\sigma(3) + s_4(-1)^{\beta_4}\sigma(4)$$

where each β_i means a positive integer and the map σ from $\{1, 2, 3, 4\}$ to $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4\}$ is a bijective function. Since $1 \otimes 1$ is the identity of multiplication on $\mathbb{H} \otimes \mathbb{E}$, we get that $(-1)^{\beta_1}\sigma(1) = 1 \otimes 1$. Note that

$$\begin{aligned} M_2 &= \mathcal{S}(0, 1, 0, 0) \\ M_3 &= \mathcal{S}(0, 0, 1, 0) \\ M_4 &= \mathcal{S}(0, 0, 0, 1) \end{aligned}$$

are the matrix representations corresponding to $(-1)^{\beta_2}\sigma(2)$, $(-1)^{\beta_3}\sigma(3), (-1)^{\beta_4}\sigma(4)$, respectively. Since $\boldsymbol{b}_2^2 = 1 \otimes 1$, and $\boldsymbol{b}_3^2 = \boldsymbol{b}_4^2 = -(1 \otimes 1)$, by

$$\sigma(2), \sigma(3), \sigma(4) \in \{\boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4\}$$

there is exactly one $\sigma(i)$ among $\sigma(2), \sigma(3), \sigma(4)$ such that $\sigma(i)^2 = \boldsymbol{b}_1 = 1 \otimes 1$ and the others have squared value equal to $-(1 \otimes 1)$. Suppose that $\sigma(2)^2 = 1 \otimes 1$ (i.e., $\sigma(2) = \boldsymbol{b}_2$). Thus,

$$M_2^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ * & 0 & * & * \\ * & * & 0 & * \\ * & * & * & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that

$$M_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Therefore, we obtain $\mathcal{S}\,(s_1, s_2, s_3, s_4)$ as

$$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_1 & * & * \\ * & * & s_1 & s_2 \\ * & * & s_2 & s_1 \end{pmatrix} \text{ or } \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_1 & * & * \\ * & * & s_1 & -s_2 \\ * & * & -s_2 & s_1 \end{pmatrix}.$$

Similarly, by $M_3^2 = -(1 \otimes 1)$ (e.g., $\sigma(3) = \boldsymbol{b}_3$) we have

$$M_3^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and hence,

$$M_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Now, using the fact that

$$(-1)^{\beta_2}\sigma(2)(-1)^{\beta_3}\sigma(3) = (-1)^{\beta_3}\sigma(3)(-1)^{\beta_2}\sigma(2)$$

we get

$$M_2 M_3 = M_3 M_2.$$

It follows that (M_2, M_3) should be

$$\left(\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}\right)$$

or

$$\left(\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}\right).$$

In other words, we have $\mathcal{S}(s_1, s_2, s_3, s_4)$ as

$$\begin{pmatrix}s_1&s_2&s_3&s_4\\s_2&s_1&*&s_3\\-s_3&*&s_1&s_2\\*&-s_3&s_2&s_1\end{pmatrix}\text{ or }\begin{pmatrix}s_1&s_2&s_3&s_4\\s_2&s_1&*&-s_3\\-s_3&*&s_1&-s_2\\*&s_3&-s_2&s_1\end{pmatrix}.$$

Furthermore, in virtue of

$$(-1)^{\beta_2}\sigma(2)(-1)^{\beta_3}\sigma(3) = \pm(-1)^{\beta_4}\sigma(4)$$

we obtain $M_2M_3 = \pm M_4$. Following the same procedures we summarize that $\mathcal{S}(s_1, s_2, s_3, s_4)$ is of the forms

$$\begin{pmatrix}s_1&s_2&s_3&s_4\\s_2&s_1&s_4&s_3\\-s_3&-s_4&s_1&s_2\\-s_4&-s_3&s_2&s_1\end{pmatrix}\text{ or }\begin{pmatrix}s_1&s_2&s_3&s_4\\s_2&s_1&-s_4&-s_3\\-s_3&s_4&s_1&-s_2\\-s_4&s_3&-s_2&s_1\end{pmatrix}.$$

