On the Errors and Comparison of Vega Estimation Methods

SAN-LIN CHUNG*
MARK SHACKLETON

This article discusses convergence problems when calculating Vega (option sensitivity to volatility) that arise from discretization errors embedded in the lattice approach. Four alternative improvements to the traditional binomial method are discussed and investigated for performance. We also propose a new Modified Binomial (MB) Method to calculate Vegas. Numerical results show that although the MB is not the most price accurate of the models, due to its error structure as a function of volatility, it produces the most accurate and fastest Vega estimates. © 2005 Wiley Periodicals, Inc. Jrl Fut Mark 25:21–38, 2005

INTRODUCTION

Once the price of an option position has been negotiated and the position established, tracking and maintenance of the hedging properties are essential tasks not only for the buyer but especially for the option seller.

Thanks go to the Editor and Referee for their helpful comments. Financial support from the National Science Council of Taiwan is acknowledged.

*Correspondence author, National Taiwan University, 106 Taipei, Taiwan, Republic of China; e-mail: chungs@mba.ntu.edu.tw

Received July 2003; Accepted April 2004

- *San-Lin Chung is in the Department of Finance at National Taiwan University in Taipei, Taiwan, Republic of China.*
- *Mark Shackleton is in the Department of Accounting and Finance in the Management School at Lancaster University, United Kingdom.*

Published online in Wiley InterScience (www.interscience.wiley.com). DOI:10.1002/fut.20127

This is because the option's risk characteristics change dynamically as the stock price and time to maturity change. Therefore, the so-called "Greeks" (or partial differentials) with respect to model variables must be calculated accurately and repeatedly.

For many different option models and numerical estimation methods, there is a body of literature concerning the properties of these Greeks that annotates the problems associated with their numerical procedures. While producing option sensitivities that converge in the number of time steps (grid points or tree nodes), numerical differentiation is hazardous because results converge in an oscillatory fashion (Pelsser & Vorst, 1994; Chung & Shackleton, 2002, for details).[1]

In addition to the first (second) differentials with respect to price and time *variables,* Delta, (Gamma), and Theta, there are Greeks that represent differentials with respect to option *parameters*. The first differential with respect to volatility (and interest rate), Vega (Rho), are also calculated and often quoted even though many option models assume these factors to be constant. This is because these are used in a slightly different, but no less important manner by purchasers and hedgers of options.

There are several reasons why an option's sensitivity to a fixed (or potentially nondynamic) parameter needs to be considered and calculated repeatedly and accurately. It is also clear for option models that explicitly model stochastic volatility and treat it as a variable, why Vega sensitivity is important.

First, even if an option model has a fixed volatility, there is the danger of parameter mis-estimation. It may be the case that volatility may be constant but just difficult to estimate accurately with limited data. This means that the sensitivity of a model's volatility assumption needs to be examined through its Vega to test the sensitivity to any one specific chosen volatility value.

Second, even for constant volatility models, potential uncertainty about the exact model specification means that different models may be used not only to estimate the option's price but also to estimate the impact of the same volatility assumption in different models.

Third, for tractability, many practitioners may assume that these last two variables (volatility and interest rates) are known and constant, in the knowledge that this is only an approximation. If the validity of this assumption is questionable, they may use two option positions with

[1]This is largely due to the fact that changing the number of time steps (or another numerical parameter) changes the number of nodes on either side of the exercise point, therefore the estimated price tends to oscillate around the true price as it converges.

opposite Vega in an attempt to eliminate their volatility risk. Essentially they may estimate Vega from each model with constant volatility, hoping that if volatility actually changes that it will affect both of the mis-specified model prices to the same degree and due to the supposed Vega neutrality—cancel each other out.

Finally, there are an increasing number of models that include stochastic rather than fixed volatility. Although in this paper we apply numerical methods to the estimation of Vega sensitivity with fixed volatility, our results relate to other more complex Vega calculation problems.

We first review the five methods that are currently used or discussed in the literature. Then we propose a new method, named the Modified Binomial (MB) method and go on to assess the efficiency of each of the five existing methods as well as the new method proposed. We also examine the root mean square error against computational speed (efficiency) of each method. In the final section we present our conclusions.

VEGA ESTIMATION

The numerical error of Vega estimates is mainly due to the discretization embedded in the binomial lattice approach, particularly at option expiry where the positioning of nodes with respect to the exercise price is critical. Within the literature, there are at least four proposed solutions that reduce the discretization errors or enhance the rate of convergence compared to the standard tree structure. First, addition of one or more small sections of fine high-resolution lattice within a tree with coarser time and price steps (Figlewski & Gao, 1999). Second, adjustment of the discrete-time solution prior to maturity (Broadie & Detemple, 1996; Heston & Zhou, 2000) or smoothing of the payoffs at maturity (Heston & Zhou, 2000). Finally, allocation and number (the shape and span) of tree nodes (Ritchken, 1995; Tian, 1999; Widdicks, Andricopoulos, Newton, & Duck, 2002). In this section we review the motivation and construction of each method.

Binomial Method

In the lattice approach to option pricing and hedging (for example, see Hull, 2000), it is common wisdom in calculating Vega to make a small change, $\Delta\sigma$, to the volatility σ and construct a new tree to obtain a new value of the option. This is then used to calculate a numerical derivative. Since the magnitude of the pricing error implicit in the discrete tree depends on the number of steps in the tree (as well as other things), usually the same number of steps is used so as to minimize potential errors involved in this numerical differentiation. The estimate of Vega is then

$$V(\sigma, \Delta\sigma, n) = \frac{P(\sigma + \Delta\sigma, n) - P(\sigma, n)}{\Delta\sigma} \tag{1}$$

where $P(\sigma + \Delta\sigma, n)$ and $P(\sigma, n)$ are the estimates of the option price from the original and the new tree, respectively (both with n steps). Note that this is only a numerical differential of the true sensitivity $V(\sigma, 0, n) = \lim_{\Delta\sigma \to 0} V(\sigma, \Delta\sigma, n)$ and as such will contain estimation error for the Vega that depends on the size of $\Delta\sigma$ and the number of tree steps (n).

