Optical Orthogonal Codes With Nonideal Cross Correlation

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***Abstract*—For optical code division multiple access (OCDMA) networks, many optical orthogonal codes (OOCs) with ideal auto- and cross-correlation properties had been studied widely. In this paper, we relax the cross-correlation constraint slightly and propose a new code family based on perfect difference codes. Given the same code weight and code length, the size of new codes may increase 10 times more than that of ideal OOCs. Although the maximum cross correlation of new codes is larger than one, the cross correlation is less than or equal to one, for the most part. Consequently, the performance of new codes approaches that of ideal OOCs. Numerical results show that the performance of proposed codes was almost the same as that of conventional OOCs under the same code length and code weight.**

***Index Terms*—Maximal system, multiuser interference, optical code division multiple access (OCDMA), optical orthogonal code, perfect difference set.**

I. Introduction

RECENTLY, the construction and performance analysis of OOCs for optical code division multiple access (OCDMA) systems have been investigated widely [1]–[14]. A $(v, w, \lambda_a, \lambda_c)$-OOC is a family of (0, 1) sequences with code length v, code weight w, the maximum value of off-peak autocorrelation λ_a, and the maximum value of cross correlation λ_c. Most studies paid attention to $(v, w, 1)$-OOCs with $\lambda_a = \lambda_c = 1$ for the sake of synchronization and minimizing interference. However, under the constraint of the ideal auto- and cross-correlation properties, the code size is upper bounded by $\lfloor (1/w) \lfloor (v-1)/(w-1) \rfloor \rfloor$ [5], which is linear to the code length, where $\lfloor x \rfloor$ denotes the maximal integer not larger than x. Therefore, the code size is very sparse with respect to the code length. In order to obtain a larger code size, we should relax the constraint.

In [5], Chung and Kumar constructed optimal $(p^{2m} - 1, p^m + 1, 2)$-OOCs, where p is any prime and the family size is $p^m - 2$. In [12], Yang and Fuja investigated $(v, w, 2, 1)$- OOCs and proved that it is impossible to get more than $2(v-1)/(w^2 - w)$ code words whose code size is twice the upper bound of $(v, w, 1)$-OOCs. In [8], Yang also constructed $(v, w, 1, 2)$-OOCs, and the code size is w times (for even w) or $w - 1$ times (for odd w) the size of $(v, w, 1)$-OOCs, when w is less than eight. To the best of our knowledge, there is no code family with code size larger than the code length. In this paper, we relax the constraint of $\lambda_c = 1$ to $\lambda_c = 2$. On the other hand, we maintain the value of λ_a to 1 for the sake of synchronization between the receiver and transmitter. In [15], we proposed a code family named *perfect difference codes* based on $(v, k, 1)$-perfect difference sets [16] for synchronous OCDMA systems. A perfect difference code with code weight k $(=p+1, p$ is a power of a prime) and code length v $(=k^2 - k + 1)$ has an interesting property; the off-peak autocorrelation is exactly one. For synchronous OCDMA systems, we may cyclically shift such code $(v-1)$ times to get other $(v-1)$ codes. Thus, the code size is the same as the code length. We also showed that it is easy to cancel the multiuser interference (MUI). However, from the viewpoint of asynchronous systems, these codes are basically identical because they are cyclically shifted with one another. In other words, we have only one code for asynchronous systems and the code has the property of $\lambda_a = 1$. Therefore, we must modify the perfect difference codes to get a larger code size. To do so, we consider the case of $\lambda_c = 2$ and observe the cross correlation between the two identical perfect difference codes. When the two codes are not aligned—that is, they are cyclically shifted with each other—the cross correlation between the two codes is exactly one. In such a situation, it does not violate the constraint of $\lambda_c = 2$. However, when there is no cyclic shifting between them, the value of the cross correlation is k, which is far from the constraint. To overcome this, we can drop some marks appropriately from k marks of each code, such that the modified code weight becomes w and the code length is still equal to v. Moreover, the cross-correlation property should satisfy the constraint of $\lambda_c = 2$, even when the two codes are aligned with each other. For example, the set $\{0, 1, 3, 7, 15, 31, 36, 54, 63\}$ is a (73, 9, 1)-perfect difference set, where each element means the mark position. We can drop the first and the last five elements from the set, respectively, to form two subsets. Therefore, the two subsets $\{31, 36, 54, 63\}$ and $\{0, 1, 3, 7\}$ form a family of (73, 4, 1, 2)-OOCs. In fact, we can obtain more than two codes satisfying the constraint of $\lambda_c = 2$, as long as we choose the subsets appropriately. Therefore, the question is how many codes we can get, so that each code is formed by reserving some w marks from the original k marks of a perfect difference code, and the cross correlation between any two distinct codes is not larger than two. In other words, how many different w subsets of a k set, such that any two distinct subsets share, at most, two elements, can we obtain? Consider a set E with k elements and let w be a positive integer satisfying $3 \le w \le k$. This question is the same as constructing an $m(w, 3, k)$ maximal system of w-tuples (subsets of E having w elements each),

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such that each triple of elements is contained in at most one w-tuple of the system [17], [18]. Based on the maximal system, we will demonstrate that it is possible to construct a family of $(v, w, 1, 2)$-OOCs, such that the family size is larger than the code length. However, the maximal value of cross correlation is two. This maximal value occurs only when the two codes are aligned with each other (the probability is only $1/v$) and they overlap at two marks. That is, for the most part, the value of cross correlation is not larger than one. Therefore, the new code family is essentially similar to $(v, w, 1)$-OOCs, but the code size of the former is much larger than that of the latter. For example, consider a (757, 28, 1)-perfect difference set; we can obtain 819 four-tuples to form a family of (757, 4, 1, 2)-OOCs with family size equal to 819. In fact, we can reverse all 819 codes to obtain another 819 codes, such that all 1638 codes also form a family of (757, 4, 1, 2)-OOCs. On the other hand, an optimal family of (757, 4, 1)-OOCs only contains 63 code words.

