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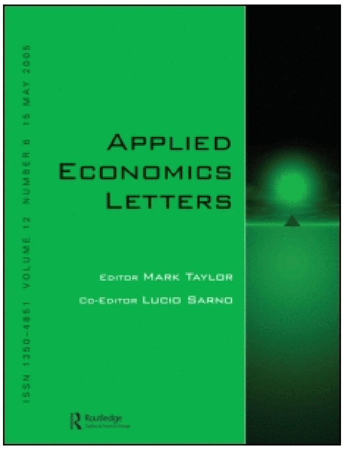

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The simplest American and Real Option approximations: Geske-Johnson interpolation in maturity and yield

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The simplest American and Real Option approximations: Geske–Johnson interpolation in maturity and yield

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The American early exercise feature of the Real Option to invest in a new project is important in capital budgeting and project valuation. Closed form solutions for American, and therefore Real, Options are known for two special cases; an infinite horizon generates the Merton (*Bell Journal of Economics*, **4**, 141–83, 1973) solution while a zero dividend yield on the project generates Black-Scholes (*Journal of Political Economy*, **81**, 637–59, 1973) prices since early exercise is never optimal. Geske–Johnson (*Journal of Finance*, **39**, 1511–24, 1984) approximation is extended to a bivariate case by assuming various forms of separability for option prices as a function of time to maturity and yield to produce fully explicit and asymptotically correct approximations. These methods are compared with another simple approximation method due to Barone-Adesi and Whaley (*Journal of Finance*, **42**, 301–20, 1987) and MacMillan (*Advances in Futures Options and Research*, **2**, 117–42, 1987) and the estimated error these expressions contain compared to an accurate numerical benchmark technique.

I. INTRODUCTION

Since Black and Scholes (1973) solved the European case, American option pricing has remained a challenge in finance. Discussion of differing pricing techniques has persisted in the literature for more than 25 years because no convenient solution form is known to the early exercise, free boundary problem other than the Merton solution (1973) for American options of infinite maturity and the European case that represents the optimal American strategy when underlying dividends are zero.

The American option pricing problem is important in capital budgeting when evaluating investments because many projects have future flexibility that can significantly augment the value above the no flexibility value. Examples of operating flexibility include real options to open new or close existing business lines (see Dixit and Pindyck (1994) for the large literature has developed in this area).

Analytical approximations to the American pricing problem include Johnson (1983), Barone-Adesi and Whaley (1987), MacMillan (1986) (BAWM hereafter) and Ju and Zhong (1999) (approximating the differential equation) and Omberg (1987) and Ju (1998) (utilizing boundary approximations). This note shows that although BAWM prices converge to the Merton perpetual solution, they can do so from the wrong side. In contrast the proposed method in this note will converge to the Merton perpetual solution correctly.

Geske and Johnson (1984) (GJ hereafter) provided both an analytic solution *and approximation scheme* to the American option pricing problem. In their explicit solution however, the hedge portfolio (stock and bond weights) that replicate the option are derived from an infinite series of terms whose elements although eventually becoming insignificant, require increasing computational power to evaluate due to multiple integrals

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of higher order.[1] As a result, their more useful practical contribution was to consider American options as the limit of a series of options with increasing exercisability (decreasing inter-exercise time) and apply Richardson extrapolation to estimate the series limit from the first few terms. Their resultant two-point scheme uses two prices to estimate the American price; a twice-exercisable option and a once exercisable (Black–Scholes) option for which a closed-form exists. Although simple, this Geske–Johnson scheme works remarkably well for short times to maturity. However it breaks down for long time to maturity and does not converge to the Merton price (infinite time to maturity).

This note contributes to the literature on American and Real options approximations in two ways. First, based on the GJ method, it establishes a new asymptotically correct approximation scheme for long maturity options or options where the yield on the underlying is low. Second the approximation scheme can be used to infer an asymptotically correct formula that describes the path of the early exercise boundary for long maturities or options where the underlying yield is low.

Section II puts American options onto a surface, for which limiting cases are known, Section III shows how GJ approximation can be used to yield new approximation forms. The assumptions utilized in the work of Barone-Adesi and Whaley (1987) and MacMillan (1986) are summarized in Section IV which also compares the equations that give the asymptotic properties of the exercise boundaries under the method discussed. Section V shows numerical results and Section VI concludes.