However, there is a convergence problem similar to that of pricing barrier-type options (Boyle & Lau, 1994; Ritchken, 1995). This method for calculating Vega also faces discretization errors embedded within the lattice approach, particularly with respect to the positioning of final nodes around the exercise threshold. The problem is serious because the two estimates of the option price are obtained using two different trees where the final nodes are *noncoincident* (i.e., same number but differently placed nodes) so that discretization errors in each estimate are imperfectly correlated and do not cancel.

$$\begin{aligned} P_{\text{est.}}(\sigma, n) &= P_{\text{true}}(\sigma, n) + \varepsilon(\sigma, n) \\ P_{\text{est.}}(\sigma + \Delta\sigma, n) &= P_{\text{true}}(\sigma + \Delta\sigma, n) + \varepsilon'(\sigma + \Delta\sigma, n) \end{aligned} \tag{2}$$

$$\begin{aligned} V(\sigma, \Delta\sigma, n) &= \frac{P_{\text{est.}}(\sigma + \Delta\sigma, n) - P_{\text{est.}}(\sigma, n)}{\Delta\sigma} \\ &= \frac{P_{\text{true}}(\sigma + \Delta\sigma, n) - P_{\text{true}}(\sigma, n)}{\Delta\sigma} + \frac{\varepsilon'(\sigma + \Delta\sigma, n) - \varepsilon(\sigma, n)}{\Delta\sigma} \end{aligned} \tag{3}$$

Thus with imperfectly correlated errors, Vega is always estimated with error. This total error will depend on the way the second error depends on the first. If the best-fit line between them is of unit slope and zero intercept, then their cancellation can be expected and the variance of the residual error will depend on the goodness of fit between the two errors. Hence, investigation of the error dependency is a crucial part of the analysis of Vega estimation. This is conducted in the Numerical Results section.

Adaptive Mesh Model

To improve price convergence Figlewski and Gao (1999) propose a method termed the *Adaptive Mesh Model* (AMM) to reduce the convexity error at the terminal boundary. In their method, one or more small sections of fine high-resolution lattice are added into a tree with coarser time and price steps. This incorporates a finer mesh of values around the critical nodes in the coarse tree and thus achieves greater accuracy by reducing the nonlinearity error.

This method has been shown to reduce *pricing* errors considerably. The cost is that the tree now has different structures in different regions and so is more complex to implement. Furthermore, although the *location* of the maximum convexity is know (near the money at maturity), the choice of *span* around this area is arbitrary. Any number of extra nodes between two and the number of steps already used could be added (in the latter case the tree is again a regular one with twice as many steps).

Thus, this method partially smooths the option price with respect to a perturbation of its parameters. With the increased node density around the exercise threshold, price oscillation in n is also reduced. However, in light of Equation (2) the correlation of these errors is shown to be as important as the magnitude, and so the correlation structure of the price errors should also be considered.

As will be seen in a latter section a lack of error correlation structure, means that although individually accurate for prices, the AMM method does not actually perform well for Vega estimation.

Binomial Black Scholes

The next method is to adjust the discrete-time solution one period prior to maturity. For instance, Broadie and Detemple (1996) proposed the so-called *binomial Black Scholes (BBS) method.* They augmented the binomial tree method through the addition of Black Scholes prices at the penultimate pricing node, arguing that since one period before maturity the option will revert to either its European value or payoff, these closed-form expressions would be useful in increasing the accuracy of American or other option types. This is akin to using a continuum of new nodes over the last interval to generate a smooth function.

Chung and Shackleton (2002) have shown that the numerical differentiation of the BBS prices produces very accurate estimates for option Deltas and Thetas because the method itself contains a smooth and not discrete function of the stock price and time to maturity. Therefore, it produces option values that are also a smooth function of the initial price. This is especially important when employing numerical differentiation over arbitrarily small changes in a time or price parameter.

In this article we further investigate the accuracy of numerical Vegas using the BBS method and show that smoothness in time and stock price space does not necessarily assist estimation of other option sensitivities such as Vega.

Heston Zhou Method

The second method, Heston Zhou (HZ), to smooth the option's payoff at maturity is from Heston and Zhou (2000). They first show that the accuracy or rate of convergence of the binomial method depends crucially on the smoothness of the payoff function. Intuitively, if the payoff function at singular (infinite convexity) points can be smoothed, the binomial recursion will be more accurate. They propose an approach that smooths the payoff function of a European option. If $g(x)$ is the payoff function, they suggest setting the smoothed payoff function $G(x)$ as follows

$$G(x) = \frac{1}{2\Delta x}\int_{-\Delta x}^{\Delta x} g(x-y)\,dy$$

where Δx is the step size of the binomial tree.[2] The above transformation is called rectangular smoothing of $g(x)$. The smoothed function $G(x)$ can be easily computed analytically for most payoff functions used in practice. Applying the binomial model to $G(x)$ instead of $g(x)$ yields a rather surprising and interesting result. The associated binomial prices converge now at a rate of $1/n$ to the continuous-time limit and this convergence is uniform across the nodes of the binomial tree (in the same paper Heston and Zhou show that without this correction the rate goes with $1/\sqrt{n}$). Against this convergence benefit, this method may suffer greater initial price error, especially for small n.