Due to good property of the proposed codes, the bit error rate (BER) of the proposed codes is almost the same as that of ideal codes under the same code length, code weight, and number of simultaneous users. Moreover, the code size of the former is 10 times (which depends on the values of k and w) more than ideal codes. If it is needed to increase the number of simultaneous users under BER$\leq 10^{-9}$, there are two possible ways to achieve it. One is increasing the code length and reducing the bit rate. Another way is increasing the code weight and simultaneously reducing the code size. Numerical results show that the number of simultaneous users with the proposed (6643, 10, 1, 2)-OOCs is 23 times more than that with the ideal (6643, 3, 1)-OOCs under BER$\leq 10^{-9}$. The code size of the proposed (6643, 10, 1, 2)-OOCs is still larger than the ideal (6643, 3, 1)-OOCs. Numerical results also show that the performance of the proposed codes is better than that of Yang's $(v, w, 1, 2)$ codes.

The remainder of this paper is organized as follows. In Section II, we construct a family of $(v, w, 1, 2)$-OOCs. In Section III, we analyze the BER performance of the systems using the proposed codes with double hard limiters. The numerical results are given in Section IV. We also compare the performances of the proposed codes, the ideal OOCs, and Yang's $(v, w, 1, 2)$ codes. The conclusion is given in Section V.

II. OOCs With $\lambda_c = 2$ Based on Perfect Difference Codes

Because $(v, w, 1, 2)$-OOCs are based on perfect difference codes, we first introduce perfect difference sets. Let W be the v-set of integers $0, 1, \ldots, v-1$ modulo v. A set $D = \{d_1, d_2, \ldots, d_k\}$ is a k subset of W. For every $a \neq 0 \pmod v$, there are exactly λ ordered pairs $(d_i, d_j), i \neq j$, such that

$$d_i - d_j \equiv a (\mathrm{mod}\ v). \tag{1}$$

A set D satisfying these requirements is called a (v, k, λ)-perfect difference set. A special type of perfect difference sets is the $(v, k, 1)$-perfect difference set. The existence of the $(q^2 + q + 1, q + 1, 1)$-perfect difference set, where q is a power of a prime, has been proved and constructed by Singer [16]. We can construct a perfect difference code $C = \{c_0, c_1, \ldots, c_i, \ldots, c_{(v-1)}\}$ based on the perfect difference set D with the rule

$$c_i = \begin{cases} 1, & \text{if } i \in D \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The code weight and code length are k and v, respectively, where $v = k^2 - k + 1$.

To achieve the maximal code size of $(v, w, 1, \lambda_c)$-OOCs based on perfect difference codes, we should choose w-tuples from a $(v, k, 1)$-perfect difference set, which contains k elements, such that the same elements between any two w subsets is not more than λ_c. The question is the same as constructing an $m(w, \lambda_c + 1, k)$ maximal system such that every $(\lambda_c + 1)$-tuple is contained in at most one set of the system. Assume the number of w-tuples in the system is T. Every w-tuple contains $\binom{w}{\lambda_c + 1}$ distinct $(\lambda_c + 1)$-tuples. The total number of $(\lambda_c + 1)$-tuples from k elements is $\binom{k}{\lambda_c + 1}$. Therefore, the upper bound of T is

$$T \leq \frac{\binom{k}{\lambda_c + 1}}{\binom{w}{\lambda_c + 1}} = \frac{k(k-1)\ldots(k-\lambda_c)}{w(w-1)\ldots(w-\lambda_c)} \tag{3}$$

or more tightly [18]

$$T \leq \left\lfloor \frac{k}{w} \left\lfloor \frac{k-1}{w-1} \right\rfloor \cdots \left\lfloor \frac{k-\lambda_c+1}{w-\lambda_c+1} \times \left\lfloor \frac{k-\lambda_c}{w-\lambda_c} \right\rfloor \right\rfloor \cdots \right\rfloor \right\rfloor. \tag{4}$$

A special case is when $\lambda_c = 1$, then

$$T \leq \left\lfloor \frac{k}{w} \left\lfloor \frac{k-1}{w-1} \right\rfloor \right\rfloor \leq \left\lfloor \frac{1}{w} \left\lfloor \frac{k^2-k}{w-1} \right\rfloor \right\rfloor = \left\lfloor \frac{1}{w} \left\lfloor \frac{v-1}{w-1} \right\rfloor \right\rfloor \tag{5}$$

which is the upper bound of $(v, w, 1)$-OOCs. Thus, we may use a perfect difference code to construct optimal $(v, w, 1)$-OOCs. For example, the set $\{0, 1, 3, 7, 15, 31, 36, 54, 63\}$ is a (73, 9, 1)-perfect difference set. We choose three-tuples from the set appropriately to get an $m(3, 2, 9)$ maximal system with 12 subsets: $\{0, 1, 3\}$; $\{7, 15, 31\}$; $\{36, 54, 63\}$; $\{0, 7, 36\}$; $\{1, 15, 54\}$; $\{3, 31, 63\}$; $\{0, 15, 63\}$; $\{1, 31, 36\}$; $\{3, 7, 54\}$; $\{0, 31, 54\}$; $\{3, 15, 36\}$; and $\{1, 7, 63\}$. The 12 subsets form 12 code words with the rule of (2), and they are optimal (73, 3, 1)-OOCs.

Because many papers have discussed the optimal $(v, w, 1)$-OOCs, and the code sizes are very small with respect to the code length, we do not consider the ideal cross-correlation case in this paper. We will focus on the case with $\lambda_c = 2$, that is, $(v, w, 1, 2)$-OOCs. In this situation, the code size is upper bounded by

$$T \leq \left\lfloor \frac{k}{w} \left\lfloor \frac{k-1}{w-1} \left\lfloor \frac{k-2}{w-2} \right\rfloor \right\rfloor \right\rfloor = O(v^{3/2}). \tag{6}$$

In other words, the optimal code size is no longer linearly proportional to the code length, and it is possible to construct a code family whose code size is larger than the code length. Because the upper bound in (6) is $\lfloor (k-2)/(w-2) \rfloor$ times that in (5), it is possible to construct a family of $(v, w, 1, 2)$-OOCs such that the family size is $\lfloor (k-2)/(w-2) \rfloor$ times the size of an optimal $(v, w, 1)$-OOCs.