II. AMERICAN PRICES IN TIME AND YIELD

Under a risk neutral geometric Brownian motion process S for a project's price or value (with volatility σ and continuous dividend yield δ and risk free rate r), no arbitrage or risk neutrality implies that the price of any option claim[2] C must satisfy a partial differential equation

$$\frac{\mathrm{d}S}{S} = (r-\delta)\mathrm{d}t + \sigma \mathrm{d}W^{Q}$$

$$\frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial S^2} + (r-\delta)\, S\frac{\partial C}{\partial S} - rC - \frac{\partial C}{\partial T} = 0$$

where $C(S, X, \delta, r, \sigma, T)$ is a function of the current value S, exercise price X, dividend yield δ, interest rate r, volatility σ and time to maturity T. Notation is simplified by suppressing the dependency on (S, X, r, σ) and by highlighting the two variables of interest alone T, δ so C_δ^T. It is assumed that the project cash yield δ is lower than the risk free rate r or equivalently that the risk neutral drift $(r-\delta)$ of the project value is positive. This is because the solution method presented is only asymptotically correct for small but positive δ (δ can be thought of as a return shortfall below the rate of return or this required rate less the capital growth gain).

The three boundary conditions for a call on the project are given in the limit as the project value goes to zero, at the optimal exercise threshold $K(T)$ (also a function of δ) as well as a further smooth pasting condition.[3]

Under special restrictions, two solutions are known for American call option prices; the perpetual Merton $(T \to \infty)$ and zero dividend $(\delta \to 0)$ Black Scholes solutions (see Merton, 1973). This means that limiting solution forms are actually known for two edges of a grid in T, δ. Table 1 has the zero dividend Black Scholes values along the top row and Merton values in the left most column. The right hand column and bottom row contain payoffs, $C_\delta^0 = C_{\delta^*}^T = (S-X)^+$ (where the critical dividend yield δ^* is defined later).

Although the interior true American option price surface C_δ^T has unknown form but limits that converge to known forms, if an assumption is made as to the shape or form of the surface separation in time and yield, GJ interpolation is possible using Black Scholes and Merton prices.

Black Scholes limit

Zero dividend Black Scholes call prices $C_{\delta\to 0}^T$ are given by

$$C_{\delta\to 0}^T = SN(d_1) - Xe^{-rT}N(d_2)$$
$$d_{1,2} = \frac{\ln(S/X) + \left(r \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

As expiry approaches $(T \to 0)$ because $d_{1,2}$ will both diverge to $+\infty$ or $-\infty$ (depending on the sign of $\ln(S/X)$), the call option clearly tends to the payoff $(S-X)^+$.

As expiry becomes increasingly distant $(T \to \infty)$ the moneyness of the option becomes irrelevant $((1/\sigma\sqrt{T})\times \ln(S/X) \to 0)$ and $Xe^{-rT} \to 0$) so the call always tends towards the project value S because d_1 always becomes positive in the limit.

$$\lim_{T\to 0} C_0^T = (S-X)^+ \qquad \lim_{T\to\infty} C_0^T = S$$

[1] In fact, for out-of-the-money options, (fourth and) higher order multivariate normal terms are consequential.
[2] C here represents a call, results for puts can be derived from Put Call Symmetry (see Carr and Chesney, 1996).
[3] The call boundary conditions are

$$\lim_{S\to 0} C_\delta^T(S) = 0 \qquad C_\delta^T(K) = (K-X)^+ \qquad \left.\frac{\partial C_\delta^T}{\partial S}\right|_K = 1$$

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Table 1. *American Call options C_δ^T by final maturity T and dividend yield δ and limiting cases (in boxes); Merton perpetuals ($T \to \infty$, C_δ^∞ left column), Black Scholes ($\delta = 0$, C_0^T top row) and Payoffs ($C_\delta^0, C_\infty^T = (S-X)^+$ right column and bottom row). The top left hand box has a value equal to the stock price $C_0^\infty = S$. The critical dividend yield δ^* is defined at the end of Section II*