[2]Heston and Zhou (2000) make a transformation of variables by setting $x = [\ln S - (r - \frac{1}{2}\sigma^2)t]$ and thus model log prices. This smoothing procedure helps numerical differentiation but leads to a price bias. Similar to Jensen's Inequality, the expectation of a less convex payoff is higher than the more convex payoff.

Ritchken's Trinomial Method

It is possible to reduce the discretization errors embedded in the lattice approach by allocating the nodes of the binomial or trinomial tree so that they match the payoff of the option or satisfy some other specific requirement (e.g., *overlapping* nodes of two trees as required in calculating Vegas). For example, Ritchken (1995) takes advantage of the flexibility (additional degree of freedom) offered by the trinomial model and proposes an ingenious way to "stretch" the node separation so that one layer of price nodes coincides exactly with the barrier or final exercise price that is problematic. The idea can be applied in constructing trinomial trees for two different volatilities so that their nodes are *always coincident.* In a standard trinomial tree, the asset price at any given time, can move into three possible states, up, down, or middle, in the next period. If S_t denotes the asset price at time t, then at time $t + \Delta t$ the prices will be uS_t, mS_t, or dS_t. Parameters are defined as follows

$$\begin{aligned} u &= e^{\lambda\sigma\sqrt{\Delta t}} \\ m &= 1 \\ d &= e^{-\lambda\sigma\sqrt{\Delta t}} \end{aligned}$$

where $\lambda \geq 1$ (a dispersion parameter generated by the extra degree of freedom) is chosen freely as long as the resulting probabilities are positive. For fastest convergence, Omberg (1988) suggested setting the free parameter $\lambda = \frac{\sqrt{2\pi}}{2}$ (according to a normal density condition).

Matching the first two moments (i.e., per period mean M and variance Σ) of the risk-neutral returns distribution leads to the probabilities associated with these states

$$\begin{aligned} P_u &= \frac{u(\Sigma + M^2 - M) - (M - 1)}{(u - 1)(u^2 - 1)} \\ P_m &= 1 - P_u - P_d \\ P_d &= \frac{u^2(\Sigma + M^2 - M) - u^3(M - 1)}{(u - 1)(u^2 - 1)} \end{aligned}$$

where[3] $M = e^{(r-q)\Delta t}$ and $\Sigma = M^2(e^{\sigma^2\Delta t} - 1)$. For the trinomial trees of two different volatilities (σ and $\sigma + \Delta\sigma$) to have exactly the same nodes

[3] q is the dividend yield.

positions, λ should be chosen differently for the σ and $\sigma + \Delta\sigma$ trees so that

$$e^{\lambda\sigma\sqrt{\Delta t}} = e^{\lambda^*(\sigma+\Delta\sigma)\sqrt{\Delta t}} \tag{4}$$

Therefore, following Omberg we set $\lambda = \frac{\sqrt{2\pi}}{2}$ but determine λ^* from σ and $\Delta\sigma$ accordingly.

$$\lambda^* = \frac{\lambda\sigma}{\sigma + \Delta\sigma} = \frac{\sqrt{2\pi}}{2}\frac{\sigma}{\sigma + \Delta\sigma}$$

Now the final σ and $\sigma + \Delta\sigma$ tree nodes are always located at the same points although the probabilities for each branch in the trees are different. This reduces discretization error where nodes and critical boundaries oscillate as a function of step number n.

The next section proposes a sixth and new method that also exploits careful node placing to reduce discretization error.

MODIFIED BINOMIAL METHOD

We propose a new Modified Binomial (MB) method to calculate Vegas. Our idea is derived from Amin (1991), who suggested a binomial method to price options where the underlying asset has a time-varying (or functional) volatility.

Amin (1991) allowed time step sizes of varying magnitude to cancel out the variable volatility term. Similarly, we increase the number of steps in the $\sigma + \Delta\sigma$ tree by two above the σ tree so that at maturity all but the two new nodes in the new tree are common and coincident. In other words, if we denote the time step sizes of the σ and $\sigma + \Delta\sigma$ trees as Δt, $\Delta t'$, we set

$$\Delta t = \frac{T}{n}$$

$$\Delta t' = \frac{T}{n+2}$$

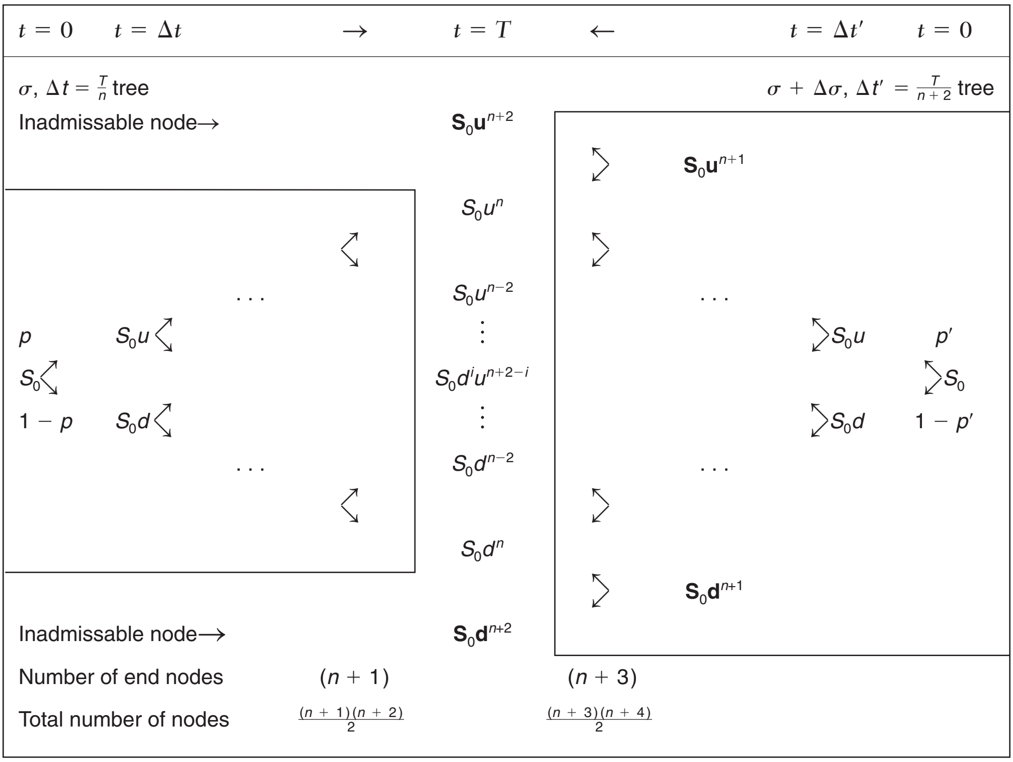
$$u = e^{\sigma\sqrt{\Delta t}} = e^{(\sigma+\Delta\sigma)\sqrt{\Delta t'}}$$