The determination of the maximal code size is still an unsolved problem [18]. However, many useful results had been reported [17]–[21]. We describe some results directly related to this paper in the following. Consider a Galois field GF (q^r) where q is the power of a prime and r is a positive integer. The extended field F is obtained by adding an element ∞ into GF (q^r), that is, $F = \mathrm{GF}\,(q^r) \cup \{\infty\}$. A linear transformation

$$\eta = T(\xi) = \frac{\alpha\xi + \beta}{\gamma\xi + \delta} \tag{7}$$

is one-to-one and forms a group, where $\{\alpha, \beta, \gamma, \delta\} \subset \mathrm{GF}(q^r)$, $\{\eta, \xi\} \subset F$, and $\alpha\delta - \beta\gamma \neq 0$. The image of ξ is defined as the cross ratio

$$(\xi, \xi_2, \xi_3, \xi_4) = \frac{\xi - \xi_2}{\xi - \xi_4} \bigg/ \frac{\xi_3 - \xi_2}{\xi_3 - \xi_4} \tag{8}$$

which is a linear transformation, and it carries ξ_2, ξ_3, and ξ_4 into 0, 1, and ∞ of GF $(q) \cup \{\infty\}$ respectively [19]. For any four distinct elements ξ_1, ξ_2, ξ_3, and ξ_4 of S, a subset S of F is called a circle if $(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathrm{GF}\,(q)$ and there is no set properly containing S as its property. With the systems of circles, the extended field F forms a finite Möbius geometry of MG (q, r), which has $q^r + 1$ elements. Any triple of elements in MG (q, r) is included in exactly one circle and every circle has $q + 1$ elements [17]. Based on this construction, it is easy to form an $m(q+1, 3, q^r+1)$ maximal system and the number of w-tuples (or circles) achieves the upper bound in (6). As a result, we can construct a code family of $(v, w, 1, 2)$-OOCs with $w = q + 1, k = q^r + 1$, and $v = k^2 - k + 1 = q^{2r} + q^r + 1$ by the following algorithm.

1) Construct a $(v, k, 1)$ perfect difference set with k elements according to [16], where $k = q^r + 1$ and $v = q^{2r} + q^r + 1$.
2) Each element of an extended field $F = \mathrm{GF}\,(q^r) \cup \{\infty\}$, which also contains k elements, is related to one element of the perfect difference set.
3) Choose all the possible subsets with w elements from F and reserve the subsets fulfilling the property of a circle, where $w = q + 1$.
4) Each circle is related to w elements of the perfect difference set. The w elements form a code word with code length v and code weight w, where each element represents the mark position. The codes form a family of $(v, w, 1, 2)$-OOCs.

The total number of codes is given by

$$\begin{aligned} T &= \left\lfloor \frac{k}{w} \left\lfloor \frac{k-1}{w-1} \left\lfloor \frac{k-2}{w-2} \right\rfloor \right\rfloor \right\rfloor \\ &= \left\lfloor \frac{q^r+1}{q+1} \left\lfloor \frac{q^r+1-1}{q+1-1} \left\lfloor \frac{q^r+1-2}{q+1-2} \right\rfloor \right\rfloor \right\rfloor \\ &= q^{r-1} \cdot \frac{q^{2r}-1}{q^2-1}. \end{aligned} \tag{9}$$

TABLE I
THE (21, 3, 1, 2)-OOCs WITHOUT REVERSED CODES

i	codes	code sets	codewords
1	C^1	{ 0 1 4 }	{110010000000000000000}
2	C^2	{ 0 1 14 }	{110000000000001000000}
3	C^3	{ 0 1 16 }	{110000000000000010000}
4	C^4	{ 0 4 14 }	{100010000000001000000}
5	C^5	{ 0 4 16 }	{100010000000000010000}
6	C^6	{ 0 14 16 }	{100000000000001010000}
7	C^7	{ 1 4 14 }	{010010000000001000000}
8	C^8	{ 1 4 16 }	{010010000000000010000}
9	C^9	{ 1 14 16 }	{010000000000001010000}
10	C^{10}	{ 4 14 16 }	{000010000000001010000}

Because any three elements in MG (q, r) determine a circle, for any two distinct elements of a circle, there are $(k - w)/(w - 2)$ circles intersected with the circle at these two distinct elements. Moreover, a circle with w elements has $w(w-1)/2$ nonordered pairs. Therefore, given any circle O^1, the total number of circles such that each circle intersects with O^1 at two elements can be expressed as

$$T_2 = \frac{w(w-1)}{2} \cdot \frac{k-w}{w-2}. \tag{10}$$

In other words, given any one code C^1, there are T_2 codes such that the maximal cross correlation with C^1 is two (this only occurs when each of them is aligned with C^1). With respect to the code C^1, we class the T_2 codes as *Group 2* (in which the maximal cross correlation with C^1 is two and only occurs once among v possible cyclic shifts with C^1) and the other $T - 1 - T_2$ codes as *Group 1* (in which the maximal cross correlation with C^1 is one).

Note that we can obtain another T codes by reversing all the T codes. It is easy to prove that all the $2T$ codes also form a family of $(v, w, 1, 2)$-OOCs. Moreover, any one code reversed from *Group 1* or *Group 2* is still in the same group. Therefore, given any one code C^1 in the $2T$ codes, there are $2T_2$ codes in *Group 2* and $2(T - 1 - T_2)$ codes in *Group 1* with respect to the code C^1. However, the reverse of the code C^1, C^1_r, has not yet been considered. Although the maximal cross correlation between C^1 and C^1_r is two, we do not class this reversed code in *Group 2* or *Group 1*. Because the maximal cross correlation does not occur only once, actually, it occurs $\binom{w}{2}$ times among v possible cyclic shifts with C^1. As an example, the (21, 3, 1, 2)-OOCs without reversed codes are presented in Table I.