C_δ^T		Time to final maturity, T: ∞	..	4	..	2	..	1	..	0
	0	$\boxed{C_0^\infty}$		$\boxed{C_0^4}$		$\boxed{C_0^2}$		$\boxed{C_0^1}$		$\boxed{C_0^0}$
	⋮									
Dividend	1%	$\boxed{C_{1\%}^\infty}$		$C_{1\%}^4$		$C_{1\%}^2$		$C_{1\%}^1$		$\boxed{C_{1\%}^0}$
	⋮									
yield	2%	$\boxed{C_{2\%}^\infty}$		$C_{2\%}^4$		$C_{2\%}^2$		$C_{2\%}^1$		$\boxed{C_{2\%}^0}$
	⋮									
δ	4%	$\boxed{C_{4\%}^\infty}$		$C_{4\%}^4$		$C_{4\%}^2$		$C_{4\%}^1$		$\boxed{C_{4\%}^0}$
	⋮									
	δ^*	$\boxed{C_{\delta^*}^\infty}$		$\boxed{C_{\delta^*}^4}$		$\boxed{C_{\delta^*}^2}$		$\boxed{C_{\delta^*}^1}$		$\boxed{C_{\delta^*}^0}$

Merton limit

Perpetual American call option prices $C_\delta^{T\to\infty}$ are given by the Merton formula

$$C_\delta^{T\to\infty} = \begin{cases} (K(\infty) - X)\left(\dfrac{S}{K(\infty)}\right)^\beta & S < K(\infty) \\ S - X & S > K(\infty) \end{cases}$$

$$K(\infty) = \frac{\beta}{\beta - 1}X \quad \beta = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

For the project call as the underlying yield becomes large ($\delta \to \infty$), the option elasticity become large ($\beta \to \infty$) and the critical exercise threshold decreases to the exercise threshold ($K(\infty) \to X$). This implies that the Merton call option reverts to its immediate payoff $(S-X)^+$ and the option will be exercised immediately if in the money or will have zero value if out of the money.

As the underlying yield becomes very small ($\delta \to 0$), the elasticity of the option decreases to one ($\beta \to 1$) but it does so at such a rate that the exercise price in the Merton formula becomes irrelevant and the project option reverts to the project value itself.[4]

$$\lim_{\delta\to\infty} C_\delta^\infty = (S-X)^+ \qquad \lim_{\delta\to 0} C_\delta^\infty = S$$

Note also that there is a finite dividend yield δ^* (as a function of remaining parameters S, X, r, σ) for which the Merton call should be exercised immediately for its payoff value (if positive). This critical dividend yield solves $C_\delta^\infty = (S-X)^+$

$$\lim_{T\to\infty} \delta^* \to \begin{cases} \dfrac{rX}{S} + \dfrac{1}{2}\sigma^2 \dfrac{X}{S-X} & S > X \\ \infty & S \leqslant X \end{cases}$$

When volatility σ is low, this critical yield tends towards a cash return consideration which says that the critical threshold $K(\infty)$ is determined by a condition comparing the yields on the project and the exercise price $\left(\delta^* S \gtrless X\right)$. If the option is at or out of the money ($S \leqslant X$) there is no finite dividend yield that can trigger exercise of a Merton option ($\delta^* \to \infty$).

Subsequently, when parameter values of $(S, X, r, \sigma) = (120, 100, 10\%, 20\%)$ are chosen, a critical yield of $\delta^* = 18\frac{1}{3}\%$ will become apparent.

Zero volatility limit

As well as considering American prices expanded as a function of δ and T, it is useful to consider their behaviour for small volatilities (near $\sigma = 0$). This is because American option prices strictly increase as a function of volatility so the zero case provides a lower bound for all other American prices. A formula for the intrinsic value of an American option is derived for use in Section IV.

[4] The perpetual can also be expressed as

$$(K(\infty) - X)\left(\frac{S}{K(\infty)}\right)^\beta = \left(\frac{X}{\beta - 1}\right)^{1-\beta}\left(\frac{S}{\beta}\right)^\beta$$

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For zero volatility, the perpetual Merton option has a critical threshold that is determined[5] by a yield criterion $K = rX/\delta$. This is in accordance with the analysis of critical dividend yield δ^* for zero volatility. Note that β is a decreasing function of σ while K is an increasing function of σ. When this boundary is reached, the yield rate of exercise δS exceeds the cost rate of exercise rX. The current value of such an exercise strategy is different depending on the current project value relative the critical threshold K. Above this threshold, waiting has no benefit and so immediate exercise is optimal. Below the threshold, it pays to wait just long enough to reach this threshold.