$$d = \frac{1}{u}$$

This implies that $\Delta\sigma = (\sqrt{\frac{n+2}{n}} - 1)\sigma$ and the volatility increment is fixed in proportion to the volatility by the size of the first tree and the number of new nodes at maturity (2), $\Delta\sigma$ can still be made arbitrarily

TABLE I

Two Binomial Trees (the Larger Reversed in Time to Aid Comparison) With Different Volatilities and Probabilities p, p' and All Final Nodes Coincident Except Two. The Four Stock Prices That Generate New Levels Are in Bold.



small by making n large. Also note that the probability of an up movement as well as the per period discount rate in the new tree are again different.[4] The detailed tree structures of the modified binomial method are shown in Table I, note the two extra nodes (inadmissible for the first tree), one at both extremes of the final asset price.

Now, as volatility is increased one new node is added above and below the exercise price so that the balance of nodes on either side is less likely to change. Since all but two final nodes are coincident with those in the initial tree, the distance between the exercise threshold and its closest node stays the same when the tree is changed and the potential for oscillation around the exercise threshold is lessened.

It is worth discussing that, in the spirit of the extended binomial tree method of Pelsser and Vorst (1994), the modified binomial method

[4]Probabilities $p' = [(e^{r\Delta t'} - d)/(u - d)]$ are smaller than the initial ones $p = [(e^{r\Delta t} - d)/(u - d)]$ because discount factors $e^{r\Delta t'}$ are smaller than $e^{r\Delta t}[(\Delta t'/\Delta t) = (n/n + 2)]$.

and the trinomial method are both essentially one-tree structures because the trees for two different volatilities share almost all of the same nodes. They share the same stock price nodes so that each does not need separate storage during computation, only the option prices need separate storage along with the two probabilities. Therefore, this method and the trinomial method will be more computationally efficiently than the other four methods.

All of the six methods discussed in this paper are not mutually exclusive as their features can be combined. For example, the idea of the BBS method can be applied to the trinomial or the modified binomial methods. The AMM method can also be added to the trinomial method to improve accuracy. However, we have considered each method separately to evaluate their individual costs and benefits.

NUMERICAL RESULTS

To investigate the convergence properties of Vega by method, numerical values were calculated for the initial and perturbed volatility ($\sigma = 40\%$, $\Delta\sigma = \sigma(\sqrt{\frac{n+2}{n}} - 1)$) and investigated using differing numbers of total steps in the tree. The same volatility perturbation was used across all methods to ensure a fair comparison.

For number of time steps n from 20 to 100 Panels A to F in Figure 1 for the six methods show dollar errors against the known theoretical value (Black Scholes). Panels A to F in Figure 2 show the dollar errors for the two volatilities in scatter graph form for the six methods. Finally, Panels A to F in Figure 3 show the resulting numerical Vega derived from the perturbation around the initial volatility for the six methods (again as a function of number of time steps).

Table II shows the root mean square errors for each method for n from 20 to 100 and 500 to 1,000.

Binomial Method

It is known from the literature that numerical values oscillate around the true limiting values (the benchmark $n \to \infty$ Black-Scholes value in this case) as n increases. The results here show that this oscillatory convergence is a problem for estimation of Vega as well as other of the so called-Greeks (partial derivative hedge ratios).

First, Panel A of Figure 1 reaffirms the result, that neighboring values (in n the number of steps) have pricing errors, which are negatively serially correlated. Panel A of Figure 2 shows that although these