III. PERFORMANCE ANALYSIS OF OOCs WITH $\lambda_C = 2$

In this section, we analyze the performance of the systems using double hard-limiters with consideration of shot noise, thermal noise, avalanche photodiode (APD) bulk, and surface leakage currents. We use the proposed codes as the signature codes. The receiver structure is shown in Fig. 1 [22]. To simplify the performance analysis, we assume that chips are synchronous among users because it is the worst case and results in the upper bound on the performance [4].

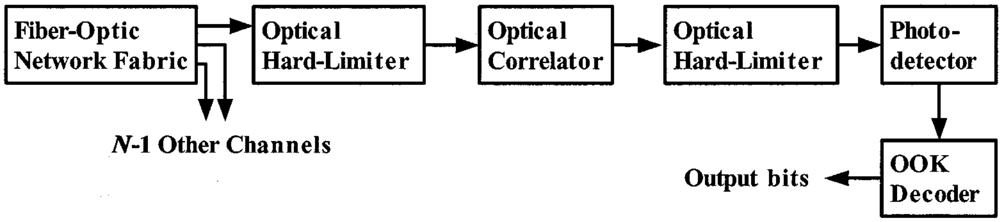

Fig. 1. The receiver structure of OCDMA systems with double hard limiters.

The average photon arrival rate λ per pulse is given by

$$\lambda = \frac{\eta P_W}{hf} \tag{11}$$

where η is APD quantum efficiency, P_W is the received signal power, h is the Planck's constant, and f is the optical frequency. There are only two states after the second hard limiter, denoted by S_1 and S_0, respectively. The average photon arrival rate of state S_1 is equal to λ, whereas the photon arrival rate of state S_0 is zero (this occurs only when the desired data bit is zero and the MUI is removed completely by the two hard limiters). For states $S_i, i \in \{0,1\}$, the probability density function of the output Y_i after the photodetector can be expressed as [23]

$$P_{Y_i}(y_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(y_i-\mu_i)^2/2\sigma_i^2} \tag{12}$$

where the mean μ_i can be expressed as

$$\mu_i = GT_c(i\lambda + I_b/e) + T_c I_s/e. \tag{13}$$

Here, G is the average APD gain, T_c is the chip duration, e is the electron charge, I_b/e is the contribution of the APD bulk leakage current to the APD output, I_s is the APD surface leakage current, and the variance σ_i^2 can be written as

$$\sigma_i^2 = G^2 F_e T_c(i\lambda + I_b/e) + T_c I_s/e + \sigma_{\mathrm{th}}^2 \tag{14}$$

where F_e is an excess noise factor given by

$$F_e = k_{\mathrm{eff}} G + (2 - 1/G)(1 - k_{\mathrm{eff}}). \tag{15}$$

Here, k_{eff} is an APD effective ionization ratio and σ_{th}^2 is the variance of thermal noise expressed as

$$\sigma_{\mathrm{th}}^2 = 2k_B T_r T_c/(e^2 R_L) \tag{16}$$

where k_B is Boltzmann's constant, T_r is the receiver noise temperature, and R_L is the receiver load resistance.

After the photodetector, the signal is fed into an ON-OFF keying (OOK) decoder. If the output Y_i is larger than the constant threshold θ, we declare that the output data bit b_o is one; otherwise, it is zero. To minimize the error probability, we set the suboptimal value of the constant threshold θ to be

$$\theta = \frac{\mu_0\sigma_1 + \mu_1\sigma_0}{\sigma_1 + \sigma_0}. \tag{17}$$

Therefore, the probability that the state S_1 (or S_0) is decoded incorrectly to be $b_o = 0$ (or $b_o = 1$) can be expressed as

$$\Pr(b_o = 0 \,|\, S_1) = \Pr(b_o = 1 \,|\, S_0) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\mu_1 - \theta}{\sqrt{2\sigma_1^2}}\right) \tag{18}$$

where $\mathrm{erfc}(\cdot)$ stands for the complementary error function, defined as

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-u^2)\, du. \tag{19}$$

The probability that the state S_1 (or S_0) is decoded correctly to be $b_o = 0$ (or $b_o = 1$) can be expressed as

$$\Pr(b_o = 1 \,|\, S_1) = 1 - \Pr(b_o = 0 \,|\, S_1) \tag{20}$$

and

$$\Pr(b_o = 0 \,|\, S_0) = 1 - \Pr(b_o = 1 \,|\, S_0) \tag{21}$$

respectively.

The performance analyses with or without the T reversed codes are similar to each other. Thus, we only derive the performance of the systems with the original T codes. That is, the total number of codes is T and none of them is reversed with each other. Without loss of generality, we consider the user U^1 assigned the code $C^1 = \{c_0^1, c_1^1, \ldots, c_{v-1}^1\}$ is the desired user and the desired data bit is b. U^2 assigned the code $C^2 = \{c_0^2, c_1^2, \ldots, c_{v-1}^2\}$ represents one of the rest $T-1$ users. If their relative cyclic shift is $j, j \in \{0, 1, \ldots, v-1\}$, the cross correlation can be expressed as

$$I_j = \sum_{i=0}^{v-1} c_i^1 c_{i\ominus j}^2 \tag{22}$$

where $\oplus$ denotes the addition modulo v. The value of I_j is given by

$$I_j = \begin{cases} 2, & \text{if } j = 0 \quad \text{and} \quad C^2 \in \mathit{Group\ 2} \\ 0 \quad \text{or} \quad 1, & \text{otherwise.} \end{cases} \tag{23}$$

Consider $C^2 \in$ *Group 2* and denote p_1 and p_2 as the probabilities that I_j is 1 and 2 respectively. The expected value of I_j is given by [6]

$$E(I_j) = p_1 + 2p_2 = \frac{w^2}{v}. \tag{24}$$

Because the value of I_j is two only when $j = 0$ and $C^2 \in$ *Group 2*, the value of p_2 is $1/v$ and then $p_1 = (w^2 - 2)/v$. Similarly, if $C^2 \in$ *Group 1*, we denote p_1' as the probability that I_j is 1. The value of p_1' is w^2/v, which is the same as that of ideal OOCs.