Effectively, the zero volatility perpetual option value is derived from a known finite risk neutral stopping (forward purchase or exercise) time $(1/(r-\delta))\ln(rX/\delta S)$.

$$C_\delta^\infty(\sigma \to 0) = \begin{cases} S - X & S > K = \dfrac{rX}{\delta} \\ \left(\dfrac{r}{\delta} - 1\right) X \left(\dfrac{\delta S}{rX}\right)^{r/(r-\delta)} & S \leqslant K = \dfrac{rX}{\delta} \end{cases}$$

Note that out of the money zero volatility calls are only priced for $r > \delta$ since if this were not the case, the project value having negative drift would never hit any upper exercise boundary.

Having established the perpetual zero volatility American option value it is now easy to generalize to the finite maturity case. The zero volatility value of a finite maturity American price is the maximized value of the payoff $\max_\tau\left[S\mathrm{e}^{-\delta\tau} - X\mathrm{e}^{-r\tau}, 0\right]$ chosen over optimal risk neutral stopping time $\tau \in [0, T]$.

If T exceeds the zero volatility stopping time defined earlier $(1/(r-\delta))\ln(rX/\delta S)$, then the optimal value is derived from exercise of a perpetual at the rX/δ threshold and stopping is chosen within the interval $[0, T]$. However, if T is less than the time to optimal exercise of a perpetual, then the value is determined as a European payoff at T. This may or may not be positive depending on whether or not sufficient time T remains to take the option from its current moneyness to a point in the money.

Combining the payoffs at T, or earlier if optimal, yields a general formula for the zero volatility or Intrinsic Value (IV or zero volatility value) of an American option of maturity T

$$\mathrm{IV} = \begin{cases} \left(\dfrac{r}{\delta} - 1\right) X \left(\dfrac{\delta S}{rX}\right)^{r/(r-\delta)} & T \geqslant \dfrac{1}{r-\delta}\ln\dfrac{rX}{\delta S} \\ \max\left[S\mathrm{e}^{-\delta T} - X\mathrm{e}^{-rT}, 0\right] & 0 < T < \dfrac{1}{(r-\delta)}\ln\dfrac{rX}{\delta S} \end{cases}$$

For a maturity T, this can help approximation methods since it forms a lower bound for estimated American prices.[6]

Critical exercise boundary in time and yield

When the American price falls below its payoff value, exercise is preferred. Thus the general critical exercise boundary where the surface of American prices intercepts the payoff plane solves $C_\delta^T(S, X) = S - X$.

Although of generally unknown form, the critical dividend yield and time to maturity pair (T, δ) that satisfy the payoff condition (for given S, X, r, σ) have two important limiting cases. As T becomes large, $\delta \to \delta^*$ as established.

The other case to consider is $T \to 0$, if out of the money ($S<X$), again there is no dividend yield that can trigger early exercise of an American and this is true at expiry also when American options expire worthless. If in the money one of two things can happen. First the option can expire and be exercised for its Black Scholes value, second it can be exercised before maturity for its American value. The former Black Scholes (not early) exercise, can only happen if $r > \delta$ and $X < S < rX/\delta$.

III. BARONE-ADESI AND WHALEY AND MACMILLAN

Initially without loss of generality, Barone-Adesi and Whaley (1987), MacMillan (1986) (BAWM) decomposed the early exercise premium of an American (with dividends) over the corresponding European (with dividends) into a separable function of the time to maturity and the project value and time to maturity

$$C_\delta^T - S\mathrm{e}^{-\delta T}N(d_1) + X\mathrm{e}^{-rT}N(d_2) = j(T)k\left(S, 1 - \mathrm{e}^{-rT}\right)$$

[5] This can be shown by evaluating β as $\sigma \to 0$. Since β can be rewritten as a square plus a term that remains finite as $\sigma \to 0$, its limit can be established.

$$\beta = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{(r-\delta)}{\sigma^2} + \frac{r+\delta}{2(r-\delta)}\right)^2 - \left(\frac{r+\delta}{2(r-\delta)}\right)^2 + \frac{1}{4}}$$

$$\lim_{\sigma\to 0}\beta = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \frac{(r-\delta)}{\sigma^2} + \frac{r+\delta}{2(r-\delta)} = \frac{r}{r-\delta}$$

$$K = \frac{\beta}{\beta-1}X = \frac{r}{\delta}X$$

[6] Note that for the perpetual call, if the current project value is already above the yield threshold $S > rX/\delta > X$ then there is an lower limit for volatility that will trigger immediate exercise because the benefit of waiting beyond rX/δ decreases with decreasing σ.