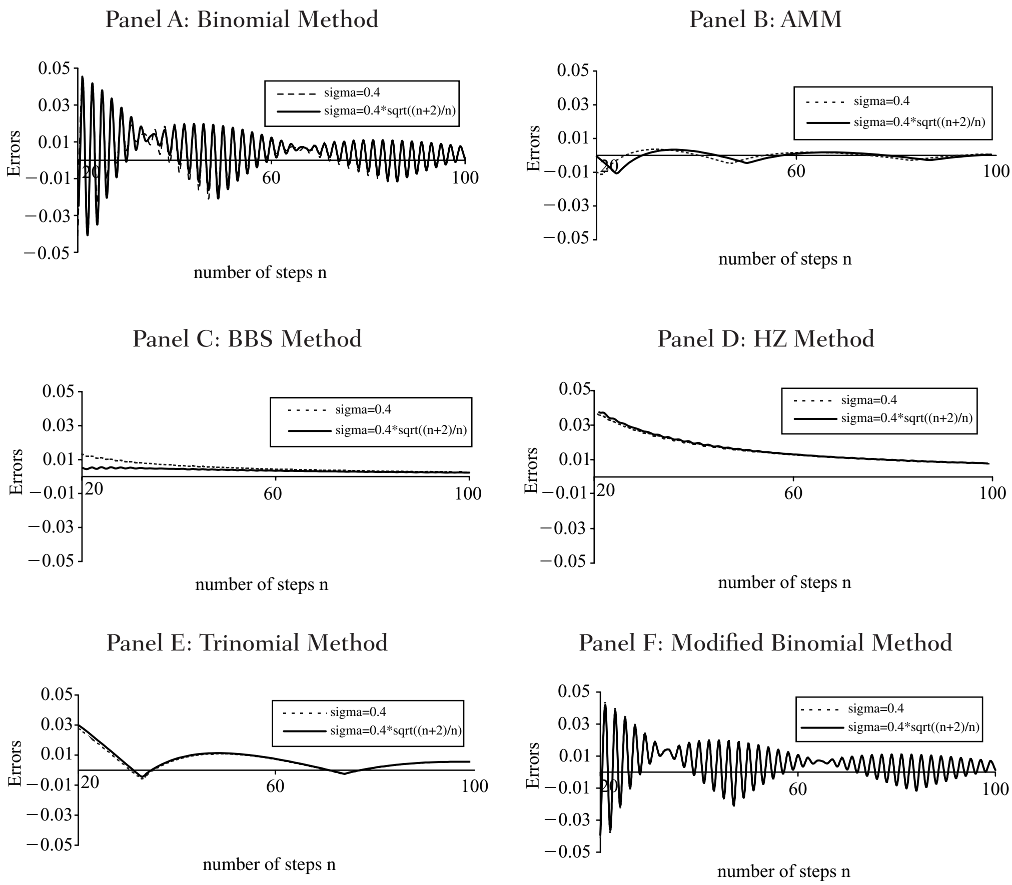


FIGURE 1
Pricing errors as a function of steps.

errors are correlated, they are not perfectly correlated and that there is considerable dispersion around the 45 degree line.

Consequently, when (for a fixed level of time steps n) numerical differentiation is employed via a perturbation $\Delta\sigma$ in σ [Equation (1)], the resulting Vega oscillates around its Black Scholes theoretical value (Panel A of Fig. 3). Convergence is slow in n and the envelope for Vega is roughly a tenth of its level (for small n). Thus as noted by many other authors for option Deltas, the binomial method is also problematic for Vega estimation because of oscillatory convergence.

Adaptive Mesh Model Method

The Adaptive Mesh Model (AMM) reduces the pricing error by adding more nodes to the tree in the region near exercise. As can be seen from Panel B of Figure 1, the resulting prices are indeed considerably more

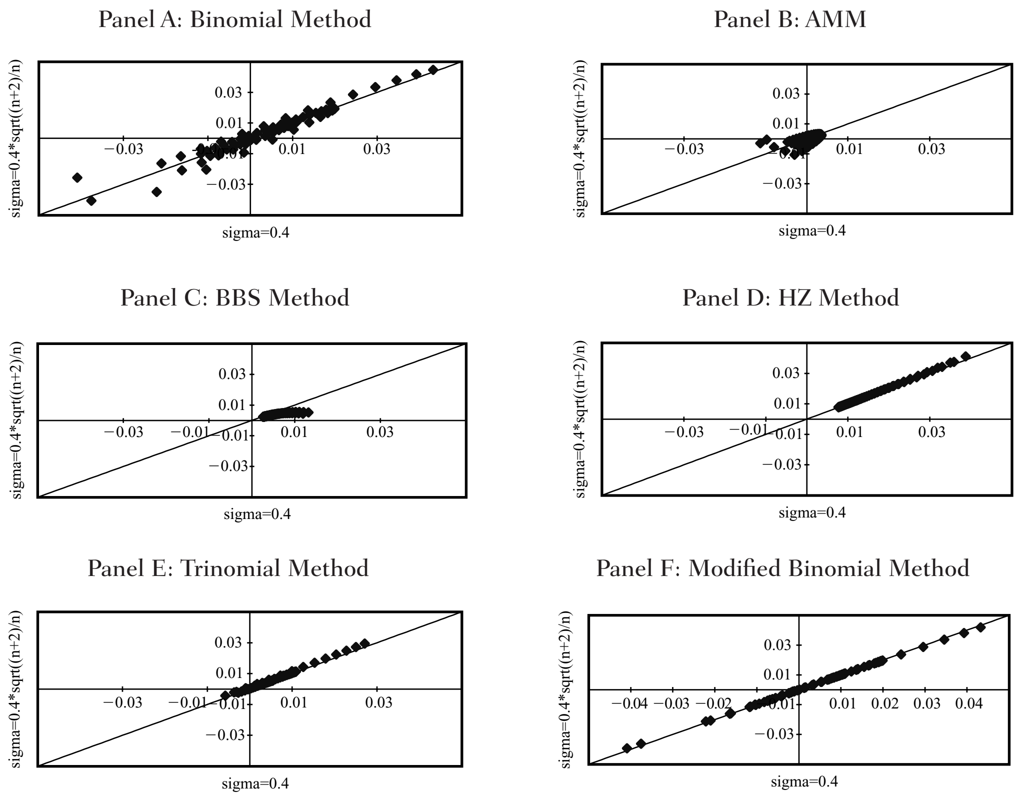


FIGURE 2
Error correlation across volatilities.

accurate than the Binomial method. For both option estimates (the unperturbed and perturbed volatility) the error envelope is considerably smaller than in the previous case and more importantly, the errors are no longer so negatively serially correlated (across n).