The probabilities that U^2 contributes one or two pulse positions are given by

$$q_1 = \Pr(I_j = 1 \,|\, C^2 \in \mathit{Group\ 2}) = p_1/2 \tag{25}$$

$$q_2 = \Pr(I_j = 2 \,|\, C^2 \in \mathit{Group\ 2}) = p_2/2 \tag{26}$$

and

$$q_1' = \Pr(I_j = 1 \,|\, C^2 \in \mathit{Group\ 1}) = p_1'/2 \tag{27}$$

where the factor $1/2$ means equiprobable ON–OFF data bits.

Given the number of simultaneous users N, the probability that there are n_2 users from *Group 2* can be expressed as

$$\Pr(n_2 \,|\, N) = \frac{\begin{pmatrix} T_2 \\ n_2 \end{pmatrix} \begin{pmatrix} T-1-T_2 \\ N-1-n_2 \end{pmatrix}}{\begin{pmatrix} T-1 \\ N-1 \end{pmatrix}}. \tag{28}$$

Among the n_2 users, the probability of l_1 users interfering at one pulse position and l_2 users interfering at two pulse positions is a trinomial distribution with parameters n_2, q_1, and q_2. The probability can be expressed as

$$\Pr(l_1, l_2 \,|\, n_2, N) = \frac{n_2!}{l_1! l_2! (n_2 - l_1 - l_2)!} \times q_1^{l_1} q_2^{l_2} (1 - q_1 - q_2)^{n_2 - l_1 - l_2}. \tag{29}$$

On the other hand, among the other $N-1-n_2$ users from the *Group 1*, the probability of l_1' users interfering at one pulse position is a binomial distribution with parameters $N-1-n_2$ and q_1', and it can be expressed as

$$\Pr(l_1' \,|\, n_2, N) = \begin{pmatrix} N-1-n_2 \\ l_1' \end{pmatrix} q_1'^{l_1'} (1-q_1')^{N-1-n_2-l_1'}. \tag{30}$$

The performance analysis of the system using double hard limiters and the proposed codes is slightly different from that of the system with ideal OOCs because each interfered user interferes at one pulse position in the latter system. In the former system, each interfered user may have contributed one or two pulses. Therefore, we should determine the value of l_2 and $l_1 + l_1'$. Because the value of q_2 is much smaller than q_1 and q_1', we first determine the pattern of l_2 interfering users and then determine that of $l_1 + l_1'$ interfering users. For example, if we know that there are only $l_1 + l_1'$ users with one-pulse contribution and no other users with two-pulse contribution, the performance analysis is the same as the systems with ideal OOCs. However, if there is exactly one user with two-pulse contribution ($l_2 = 1$) and $l_1 + l_1'$ users with one-pulse contribution, the probability that the interference cannot be canceled by the second hard limiter is the same as the probability of each one of the remaining $w-2$ mark positions interfered by at least one of the $l_1 + l_1'$ users.

Let the l_2 users totally interfere at t pulse positions among the w marks of the desired user. Without loss of generality, we assume that the other noninterfered $w-t$ marks locate at the first $w-t$ marks. Provided there are l users among $l_1 + l_1'$ users, such that all of them interfere at the first $w-t$ marks. We denote k_t as the total number of marks interfered by the l users among the first $(w-t)$ marks, where $0 \le k_t \le w-t$. Applying the principle of inclusion and exclusion, the probability of each one of the remaining $w-t$ marks is interfered by at least one of the l users can be expressed as

$$\begin{aligned} \Pr(k_t = w-t \,|\, l) &= 1 - \sum_{i=1}^{w-t-1} (-1)^{i-1} \begin{pmatrix} w-t \\ i \end{pmatrix} \left(1 - \frac{i}{w-t}\right)^l \\ &= \sum_{i=0}^{w-t-1} (-1)^i \begin{pmatrix} w-t \\ i \end{pmatrix} \left(1 - \frac{i}{w-t}\right)^l. \end{aligned} \tag{31}$$

Therefore, the probability of $k_t = w-t$ given $l_1 + l_1'$ can be expressed as

$$\begin{aligned} \Pr(k_t = w-t \,|\, l_1 + l_1') &= \sum_{l=0}^{l_1+l_1'} \Pr(k_t = w-t \,|\, l) \Pr(l \,|\, l_1 + l_1') \\ &= \sum_{l=0}^{l_1+l_1'} \Pr(k_t = w-t \,|\, l) \begin{pmatrix} l_1 + l_1' \\ l \end{pmatrix} \\ &\quad \times \left(\frac{w-t}{w}\right)^l \left(\frac{t}{w}\right)^{l_1+l_1'-l}. \end{aligned} \tag{32}$$

The probability that the state is S_1 after the second hard limiter, given N and the desired data bit $b=0$, is given by

$$\begin{aligned} \Pr(S_1 \,|\, N, b=0) = &P_S^0(S_1, l_2 = 0 \,|\, N, b=0) \\ &+ P_S^1(S_1, l_2 = 1 \,|\, N, b=0) \\ &+ P_S^2(S_1, l_2 = 2 \,|\, N, b=0) \\ &+ P_S^3(S_1, l_2 \ge 3 \,|\, N, b=0). \end{aligned} \tag{33}$$