$$\sigma^* = \sqrt{\frac{2(S-X)(\delta S - rX)}{SX}}$$

Since each of these claims must satisfy the partial differential equation pricing equation, there are restrictions on the forms of functions j, k. They then made the specific restrictive approximation that in the resultant differential equation, the time dependence of the $k(S, 1-e^{-rT})$ term is small[7] so that the separable (approximate) solution can be recovered explicitly.

$$j(T)\,k\left(S, 1-e^{-rT}\right) \approx \left(1-e^{-\delta T}N(d_1(K(T), X, \delta, T))\right)\frac{K(T)}{\gamma(T)}\left(\frac{S}{K(T)}\right)^{\gamma(T)}$$

$$\gamma(T) = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{(1-e^{-rT})\sigma^2}} > \beta$$

A new time dependent elasticity parameter $\gamma(T)$ that is greater than β is used. The early exercise level $K(T)$ is determined by a boundary condition which reduces to

$$\frac{K(T)}{X} = \frac{1-e^{-rT}N(d_2(K(T), \delta, T))}{1-e^{-\delta T}N(d_1(K(T), \delta, T))}\frac{\gamma(T)}{\gamma(T)-1}$$

Note that because of the multiple dependence on $K(T)/X$ this must be solved numerically since this critical value is required before any further option calculations are made, $\gamma(T)$ is also time dependent.

BAWM critical threshold

For the case $r > \delta$, the BAWM threshold has a finite critical time T^* where its critical threshold $K(T^*)$ is equal to the Merton perpetual threshold (because of the term that was assumed to be close to zero). This can be seen by setting $K(T^*)/X = \beta/(\beta-1)$ and noting that $\gamma(T) > \beta$ and $\gamma(T)/(\gamma(T)-1) < \beta/(\beta-1)$ so that

$$\frac{\beta}{\beta-1}\frac{\gamma(T^*)-1}{\gamma(T^*)} = \frac{1-e^{-rT^*}N(d_2(K, \delta, T^*))}{1-e^{-\delta T^*}N(d_1(K, \delta, T^*))} = h(T^*) > 1$$

$$e^{-rT^*}N(d_2(K, \delta, T^*)) < e^{-\delta T^*}N(d_1(K, \delta, T^*))$$

since $\gamma(T^*) > \beta$ and therefore $\gamma(T^*)/(\gamma(T^*)-1) < \beta/(\beta-1)$. Since $N(d_1) > N(d_2)$ there is a critical time T^* where the BAWM threshold crosses the Merton threshold and BAWM will yield an inconsistent critical price. Note that this result does not hold if $r < \delta$ when there is no finite time that satisfies the critical equation.

Figure 2 shows the BAWM critical threshold as a function of time T for $(X, r, \delta, \sigma) = (100, 10\%, 4\%, 20\%)$. Note that it crosses the Merton perpetual threshold at about 5 years. Therefore it will yield prices higher than the Merton formula that only converge towards it *from above* (shown in Fig. 1, these cross the Merton level at $T = 10$). This would make BAWM of no practical value in assessing stopping levels for real options of long maturity since it would seemingly be advantageous to wait beyond the perpetual option stopping level!

IV. GESKE–JOHNSON APPROXIMATION

Linear separation

Using other option prices that are simpler to evaluate, Geske–Johnson (1984) produced an approximation technique for American options. The method effectively assumes that the American option is the limit of a Bermudan