Similarly, for the cases of $\sigma(3)^2 = 1\otimes 1$ and $\sigma(4)^2 = 1\otimes 1$, we get $\mathcal{S}(s_1, s_2, s_3, s_4)$ as

$$\begin{pmatrix}s_1&s_2&s_3&s_4\\-s_2&s_1&-s_4&s_3\\s_3&s_4&s_1&s_2\\-s_4&s_3&-s_2&s_1\end{pmatrix}\text{ or }\begin{pmatrix}s_1&s_2&s_3&s_4\\-s_2&s_1&s_4&-s_3\\s_3&-s_4&s_1&-s_2\\-s_4&-s_3&s_2&s_1\end{pmatrix}$$

and

$$\begin{pmatrix}s_1&s_2&s_3&s_4\\-s_2&s_1&s_4&-s_3\\-s_3&s_4&s_1&-s_2\\s_4&s_3&s_2&s_1\end{pmatrix}\text{ or }\begin{pmatrix}s_1&s_2&s_3&s_4\\-s_2&s_1&-s_4&s_3\\-s_3&-s_4&s_1&s_2\\s_4&-s_3&-s_2&s_1\end{pmatrix}$$

respectively.

Up to now we have successfully determined the distributions of positive and negative entries in $\mathcal{S}$. To deal with the complex conjugate, we remind that the definition of a quasi-orthogonal design is that the four transmission matrix columns are able to be divided into two groups, while the columns within each group are not orthogonal to each other, different groups are orthogonal to each other. Taking one of the above results

$$\mathcal{S}(s_1, s_2, s_3, s_4) = \begin{pmatrix}s_1&s_2&s_3&s_4\\-s_2&s_1&s_4&-s_3\\s_3&-s_4&s_1&-s_2\\-s_4&-s_3&s_2&s_1\end{pmatrix}$$

as an instance, it means that the nonorthogonal pairs must be (column 1, column 3) and (column 2, column 4) due to the distribution of negative entries. Recall that two column vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal if and only if

$$\boldsymbol{v}_1^\dagger \boldsymbol{v}_2 = 0.$$

From the fact that column 1 and column 2 are orthogonal, we get the entries $(-s_2)$ on coordinate position $(2,1)$ and s_1 on coordinate position $(2,2)$ must be added conjugates, and from the fact that column 3 and column 4 are orthogonal, the entries on coordinate positions $(2,3)$ and $(2,4)$ must be added conjugates. Likewise, all the entries on the forth row must also be added conjugates. If the entry on coordinate position $(3,1)$ were also added a conjugate, column 1 and column 4 would be no longer orthogonal. Similarly, all the other entries on the third row must not be added any conjugate. The corresponding matrix becomes

$$\begin{pmatrix}s_1&s_2&s_3&s_4\\-s_2^\dagger&s_1^\dagger&s_4^\dagger&-s_3^\dagger\\s_3&-s_4&s_1&-s_2\\-s_4^\dagger&-s_3^\dagger&s_2^\dagger&s_1^\dagger\end{pmatrix}.$$

Projecting the same methods on the other five matrices, we conclude that as longer as the basis of $\mathbb{A}$ is $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4\}$, a 4×4 complex STBCs can be identified with $\mathbb{H}\otimes\mathbb{E}$ only if it is one of those listed in Fig. 1.

∘ *Case ii)*: the basis of $\mathbb{A}$ is $\{\boldsymbol{b}_1, \boldsymbol{b}_3, \boldsymbol{b}_5, \boldsymbol{b}_7\}$.

Applying all the same methods of dealing with *i)* we again obtain that a 4×4 complex STBCs can be identified with $\mathbb{H}\otimes\mathbb{E}$ only if it is one of those listed in Fig. 2 as longer as the basis of $\mathbb{A}$ is $\{\boldsymbol{b}_1, \boldsymbol{b}_3, \boldsymbol{b}_5, \boldsymbol{b}_7\}$.