The first three conditional probabilities of the right-hand side in (33) can be expressed as

$$\begin{aligned} &P_S^i(S_1, l_2 = i \,|\, N, b=0) \quad i = 0, 1, 2. \\ &\quad = \sum_{n_2} \Pr(S_1, l_2 = i \,|\, n_2, N, b=0) \Pr(n_2 \,|\, N) \end{aligned} \tag{34}$$

here

$$\begin{aligned} &\Pr(S_1, l_2 = i \,|\, n_2, N, b=0) \\ &\quad = \sum_{l_1, l_1'} \Pr(S_1 \,|\, l_1, l_1', l_2 = i, b=0) \\ &\quad\quad \times \Pr(l_1, l_2 = i \,|\, n_2, N) \Pr(l_1' \,|\, n_2, N). \end{aligned} \tag{35}$$

When $i \in \{0, 1\}$,

$$\begin{aligned} &\Pr(S_1 \,|\, l_1, l_1', l_2 = i, b=0) \\ &\quad = \Pr(k_{2i} = w - 2i \,|\, l_1 + l_1'). \end{aligned} \tag{36}$$

When $i = 2$, the conditional probability $\Pr(S_1 \,|\, l_1, l_1', l_2 = 2, b=0)$ is similar to (36). However, the $l_2 = 2$ users may contribute two, three, or four marks, i.e., t may be 2, 3, or 4. Therefore, the conditional probability can be expressed as

$$\begin{aligned} &\Pr(S_1 \,|\, l_1, l_1', l_2 = 2, b=0) \\ &\quad = \Pr(k_2 = w-2 \,|\, l_1 + l_1') \Pr(t = 2 \,|\, l_2 = 2) \\ &\quad\quad + \Pr(k_3 = w-3 \,|\, l_1 + l_1') \Pr(t = 3 \,|\, l_2 = 2) \\ &\quad\quad + \Pr(k_4 = w-4 \,|\, l_1 + l_1') \Pr(t = 4 \,|\, l_2 = 2). \end{aligned} \tag{37}$$

Because any three elements will determine a circle and a code mentioned previously, we observe the behavior among any two distinct circles of the T_2 circles and the desired circle to obtain the three conditional probabilities, given $l_2 = 2$ in (37). Consider any circle O^2 chosen from the T_2 circles; it intersects with the desired circle O^1 at two of the w elements. In fact, there are $(k-w)/(w-2)$ circles in the T_2 circles intersecting at these two elements. Therefore, among the other $T_2 - 1$ circles, there are $(k-w)/(w-2) - 1$ circles overlapping these two elements. Similarly, there are $2(w-2) \cdot (k-w)/(w-2)$

codes interfering with the desired code at one of these two elements and one of the remaining $w-2$ elements. There are $((w-2)(w-3)/2)\cdot(k-w)/(w-2)$ codes interfering with the desired code at two of the remaining $w-2$ elements. Therefore, the three conditional probabilities given $l_2=2$ in (37) are

$$\Pr(t=2\,|\,l_2=2)=\frac{\frac{k-w}{w-2}-1}{T_2-1} \quad (38)$$

$$\Pr(t=3\,|\,l_2=2)=\frac{2(w-2)\cdot\frac{k-w}{w-2}}{T_2-1} \quad (39)$$

and

$$\Pr(t=4\,|\,l_2=2)=\frac{\frac{(w-2)(w-3)}{2}\cdot\frac{k-w}{w-2}}{T_2-1} \quad (40)$$

respectively.

Finally, because it is hard to determine the interfering pattern when $l_2\geq 3$, we consider the worst case in such a situation. That is, we assume $t=\min(w,2l_2)$. Therefore, the upper bound of the last conditional probability in (33) can be expressed as

$$\begin{aligned}P_S^3(S_1,l_2\geq 3\,|\,N,b=0)&=\sum_{n_2,l_2\geq 3}\Pr(S_1,l_2\,|\,n_2,N,b=0)\Pr(n_2\,|\,N)\\&=\sum_{n_2,l_2\geq 3}\sum_{l_1,l_1'}\Pr(S_1\,|\,l_1,l_1',l_2,b=0)\\&\quad\times\Pr(l_1,l_2\,|\,n_2,N)\Pr(l_1'\,|\,n_2,N)\Pr(n_2\,|\,N)\end{aligned} \quad (41)$$

where

$$\begin{aligned}&\Pr(S_1\,|\,l_1,l_1',l_2,b=0)\\&\quad=\Pr\big(k_{\min(w,2l_2)}=w-\min(w,2l_2)\;|\,l_1+l_1'\big).\end{aligned} \quad (42)$$

Given N and $b=1$, the state is S_1 after the second hard limiter. That is, the probability of the state is S_1 and $b=1$ is obtained by

$$\Pr(S_1\,|\,N,b=1)=1. \quad (43)$$

Therefore, the probabilities that the state is S_0, given N and b $(\in\{0,1\})$, are

$$\Pr(S_0\,|\,N,b=0)=1-\Pr(S_1\,|\,N,b=0) \quad (44)$$

and

$$\Pr(S_0\,|\,N,b=1)=1-\Pr(S_1\,|\,N,b=1). \quad (45)$$

Because (41) is the upper bound of $P_S^3(S_1,l_2\geq 3\,|\,N,b=0)$, the bit error probability P_E, given N, is upper bounded by

$$\begin{aligned}P_E=&\Pr(b_o=1\,|\,S_1)\Pr(S_1\,|\,N,b=0)\Pr(b=0)\\&+\Pr(b_o=1\,|\,S_0)\Pr(S_0\,|\,N,b=0)\Pr(b=0)\\&+\Pr(b_o=0\,|\,S_1)\Pr(S_1\,|\,N,b=1)\Pr(b=1)\\&+\Pr(b_o=0\,|\,S_0)\Pr(S_0\,|\,N,b=1)\Pr(b=1).\end{aligned} \quad (46)$$

IV. Numerical Results

In this section, we present the numerical results of the systems using conventional ideal OOCs, Yang's $(v,w,1,2)$ codes, and the proposed codes. The parameters are given in the Table II. To simplify the analysis of the systems with Yang's codes, we assume that the cross-correlation property of Yang's codes is the same as that of ideal OOCs, i.e., $\lambda_c=1$. The assumption results in the lower bound of the performance of the systems with Yang's codes.