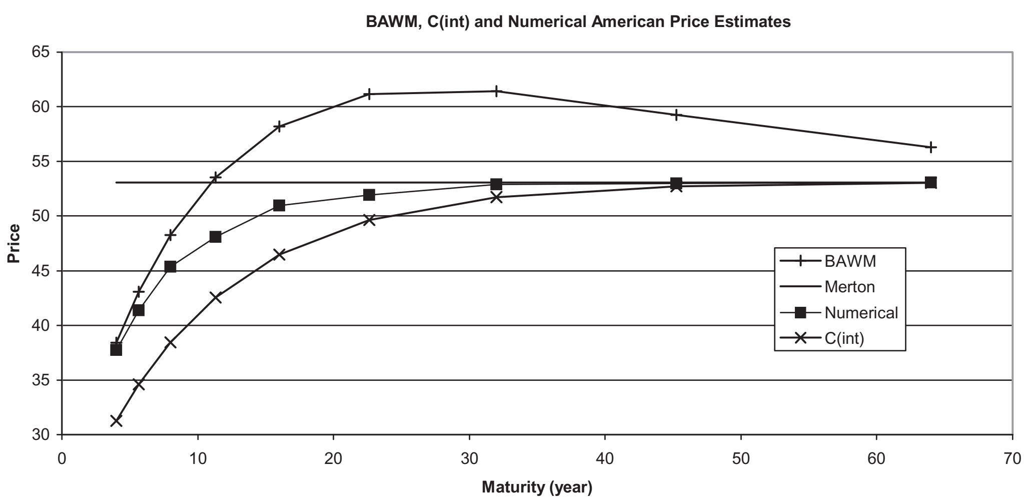

Fig. 1. *Price estimates (BAWM, Metron, Numerical and C_δ^T(int)) for (S, X, r, δ, σ) = (120, 100, 0.10, 0.04, 0.20)*

[7] $(1-e^{-rT})(\partial k/\partial(1-e^{-rT})) \approx 0$ is satisfied either if T is large, $\partial k/\partial(1-e^{-rT}) \approx 0$ or T small $(1-e^{-rT}) \approx 0$.

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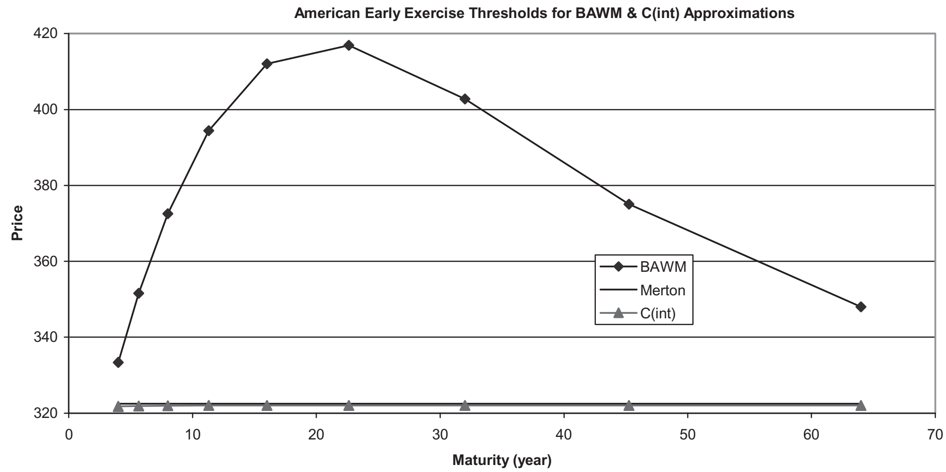

Fig. 2. *Critical excercise thresholds estimated for BAWM, Merton and C_δ^T(int) all for $(X, r, \delta, \sigma) = (100, 0.10, 0.04, 0.20)$*

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(periodically exercisable) price which itself can be priced as a separable function of its time to maturity and inter-exercise time. Two point GJ pricing further assumes that the Bermudan (periodically exercisable) price is linear in interexercise time.

Here we assume that the true American price $C_\delta^T(\text{true})$ function is approximated by a formula $C_\delta^T(\text{lin})$ that is linearly separable in two (initially arbitrary) positive functions $f(T)$, $g(\delta)$ of (T, δ)

$$C_\delta^T(\text{true}) \approx C_\delta^T(\text{lin}) = C_0^\infty - f(T) - g(\delta) \tag{1}$$

$$\lim_{T\to\infty} f(T) = 0 \qquad \lim_{\delta\to 0} g(\delta) = 0$$

f, g can be recovered by calibration from known limiting American prices at $T \to \infty$ and $\delta \to 0$

$$f(T) = C_0^\infty - C_0^T \qquad g(\delta) = C_0^\infty - C_\delta^\infty$$

thus yielding the closed form two point approximation scheme

$$\begin{aligned} C_\delta^T(\text{true}) \approx C_\delta^T(\text{lin}) &= C_\delta^\infty + C_0^T - S \\ &= \left(\frac{X}{\beta - 1}\right)^{1-\beta}\left(\frac{S}{\beta}\right)^{\beta} + SN(d_1) - Xe^{-rT}N(d_2) - S \end{aligned} \tag{2}$$

One problem with this additive method is that it may yield negative values for option prices, although exercise for a zero value would seemingly be a better strategy, this is still an unsatisfactory result so other assumptions were explored.