• *Part B:* $\mathbb{H}\otimes\mathbb{C}$

Now, the basis of $\mathbb{A}$ has only three possibilities (again, omitting symmetrical combinations)

$$\{1\otimes 1, 1\otimes\boldsymbol{i}, \boldsymbol{i}\otimes 1, \boldsymbol{i}\otimes\boldsymbol{i}\}$$
$$\{1\otimes 1, \boldsymbol{i}\otimes 1, \boldsymbol{j}\otimes 1, \boldsymbol{k}\otimes 1\}$$
$$\{1\otimes 1, \boldsymbol{i}\otimes 1, \boldsymbol{j}\otimes\boldsymbol{i}, \boldsymbol{k}\otimes\boldsymbol{i}\}.$$

Applying representation theory to the former two cases gives us exact the same results as in *Case i)* and *Case ii)* of *Part A*, respectively. If the basis of $\mathbb{A}$ is $\{1\otimes 1, \boldsymbol{i}\otimes 1, \boldsymbol{j}\otimes\boldsymbol{i}, \boldsymbol{k}\otimes\boldsymbol{i}\}$, then like the analysis of *Case i)* of *Part A*, we obtain $\sigma(2), \sigma(3), \sigma(4) \in \{\boldsymbol{i}\otimes 1, \boldsymbol{j}\otimes\boldsymbol{i}, \boldsymbol{k}\otimes\boldsymbol{i}\}$. Hence, there is exactly one $\sigma(i)$ among $\sigma(2), \sigma(3), \sigma(4)$ such that $\sigma(i)^2 = -(1\otimes 1)$ and the others have squared value equal to $1\otimes 1$. Suppose that $\sigma(2)^2 = -(1\otimes 1)$. We obtain the matrices to be

$$\begin{pmatrix}s_1&s_2&s_3&s_4\\-s_2&s_1&s_4&-s_3\\s_3&s_4&s_1&s_2\\s_4&-s_3&-s_2&s_1\end{pmatrix}$$

or

$$\begin{pmatrix}s_1&s_2&s_3&s_4\\-s_2&s_1&-s_4&s_3\\s_3&-s_4&s_1&-s_2\\s_4&s_3&s_2&s_1\end{pmatrix}.$$

However, both situations cannot reach the definition of quasi-orthogonal designs because of the distributions of their negative entries. Similar claim also holds as $\sigma(3)^2 = -(1\otimes 1)$ or $\sigma(4)^2 = -(1\otimes 1)$. It means this case constitutes nothing in our solution set.

• *Part C:* $\mathbb{H}\otimes\mathbb{F}$

Since $(\boldsymbol{h}\otimes\boldsymbol{f})^2 = 0$ for all $\boldsymbol{h}\in\mathbb{H}$, the only one meaningful basis of $\mathbb{A}$ is

$$\{1\otimes 1, \boldsymbol{i}\otimes 1, \boldsymbol{j}\otimes 1, \boldsymbol{k}\otimes 1\}.$$

Again, this would not give as any new solutions further. The desired conclusion is proved. □

Theorem 3.2: A 4×4 complex STBC from quasi-orthogonal design can be identified with $\mathbb{H}$ if and only if it is equal to one of the