TABLE II
Link Parameters

Name	Symbol	Value
Light wavelength		$1.3\mu m$
APD Quantum efficiency	η	0.6
APD gain	G	100
APD effective ionization ratio	k_{eff}	0.02
APD bulk leakage current	I_b	0.1nA
APD surface leakage current	I_s	10nA
chip duration	T_c	0.1ns
bit rate	$\frac{1}{T_b}=\frac{1}{vT_c}$	
Receiver noise temperature	T_r	300k
Receiver load resistor	R_L	1030Ω

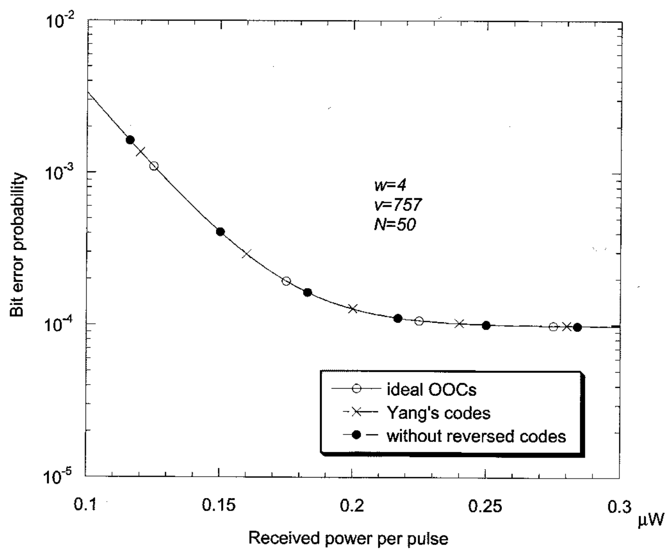

Fig. 2. The bit error probabilities versus the received power under $w=4$, $v=757$, and $N=50$.

First, we compare the performance among the ideal OOCs, Yang's codes, and the proposed nonreversed codes. Under the code length $v=757$ and code weight $w=4$, the maximum number of codes of the proposed (757, 4, 1, 2)-OOCs without reversed codes is 819, which is larger than v. However, under the same code length and code weight, the ideal OOCs only contain 63 codes and Yang's codes have 252 codes. Fig. 2 shows the bit error probability versus the received power per pulse P_W under the number of simultaneous users $N=50$. As in our prediction, the bit error probabilities of the systems with the proposed codes or the ideal OOCs are almost same. The proposed code family has 12 times the size of the ideal OOCs. This is because the probability q_2 is much smaller than q_1 or q_1'. Fig. 3 shows the bit error probability versus the number of simultaneous users under $P_W=0.4\,\mu$W. It also shows that the performances of the systems with the three classes of OOCs are almost the same.

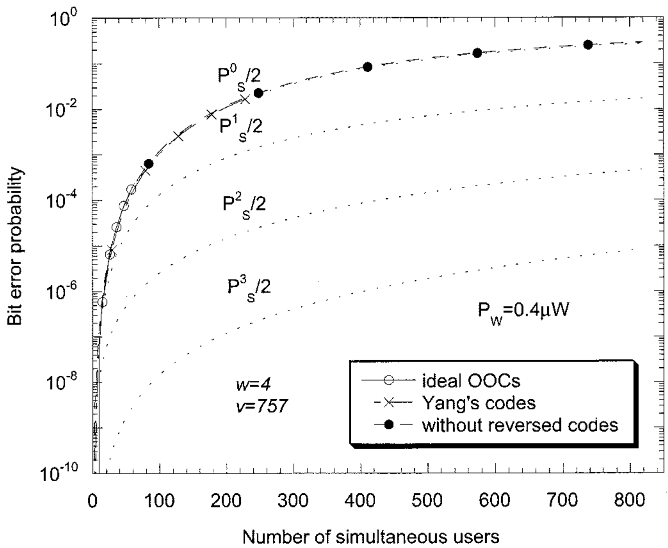

Fig. 3. The bit error probabilities versus the number of simultaneous users under $w = 4$, $v = 757$, and $P_W = 0.4\ \mu$W.

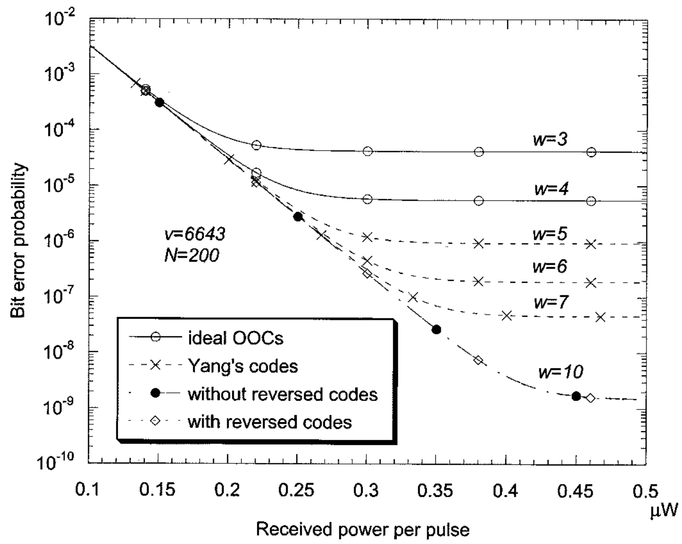

Fig. 4. The bit error probabilities versus the received power under $v = 6643$ and $N = 200$.