Furthermore, the scatter graph of pricing errors under the AMM (Panel B of Fig. 2) is much tighter to the origin, but although it has less dispersion along the 45 degree line, the correlation of these errors is smaller than for the regular Binomial method because the dispersion in the perpendicular to the 45 degree line has not been reduced by the AMM method.

Although it does reduce the pricing error, the AMM method inherits and indeed magnifies the relative dispersion of the pricing errors. Other methods may produce larger overall errors, but if these errors across two values of σ are more dependent they will be easier to eliminate.

Thus, the numerical Vega estimates in Panel B of Figure 3 (although not as oscillatory) still only tracks the true Vega within a large (but converging) envelope. This is true for this and the Binomial method

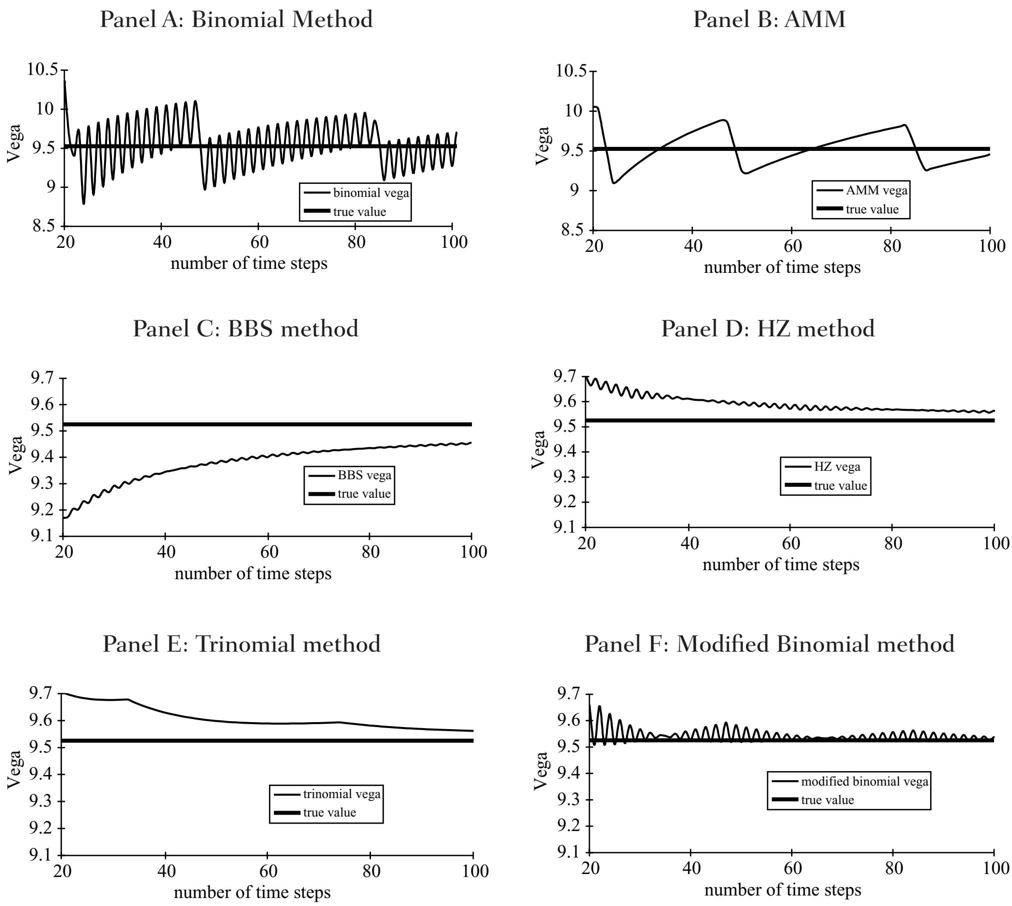


FIGURE 3
Numerical Vegas as a function of steps.

because although the effect of discretization can be reduced, its effect on the error structure and correlation cannot be eliminated.

The next pair of methods behave differently and eliminate this last problem but still prove problematic.

Binomial Black Scholes Method

The so-called Binomial Black Scholes (BBS) method smooths out the option value function through the addition of the continuous Black Scholes pricing function one period before maturity. Thus, as the number of time steps is varied there are no final nodes that can oscillate around the payoff condition.

In terms of price accuracy, the BBS performs well (Panel C of Fig. 1). Furthermore, the errors while small in magnitude are also highly correlated. However, the errors decrease at different rates as n increases,

TABLE II
RMS (Root-Mean-Squared) Errors of Six Vega Estimates

Number of steps	*20*	*40*	*60*	*80*	*100*	*500*	*1000*
Binomial	0.4817 (0:00.054)	0.2727 (0:00.086)	0.3313 (0:00.132)	0.2211 (0:00.192)	0.2254 (0:00.266)	0.0743 (0:05.083)	0.0766 (0:19.974)
Adaptive Mesh Model	0.1818 (0:00.062)	0.1311 (0:00.117)	0.1075 (0:00.204)	0.0927 (0:00.320)	0.0890 (0:00.469)	0.0406 (0:09.696)	0.0307 (0:38.093)
Binomial Black Sholes	0.2971 (0:00.132)	0.1511 (0:00.584)	0.1010 (0:02.383)	0.0759 (0:04.326)	0.0609 (0:06.243)	0.0124 (0:48.092)	0.0060 (1:49.478)
Heston Zhou	0.1452 (0:00.052)	0.0736 (0:00.082)	0.0507 (0:00.125)	0.0382 (0:00.195)	0.0303 (0:00.265)	0.0061 (0:05.085)	0.0030 (0:19.730)
Trinomial	0.1502 (0:00.057)	0.0767 (0:00.107)	0.0498 (0:00.200)	0.0378 (0:00.317)	0.0309 (0:00.458)	0.0061 (0:10.127)	0.0031 (0:39.822)
Modified Binomial	0.1118 (0:00.050)	0.0546 (0:00.078)	0.0414 (0:00.130)	0.0282 (0:00.187)	0.0231 (0:00.265)	0.0044 (0:04.984)	0.0025 (0:19.460)