The Jafarkhani 4×4 quasi-orthogonal STBC series [5]: presenting higher transmission rates and larger channel capacity	
$\mathcal{S}(s_1, s_2, s_3, s_4)$	$g(\mathcal{S}(t_1+t_2\mathbf{i}, t_3+t_4\mathbf{i}, t_5+t_6\mathbf{i}, t_7+t_8\mathbf{i}))$
$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & s_4^\dagger & -s_3^\dagger \\ s_3 & -s_4 & s_1 & -s_2 \\ -s_4^\dagger & -s_3^\dagger & s_2^\dagger & s_1^\dagger \end{pmatrix}$	$t_1\mathbf{b}_1 + t_2\mathbf{b}_7 + t_3\mathbf{b}_3 + t_4\mathbf{b}_5 - t_5\mathbf{b}_2 - t_6\mathbf{b}_8 + t_7\mathbf{b}_4 + t_8\mathbf{b}_6$
$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & -s_4^\dagger & s_3^\dagger \\ -s_3^\dagger & -s_4^\dagger & s_1^\dagger & s_2^\dagger \\ s_4 & -s_3 & -s_2 & s_1 \end{pmatrix}$	$t_1\mathbf{b}_1 + t_2\mathbf{b}_7 + t_3\mathbf{b}_3 + t_4\mathbf{b}_5 + t_5\mathbf{b}_4 + t_6\mathbf{b}_6 - t_7\mathbf{b}_2 - t_8\mathbf{b}_8$
$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_1 & -s_4 & -s_3 \\ -s_3^\dagger & s_4^\dagger & s_1^\dagger & -s_2^\dagger \\ -s_4^\dagger & s_3^\dagger & -s_2^\dagger & s_1^\dagger \end{pmatrix}$	$t_1\mathbf{b}_1 + t_2\mathbf{b}_7 - t_3\mathbf{b}_2 - t_4\mathbf{b}_8 + t_5\mathbf{b}_3 + t_6\mathbf{b}_5 + t_7\mathbf{b}_4 + t_8\mathbf{b}_6$
The TBH ABBA code series [12]: presenting full coding rates with minimal non-orthogonality	
$\mathcal{S}(s_1, s_2, s_3, s_4)$	$g(\mathcal{S}(t_1+t_2\mathbf{i}, t_3+t_4\mathbf{i}, t_5+t_6\mathbf{i}, t_7+t_8\mathbf{i}))$
$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & -s_4^\dagger & s_3^\dagger \\ s_3 & s_4 & s_1 & s_2 \\ -s_4^\dagger & s_3^\dagger & -s_2^\dagger & s_1^\dagger \end{pmatrix}$	$t_1\mathbf{b}_1 + t_2\mathbf{b}_7 + t_3\mathbf{b}_3 + t_4\mathbf{b}_5 + t_5\mathbf{b}_2 + t_6\mathbf{b}_8 + t_7\mathbf{b}_4 + t_8\mathbf{b}_6$
$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & s_4^\dagger & -s_3^\dagger \\ -s_3^\dagger & s_4^\dagger & s_1^\dagger & -s_2^\dagger \\ s_4 & s_3 & s_2 & s_1 \end{pmatrix}$	$t_1\mathbf{b}_1 + t_2\mathbf{b}_7 + t_3\mathbf{b}_3 + t_4\mathbf{b}_5 + t_5\mathbf{b}_4 + t_6\mathbf{b}_6 + t_7\mathbf{b}_2 + t_8\mathbf{b}_8$
$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_1 & s_4 & s_3 \\ -s_3^\dagger & -s_4^\dagger & s_1^\dagger & s_2^\dagger \\ -s_4^\dagger & -s_3^\dagger & s_2^\dagger & s_1^\dagger \end{pmatrix}$	$t_1\mathbf{b}_1 + t_2\mathbf{b}_7 + t_3\mathbf{b}_2 + t_4\mathbf{b}_8 + t_5\mathbf{b}_3 + t_6\mathbf{b}_5 + t_7\mathbf{b}_4 + t_8\mathbf{b}_6$

Fig. 3. The 4×4 quasi-orthogonal designs with corresponding representations of $\mathbb{H}$ *(Case i))*.

matrices listed in Figs. 1 and 2 (up to a reordering of the variables s_1, s_2, s_3, s_4). Moreover, it is isomorphic to $\mathbb{H} \otimes \mathbb{E}$.

Proof: The necessity is completed by Lemma 3.1. To show the sufficiency, we have to give each matrix in Figs. 1 and 2 a corresponding representation of $\mathbb{H} \otimes \mathbb{E}$, as shown in Figs. 3 and 4. For example, when

$$\mathcal{S}(s_1, s_2, s_3, s_4) = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_1 & s_4 & s_3 \\ -s_3^\dagger & -s_4^\dagger & s_1^\dagger & s_2^\dagger \\ -s_4^\dagger & -s_3^\dagger & s_2^\dagger & s_1^\dagger \end{pmatrix}$$

from the proof of Lemma 3.1 we can let

$$g(\mathcal{S}(1,0,0,0)) = \boldsymbol{b}_1 = 1 \otimes 1$$
$$g(\mathcal{S}(0,1,0,0)) = \boldsymbol{b}_2 = 1 \otimes \boldsymbol{e}$$
$$g(\mathcal{S}(0,0,1,0)) = \boldsymbol{b}_3 = \boldsymbol{i} \otimes 1$$

and

$$g(\mathcal{S}(0,0,0,1)) = \boldsymbol{b}_4 = \boldsymbol{i} \otimes \boldsymbol{e}.$$