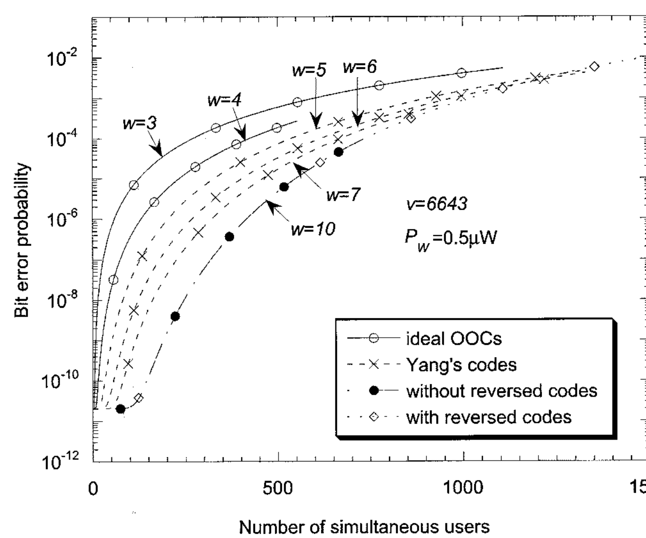

Fig. 5. The bit error probabilities versus the number of simultaneous users under $v = 6643$ and $P_W = 0.5\ \mu$W.

Under this received power, the probability that the OOK decoder will misdecode from state S_0 (or S_1) to $b_o = 1$ (or $b_o = 0$) is 10^{-9}, that is, $\Pr(b_o = 0 \,|\, S_1) = \Pr(b_o = 1 \,|\, S_0) = 10^{-9}$. Therefore, if the desired data bit is one, it contributes the bit error probability with 0.5×10^{-9}, no matter how many simultaneous users there may be. However, Fig. 3 also shows that the bit error probability grows with the number of simultaneous users. It means that the major contribution to the bit error probability is when the desired data bit is zero. This is because, if the desired data bit is one, it is free from interference due to the design of the receiver. On the other hand, if the desired data bit is zero, it may be interfered with by other users and then result in bit error, especially when the number of simultaneous users is large. Therefore, the dominant contribution to the bit error probability in (46) is the first term $\Pr(b_o = 1 \,|\, S_1) \Pr(S_1 \,|\, N, b = 0) \Pr(b = 0) \simeq (1/2) \Pr(S_1 \,|\, N, b = 0)$, where $1/2$ means the probability of $b = 0$. There are four curves, $P_S^i/2, i \in \{0, 1, 2, 3\}$, in the figure corresponding to the four conditional probabilities in (33). Note that the lowest curve, $P_S^3/2$, is the upper bound according to (41). P_E is also the upper bound of the actual bit error probability. Moreover, the dominant term of $(1/2) \Pr(S_1 \,|\, N, b = 0)$ is $P_S^0/2$, according to these four curves in the figure. Therefore, we may treat $P_S^0/2$ as the lower bound of the actual bit error probability. Fig. 3 shows that P_E is very close to $P_S^0/2$. It means that P_E is almost the same as the actual bit error probability. We can explain why the performances are similar to one another. For suitable received power, the bit error probability is the summation of $P_S^i/2, i \in \{0, 1, 2, 3\}$. However, the three conditional probabilities, $P_S^i/2, i \in \{1, 2, 3\}$, are much smaller than the first one, $P_S^0/2$, due to the very small value of q_2 with respect to q_1 or q_1'. Therefore, the bit error probability is almost equal to $P_S^0/2$, which represents the probability that $b = 0, l_2 = 0$, and the double hard limiters cannot eliminate the MUI. The meaning of $l_2 = 0$ is that any one user interfered with, at most, one pulse position. The property is the same as the ideal OOCs. That is why the bit error probabilities of the systems with ideal OOCs or the proposed codes are so close.

Because code weight and bit error probability are inversely related, we increase the code weight to lower the BER. Under the same code length $(v = 6643)$, the code sizes of ideal OOCs with $w = 3$ and $w = 4$ are 1107 and 553, respectively. The code sizes of Yang's codes with $w = 5, w = 6$, and $w = 7$ are upper bounded by 1328, 1328, and 948, respectively. The code sizes of the nonreversed codes and reversed codes with code weight $w = 10$ are 738 and 1476, respectively. Fig. 4 shows the bit error probability versus the received power per pulse of the systems using the four classes of OOCs under the number of simultaneous users $N = 200$ and code length $v = 6643$. When the received power P_W is small, these systems are power limited. When P_W is large, these systems are MUI limited. It also shows that the proposed codes with reversed codes perform better than ideal OOCs and Yang's codes. Moreover, the proposed codes have the largest code size. Fig. 5 shows the bit error probability versus the number of simultaneous users under the fixed received power $P_W = 0.5\ \mu$W. At BER $\leq 10^{-9}$, the systems with the ideal OOCs have only about eight (with $w = 3$) or 25 (with $w = 4$) simultaneous users. The systems with Yang's codes have about 52 (with $w = 5$), 83 (with $w = 6$), or 115 (with $w = 7$) simultaneous users. On the other hand, the systems with the two proposed codes accommodate about 190 users. Figs. 4 and 5 also show that the performances of the systems with the two proposed codes are almost same. This is because the $T-1$ reversed codes (excluding C_r^1) from *Group 1* or *Group 2* are still in the same group and the inversive code C_r^1 of the desired code does not affect the performance significantly. Consequently, the two proposed codes have similar properties.

V. Conclusion

In this paper, we propose two new classes of OOCs. For the first time, we show that it is possible to construct a code family with code size larger than the code length. Compared with ideal OOCs and Yang's $(v, w, 1, 2)$ codes under the same code length and code weight, the performances of the systems with the four classes of codes are almost the same as one another. However, the proposed codes accommodate the users 10 times (which depends on the values of k and w) more than ideal OOCs for larger code length. To increase the number of simultaneous users for a given BER, we may increase the code weight and reduce the code size. We show that it is possible to accommodate many more simultaneous users with the proposed codes than ideal OOCs and Yang's codes. Between the two classes of proposed codes, the code family with reversed codes has twice the code size than that of nonreversed codes. The performances of the two kinds of proposed codes are almost the same.

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