Note. RMS $= \sqrt{\frac{1}{243}\sum_{i=1}^{243} e_i^2}$ defines the root-mean-squared errors for European puts where the error $e_i = (V_i^* - V_i)$ depends on V_i is the accurate (closed form) European Vega, V_i^* the estimated Vega using σ and $\sigma((n+2)/n)^{0.5}$. There are 243 parameter sets used: $S = 40$, and combinations of $K = 35, 40, 45$, $\sigma = 0.2, 0.3, 0.4$, $T = 1, 4, 7$ months, $r = 3, 5, 7\%$, and $q = 2, 5, 8\%$. The CPU time (in minutes, seconds, hundred of seconds) required to value all 243 options is also given.

therefore the difference in errors is not expected to be zero. This leads to the situation where Equation (1) is inconsistent due to the non-zero mean error difference. The best situation for the two errors is that their best fit line is of unit slope and passes through the origin with a tight fit. Then both errors are highly dependent and their difference is much smaller. The scatter graphs in Panel C of Figure 2 contain the 45 degree line for reference.

As can be seen for Panel C of Figure 2, this is not the case for the BBS method so the numerical Vega contains error bias. Thus, the numerical Vega (in Panel C of Fig. 3) although stable across choice of n is inconsistent for all but the highest number of step. This is to say that the option price difference for the higher volatility less the lower is too low compared to the volatility difference itself.

Heston Zhou Method

Like the BBS, this method applies smoothing. The D Panels tell a similar story. Price errors themselves are large and declining almost monotonically with n (Panel D of Fig. 1). Although highly correlated across the two volatilities, these errors are not consistent. Their regression line is not of unit slope; it is greater than one so that the error difference again is not zero.

Unlike the previous case, the higher volatility price contains a higher mean error than the lower case as can be seen in Panel D of Figure 2. As a consequence, the Vega estimate is overestimated and converges from above but only slowly (Panel D of Fig. 3).

Both of these last two methods that smooth the value function (BBS and HZ) thus lead to less precise Vega estimates even though they may produce prices that are in themselves more accurate. This is because of the structure present in the errors. The next two methods seek to exploit some of the similarity properties of the tree node structure in each volatility case and therefore produce a smaller overall error.

Trinomial Method

The trinomial tree method performs quite well in Vega estimation. Price errors are highly correlated and have a best fit very close to the unit slope, zero intercept line. Thus, the errors cancel out well in the numerical Vega estimation and although upward biased (the higher volatility option has the higher error) they converge quite well as n increases.

These results are somewhat akin to those of the Binomial method since two binomial steps (with recombination) produce a very similar tree to that of a trinomial. Looking at Panel A of Figure 1, if alternate points as a function of n (say for even n) were considered the results would be similar to the trinomial tree. Indeed, the (upper) envelopes of the price and Vega curves for the Binomial Method are similar to the curves for the Trinomial method in Panel E of Figures 1 and 3.

Therefore, it may seem as if the results for the Binomial method could have been substantially improved through the use of even n ordinates only, which would be equivalent to comparing two trinomial trees similar to Ritchken's method. However, when we tested this method it only seemed to eliminate the oscillatory convergence in the special case where the initial stock price S equaled the exercise price K. For other S/K ratios, the Vega still converged in an oscillatory fashion.

Overall, as can be seen from Panel E of Figures 1 to 3, the Trinomial method while containing some bias, works quite well in terms of Vega estimation.

Modified Binomial Method

Finally, the *Modified Binomial* (MB) *Method* is presented. Because it does not add more nodes near expiry or attempt to smooth out the value function by an insertion of a functional form near expiry, the pricing errors are large and suffer the regular oscillatory convergence problem.

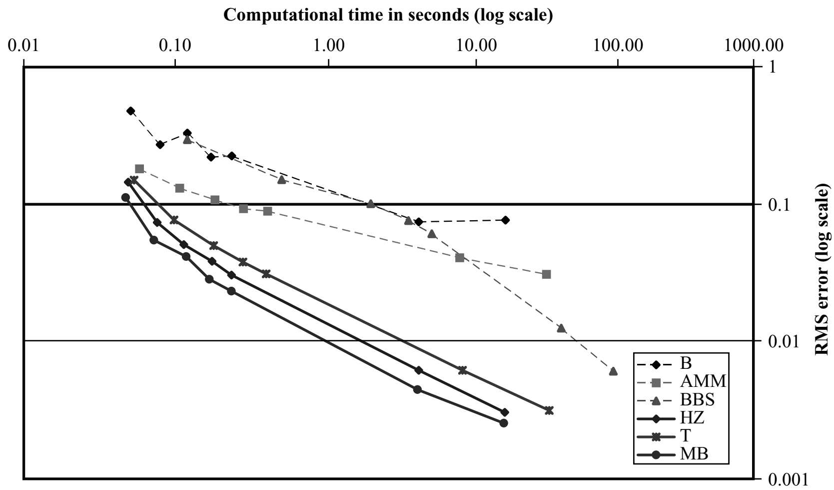


FIGURE 4
RMS error against computational time for six Vega estimation methods.