Multiplicative separation

The second method, $C_\delta^T(\text{mult})$ assumes multiplicative separation. If the log of the option price is additively separable in $f'(T)$, $g'(\delta)$ rather than the price itself, a multiplicative formula will arise that corresponds to that of Ho *et al.* (1994), (again the maximum of the option price and the payoff are preferred)

$$C_\delta^T(\text{true}) \approx C_\delta^T(\text{mult}) = f'(T)\, g'(\delta)\, C_0^\infty = \frac{C_\delta^\infty C_0^T}{S} \tag{3}$$

This formula benefits from the fact that it cannot produce negative option prices.

Interpolated linear and multiplicative form

The third and preferred possibility $C_\delta^T(\text{int})$ is to make an assumption about the slope of the surface $C_\delta^T(\text{true})$. Assuming that the derivative of the surface satisfies the following condition (slopes are a constant multiple across δ)[8]

$$C_\delta^T(\text{true}) \approx C_\delta^T(\text{int}) \tag{4}$$

$$\frac{\partial C_\delta^\tau(\text{int})}{\partial \tau} = a\frac{\partial C_0^\tau(\text{int})}{\partial \tau}$$

$$C_\delta^T(\text{int}) = C_\delta^0 + \int_0^T \frac{\partial C_\delta^\tau}{\partial \tau}\, d\tau = C_\delta^0 + a\left[C_0^T - C_0^0\right]$$

and calibrating $a = (C_\delta^\infty - C_\delta^0)/(C_0^\infty - C_0^0)$ at $T \to \infty$ yields the third, interpolated, approximation form

$$C_\delta^T(\text{int}) = \frac{C_0^\infty - C_0^T}{C_0^\infty - C_0^0} C_\delta^0 + \frac{C_0^T - C_0^0}{C_0^\infty - C_0^0} C_\delta^\infty \tag{5}$$

[8] It is also possible to assume the following

$$\frac{\partial C_D^\infty}{\partial D} = b\frac{\partial C_D^T}{\partial D}$$

which will generate another form of interpolated result.

The Zero Volatility lower bound was also used to improve this approximation result, which since $C_0^0 = C_\delta^0 = (S - X)^+$ and $C_0^\infty = S$ yields

$$C_\delta^T(\text{int}) = \max\left(\frac{C_0^T C_\delta^\infty - \max(S - X, 0)\left(C_0^T + C_\delta^\infty - S\right)}{\min(S, X)}, \text{IV}\right)$$

$$= \begin{cases} \max\left(\dfrac{C_0^T C_\delta^\infty - (S - X)\left(C_0^T + C_\delta^\infty - S\right)}{X}, \text{IV}\right) & S > X \\ \max\left(\dfrac{C_0^T C_\delta^\infty}{S}, \text{IV}\right) & S \leqslant X \end{cases}$$

Since this interpolated form $C_\delta^T(\text{int})$ has a different form for $S \gtrless X$ it has better overall performance than either the linear $C_\delta^T(\text{lin})$ or multiplicative $C_\delta^T(\text{mult})$ separation cases. Consequently in Table 2 the results against the benchmark numerical procedure are reported for it $C_\delta^T(\text{int})$ alone.

V. NUMERICAL RESULTS

Benchmark method

Numerical price estimates were obtained using a binomial tree approach with two modifications. First, Black Scholes values were inserted into the penultimate grid point before expiry allowing more accurate use of a closed form solution for one extra grid point subsuming the expiry point and thus use one fewer approximate interim point (Binomial Black–Scholes or BBS). The second amendment is to then use Richardson extrapolation (see GJ for this method) to improve American prices at each point on the grid from previously derived estimates on the grid. This Binomial Black–Scholes–Richardson extrapolation version is known as the BBSR method. Benchmark numerical values are typically estimated using 10 000 steps per year so we followed this convention for the values in this note.