Suppose that $g(\mathcal{S}(\boldsymbol{i},0,0,0)) = \boldsymbol{b}_7 = \boldsymbol{k} \otimes 1$. Then using

$$\begin{aligned} &\mathcal{S}(\boldsymbol{i},0,0,0) \cdot \mathcal{S}(0,1,0,0) \\ &= \begin{pmatrix} \boldsymbol{i} & 0 & 0 & 0 \\ 0 & \boldsymbol{i} & 0 & 0 \\ 0 & 0 & -\boldsymbol{i} & 0 \\ 0 & 0 & 0 & -\boldsymbol{i} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \boldsymbol{i} & 0 & 0 \\ \boldsymbol{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\boldsymbol{i} \\ 0 & 0 & -\boldsymbol{i} & 0 \end{pmatrix} \\ &= \mathcal{S}(0,\boldsymbol{i},0,0) \end{aligned}$$

and

$$\boldsymbol{b}_7 \cdot \boldsymbol{b}_2 = (\boldsymbol{k} \otimes 1) \cdot (1 \otimes \boldsymbol{e}) = \boldsymbol{k} \otimes \boldsymbol{e} = \boldsymbol{b}_8$$

we know that $g(\mathcal{S}(0,\boldsymbol{i},0,0)) = \boldsymbol{b}_8$. Due to the same reasons we find

$$\begin{aligned} g(\mathcal{S}(0,0,\boldsymbol{i},0)) &= g(\mathcal{S}(\boldsymbol{i},0,0,0) \cdot \mathcal{S}(0,0,1,0)\big) \\ &= g(\mathcal{S}(\boldsymbol{i},0,0,0)) \cdot g(\mathcal{S}(0,0,1,0)) \\ &= \boldsymbol{b}_5 \end{aligned}$$

and

$$\begin{aligned} g(\mathcal{S}(0,0,0,\boldsymbol{i})) &= g(\mathcal{S}(\boldsymbol{i},0,0,0) \cdot \mathcal{S}(0,0,0,1)\big) \\ &= g(\mathcal{S}(\boldsymbol{i},0,0,0)) \cdot g(\mathcal{S}(0,0,0,1)) \\ &= \boldsymbol{b}_6. \end{aligned}$$

The corresponding map

$$\begin{aligned} &g(\mathcal{S}(t_1+t_2\boldsymbol{i}, t_3+t_4\boldsymbol{i}, t_5+t_6\boldsymbol{i}, t_7+t_8\boldsymbol{i})) \\ &\quad = t_1\boldsymbol{b}_1 + t_2\boldsymbol{b}_7 + t_3\boldsymbol{b}_2 + t_4\boldsymbol{b}_8 + t_5\boldsymbol{b}_3 + t_6\boldsymbol{b}_5 + t_7\boldsymbol{b}_4 + t_8\boldsymbol{b}_6 \end{aligned}$$

where $s_1 = t_1 + t_2\boldsymbol{i}$, $s_2 = t_3 + t_4\boldsymbol{i}$, $s_3 = t_5 + t_6\boldsymbol{i}$, $s_4 = t_7 + t_8\boldsymbol{i}$, satisfies

$$\begin{aligned} &g\left(\mathcal{S}(s_1,s_2,s_3,s_4) \cdot \mathcal{S}\left(s_1', s_2', s_3', s_4'\right)\right) \\ &\quad = g(\mathcal{S}(s_1,s_2,s_3,s_4)) \cdot g\left(\mathcal{S}\left(s_1', s_2', s_3', s_4'\right)\right) \end{aligned}$$

for all $s_j, s_j' \in \mathbb{C}$. By the same argument, we get the desired result. □

Remark: In fact, there are only four elements in $\mathbb{H} \otimes \mathbb{E}$ satisfying $x^2 = 1 \otimes 1$. Thus, $\pm(1 \otimes 1)$ and $\pm(1 \otimes \boldsymbol{e})$. But in $\mathbb{H} \otimes \mathbb{C}$ we can find at least eight elements such that $x^2 = 1 \otimes 1$. This says that $\mathbb{H} \otimes \mathbb{E}$ cannot be isomorphic to $\mathbb{H} \otimes \mathbb{C}$ as an $\mathbb{R}$-algebra (even as a ring). Similarly, $\mathbb{H} \otimes \mathbb{E}$ cannot be isomorphic to $\mathbb{H} \otimes \mathbb{F}$. So this explains why all the