For this method to work the volatility perturbation has to be chosen to depend on n in a particular fashion (with $\sigma = 40\%$ and $\sigma\sqrt{\frac{n+2}{n}}$).[5]

Panel F of Figure 1 shows the price of two options. However, because of the positioning of the final nodes, these errors are both highly correlated and nearly on the unit slope zero intercept line as can be seen from the scatter plot in Panel F of Figure 2.

COMPARISON OF RMS ERRORS AND TIMES

Table II and Figure 4 show root mean squared errors for 243 option prices for each of the six methods. For all $3^5 = 243$ combinations of parameters drawn from combinations of $K \in \{35, 40, 45\}$, $T \in \{1, 4, 7\}$ months, $\sigma \in \{20, 30, 40\}\%$, $r \in \{3, 5, 7\}\%$, $q \in \{2, 5, 8\}\%$ (dividend yield) numerical Vegas were calculated using $\Delta\sigma = \sigma(\sqrt{\frac{n+2}{n}} - 1)$ for the perturbation.

The errors against the Black Scholes continuous value were squared, averaged, and square rooted to produce an aggregate RMS statistic. This was done repeatedly for differing numbers of time steps $n \in \{20, 40, \ldots, 100, 500, 1000\}$ to examine the convergence properties. The time taken for computation is also shown in minutes, seconds, and hundredths of seconds.

[5]This choice of volatility perturbation is required for the MB method to work but has also been adopted for all other methods to ensure comparability of results. It also has the useful property of decreasing as n increases in a manner that does not distort the numerical differentiation.

Each method almost always yields a lower RMS for increased n (the binomial method being the most striking exception for lower values of n). Computational time always increases with n and all methods require a similar magnitude of time to compute apart from the BBS (which requires normal integrals). However, the AMM and the trinomial method take about twice the time of the binomial, HZ and MB methods because of their extra tree complexity.

For many values of n and especially for large n, the binomial method and AMM perform the worst in terms of RMS errors. This is because of the poor correlation between the errors for differing volatilities. Surprisingly, the BBS method also performs poorly for low n as well, however, its performance improves as n increases although this comes with increased computational burden. Although the BBS price errors are well correlated across the volatility and its perturbed value, their slope is not unity so that the Vegas calculated using this method contain bias.

The final three methods (Trinomial, HZ, and MB) are quite similar in terms of RMS performance and speed, however (like the AMM method) the Trinomial calculation that contains more nodes and paths takes longer to evaluate than the HZ and MB methods. The new Modified Binomial method performs quite similarly to the HZ method but its errors are always lower for comparable computational times.

However, because of the exceptionally good correspondence of pricing errors (they fall on the unit slope line in Fig. 2F), the MB Vega estimates in Panel F of Figure 3, although oscillatory, have low error, are within a tight envelope and converge quickly. This is in contrast to all previous methods. In many applications, this MB method would be highly suitable for robust, quick, and accurate Vega calculation. This is because it uses only two extra (and uniquely different) nodes so the maximum amount of nodes are common to both trees, i.e., its simple tree design lends it more intuitive appeal. This is remarkable given the lack of price accuracy inherent in the Modified Binomial method.

CONCLUSION

In conclusion, the model that produces the most accurate option prices may not be the one that is best for calculating partial derivatives via numerical differentiation. This is because the error in the Vega depends not only on the magnitude of the errors present in the two price estimates, but critically on their correlation structure.

Models that produce individually accurate but jointly inaccurate prices such as the AMM or BBS will not be good for estimating option

Vegas (and other Greeks). Models that are tailored specifically to reduce the discretization error near a termination boundary however, do much better in terms of Vega estimation.

These models, such as the Modified Binomial (MB) presented here, are the ones that are also most likely to produce the best Vega results for other option types such as American and Barrier, where no closed form benchmark formulae are available.

BIBLIOGRAPHY

Amin, K. I. (1991). On the computation of continuous time option prices using discrete approximations. Journal of Financial and Quantitative Analysis, 26, 477–495.

Boyle, P., & Lau, S. H. (1994). Bumping up against the barrier with the binomial model. Journal of Derivatives, 1, 6–14.

Broadie, M., & Detemple, J. (1996). American option valuation: New bounds, approximations, and a comparison of existing methods. Review of Financial Studies, 9, 1211–1250.

Chung, S. L., & Shackleton, M. (2002). The Binomial Black-Scholes Model and the Greeks. Journal of Futures Markets, 22, 143–153.

Figlewski, S., & Gao, B. (1999). The Adaptive Mesh Model: A new approach to efficient option pricing. Journal of Financial Economics, 53, 313–351.

Heston, S., & Zhou, G. (2000). On the rate of convergence of discrete-time contingent claims. Mathematical Finance, 10(1), 53–75.

Hull, J. (2000). Options, futures and other derivatives (4th ed.). Englewood Cliffs, NJ: Prentice Hall.

Omberg, E. (1988). Efficient discrete time jump process models in option pricing. Journal of Financial Quantitative Analysis, 23, 161–174.

Pelsser, A., & Vorst, T. (1994). The Binomial Model and the Greeks. Journal of Derivatives, 1, 45–49.

Ritchken, P. (1995). On pricing barrier options. Journal of Derivatives, 3, 19–27.

Tian, Y. (1999). A flexible binomial option pricing model. The Journal of Futures Markets, 19, 817–843.

Widdicks, M., Andricopoulos, A. D., Newton, D. P., & Duck, P. W. (2002). On the enhanced convergence of standard lattice methods for option pricing. The Journal of Futures Markets, 22, 315–338.