Results

Table 2 shows the benchmark prices with the percentage error of the approximate methods of BAWM (upper) and this note ($C_\delta^T(\text{int})$ lower). Both methods converge to the true price scheme in the zero dividend and infinite maturity cases, However, the new method performs better than BAWM in the long maturity and low yield cases. In the columns for $T = 32$, 64 years the percentage errors are lower for the new method cases compared to BAWM.

Thus the new formula is more appropriate than BAWM for approximating the price of American option situations where the time to maturity is greater than about 16 years.

Figure 1 shows one cross-section from Table 2 ($\delta = 4\%$). It clearly indicates that the BAWM yields prices that can exceed the Merton perpetual solution. Furthermore, Fig. 2 shows that using the BAWM method to solve for the critical boundary can yield thresholds that are above the perpetual boundary. The proposed new method

Table 2. *Numerical price estimates and BAWM, $C_\delta^T(\text{int})$ percentage price errors by time and yield*

Time Dividend yield	Infinite	$T=64$	$T=32$	$T=16$	$T=8$	$T=4$	$T=2$
$\delta = 0\%$	120.00	119.83	115.93	99.98	75.74	54.11	36.26
BAWM	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
$C_\delta^T(\text{int})$	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
$\delta = \frac{1}{2}\%$	101.5	101.5	100.16	90.83	71.15	51.86	38.15
BAWM	0.0%	11.8%	9.7%	3.1%	0.4%	0.0%	0.0%
$C_\delta^T(\text{int})$	0.0%	−0.1%	−2.4%	−6.6%	−8.4%	−8.1%	−6.7%
$\delta = 1\%$	89.52	89.52	89.01	82.67	66.75	49.66	37.06
BAWM	0.0%	14.5%	15.0%	6.3%	1.1%	0.1%	0.0%
$C_\delta^T(\text{int})$	0.0%	−0.1%	−2.6%	−8.5%	−12.0%	−12.0%	−9.9%
$\delta = 2\%$	73.36	73.36	73.05	69.31	58.62	45.44	34.92
BAWM	0.0%	12.9%	19.0%	11.0%	3.0%	0.4%	0.0%
$C_\delta^T(\text{int})$	0.0%	−0.1%	−2.6%	−9.6%	−15.1%	−15.9%	−13.3%
$\delta = 4\%$	53.09	53.09	52.89	50.94	45.37	37.76	30.86
BAWM	0.0%	6.1%	16.1%	14.3%	6.3%	1.7%	0.3%
$C_\delta^T(\text{int})$	0.0%	−0.1%	−2.2%	−8.8%	−15.3%	−17.1%	−14.5%
$\delta = 8\%$	32.31	32.31	32.25	31.67	29.93	27.20	24.39
BAWM	0.0%	0.6%	4.5%	7.7%	5.9%	3.0%	1.2%
$C_\delta^T(\text{int})$	0.0%	−0.1%	−1.4%	−5.8%	−10.2%	−11.0%	−8.3%
$\delta = 16\%$	20.38	20.38	20.38	20.38	20.34	20.21	20.05
BAWM	0.0%	0.1%	−0.1%	−0.4%	−0.4%	−0.2%	−0.1%
$C_\delta^T(\text{int})$	0.0%	0.1%	−0.1%	−0.4%	−0.6%	−0.4%	0.1%

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C_δ^T(int), also shown on both graphs, does not suffer from this problem.

VI. CONCLUSION

American and Real option pricing problems although generally difficult to solve have two well known asymptotic solutions. GJ interpolation can use these to estimate the unknown interior of the price surface. The approximation forms derived are all analytic and asymptotically correct for large T or small δ.

The comparative statistics of these approximations are easily derived because they are in explicit closed for unlike the BAWM method. Furthermore the methods derived here do not suffer from the same problem as the BAWM method, they always converge from the correct side, unlike BAWM when $r > \delta$. For long maturities (about five years), they thus offer ease of implementation compared to both BAWM and numerical techniques and advantages of accuracy compared to the former.

Finally, whereas most numerical methods infer the price of longer maturity American options from shorter ones, the asymptotic long maturity approximation here could be used as a seed to numerically estimate shorter maturity prices from longer ones! Any error introduced by this asymptotic approximation of the long maturity seed would be no more onerous than that introduced when the first finite American option price at the first time step is estimated from a discrete step to maturity. Increased accuracy would demand either a smaller first time step when working from short to long maturity or a longer initial maturity when working from long to short maturities.

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