See [4] for their performance charts	
$\mathcal{S}(s_1, s_2, s_3, s_4)$	$g(\mathcal{S}(t_1 + t_2\mathbf{i}, t_3 + t_4\mathbf{i}, t_5 + t_6\mathbf{i}, t_7 + t_8\mathbf{i}))$
$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & -s_4^\dagger & s_3^\dagger \\ -s_3 & s_4 & s_1 & -s_2 \\ -s_4^\dagger & -s_3^\dagger & s_2^\dagger & s_1^\dagger \end{pmatrix}$	$t_1\mathbf{b}_1 + t_2\mathbf{b}_6 - t_3\mathbf{b}_3 + t_4\mathbf{b}_8 - t_5\mathbf{b}_5 + t_6\mathbf{b}_2 + t_7\mathbf{b}_7 + t_8\mathbf{b}_4$
$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & -s_4^\dagger & s_3^\dagger \\ -s_3^\dagger & s_4^\dagger & s_1^\dagger & -s_2^\dagger \\ -s_4 & -s_3 & s_2 & s_1 \end{pmatrix}$	$t_1\mathbf{b}_1 + t_2\mathbf{b}_8 + t_3\mathbf{b}_3 + t_4\mathbf{b}_6 - t_5\mathbf{b}_5 + t_6\mathbf{b}_4 - t_7\mathbf{b}_7 + t_8\mathbf{b}_2$
$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2 & s_1 & -s_4 & s_3 \\ -s_3^\dagger & s_4^\dagger & s_1^\dagger & -s_2^\dagger \\ -s_4^\dagger & -s_3^\dagger & s_2^\dagger & s_1^\dagger \end{pmatrix}$	$t_1\mathbf{b}_1 + t_2\mathbf{b}_4 - t_3\mathbf{b}_3 + t_4\mathbf{b}_2 + t_5\mathbf{b}_5 + t_6\mathbf{b}_8 - t_7\mathbf{b}_7 + t_8\mathbf{b}_6$
$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & s_4^\dagger & -s_3^\dagger \\ -s_3 & -s_4 & s_1 & s_2 \\ -s_4^\dagger & s_3^\dagger & -s_2^\dagger & s_1^\dagger \end{pmatrix}$	$t_1\mathbf{b}_1 + t_2\mathbf{b}_6 - t_3\mathbf{b}_3 + t_4\mathbf{b}_8 + t_5\mathbf{b}_5 - t_6\mathbf{b}_2 + t_7\mathbf{b}_7 + t_8\mathbf{b}_4$
$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^\dagger & s_1^\dagger & s_4^\dagger & -s_3^\dagger \\ -s_3^\dagger & -s_4^\dagger & s_1^\dagger & s_2^\dagger \\ -s_4 & s_3 & -s_2 & s_1 \end{pmatrix}$	$t_1\mathbf{b}_1 + t_2\mathbf{b}_8 + t_3\mathbf{b}_3 + t_4\mathbf{b}_6 - t_5\mathbf{b}_5 + t_6\mathbf{b}_4 + t_7\mathbf{b}_7 - t_8\mathbf{b}_2$
$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2 & s_1 & s_4 & -s_3 \\ -s_3^\dagger & -s_4^\dagger & s_1^\dagger & s_2^\dagger \\ -s_4^\dagger & s_3^\dagger & -s_2^\dagger & s_1^\dagger \end{pmatrix}$	$t_1\mathbf{b}_1 + t_2\mathbf{b}_4 + t_3\mathbf{b}_3 - t_4\mathbf{b}_2 + t_5\mathbf{b}_5 + t_6\mathbf{b}_8 - t_7\mathbf{b}_7 + t_8\mathbf{b}_6$

Fig. 4. The $\mathbf{4} \times \mathbf{4}$ quasi-orthogonal designs with corresponding representations of $\mathbb{H}$ *(Case ii))*.

4×4 complex STBCs that can be identified with a linear extension of $\mathbb{H}$ are indeed the only ones that can be identified with $\mathbb{H} \otimes \mathbb{E}$.

IV. Concluding Remarks

We have combined all the various 4×4 complex STBCs from optimal nonorthogonality designs into a mono-algebraic number system. Since the conventional doubling procedures on hypercomplex numbers, e.g., complex numbers to quaternionic numbers, reduced biquaternions, and further to octonionic numbers, are all the special cases inside bilinear algebras, we choose tensor product, the most formal consubstantiality of those scenarios, to deal with the problem in this correspondence. Besides, it opens the possibility of searching for future algorithms for processing those STBCs on the platform of hypercomplex filters, instead of the exhaustive matrix operations.

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