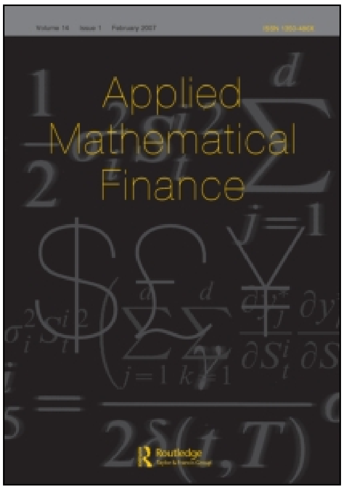

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Pricing Quanto Equity Swaps in a Stochastic Interest Rate Economy

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ABSTRACT *This paper derives a pricing model for a quanto foreign equity/domestic floating rate swap in which one party pays domestic floating interest rates and receives foreign stock returns determined in the foreign currency, but is paid in the domestic currency. We use the risk-neutral valuation technique developed by Amin and Bodurtha to generate an arbitrage-free pricing model. A closed-form solution is obtained under further restrictions on the drift rates of the asset price processes. Pricing formulae show that the value of a quanto equity swap at the start date does not depend on the foreign stock price level, but rather on the term structures of both countries and other parameters. However, the foreign stock price levels do affect the swap value times between two payment dates. The numerical implementations indicate that the domestic and foreign term structures, the correlation between the foreign interest rate and the exchange rate, and the correlation between the exchange rate and the foreign stock are more important factors in pricing a quanto equity swap than other correlations.*

KEY WORDS: Equity swaps, term structure of interest rates, risk-neutral valuation, arbitrage-free pricing model

Introduction

An equity swap is a transaction that allows an investor to exchange the rate of return on an equity investment (an individual share, a basket of shares, or an index) for the rate of return on another non-equity (e.g. fixed or floating rate) or equity investment. Like its currency and interest rate swap predecessors, equity swaps have been well received by the financial markets, because they make it possible to achieve outcomes more efficiently. For example, equity swaps are widely used to synthesize equity, convert the return character of fixed-income portfolios, bypass transfer and withholding taxes, reduce transaction costs, avoid jurisdictional dilemmas, and much more.

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There are various types of equity swaps traded in the over-the-counter market. Among them, quanto equity swaps are common in a variety of situations for practitioners. A typical function of a quanto equity swap is to convert a percentage equity return in a non-domestic equity into the base currency of the receiver so as to provide currency protection.[1] For example, if the Japanese equity market is attractive to a US investor who owns Eurodollar deposits, he could enter into a quanto foreign equity/domestic floating rate swap in which he agrees to make a series of payments based on the US.$ LIBOR rate, and in return he receives the return of the Japanese equity market determined in Japanese Yen, but is paid in US$.[2] After the transaction, he acquires the return on Japanese equity without directly incurring an exchange rate risk.

The amount of literature on pricing and hedging equity swaps, not to mention quanto equity swaps, is few. Marshall *et al.* (1992) and Rich (1995) provide pricing models using expectations operators. However, both of them do not provide arbitrage-free valuation formulae. Jarrow and Turnbull (1996) derive a preference-free formula for the fixed rate of an equity swap at the start date only. Chance and Rich (1998) use arbitrage-free replicating portfolios to derive valuation formulae for a variety of equity swaps at the start date and during the life of the swaps. They find that the values of equity swaps generally depend on the stock price level, but at the start date or immediately after the payment is made, the values of equity swaps are independent of stock prices. Similar to them, our pricing formulae show that the value of a quanto equity swap at the start date does not depend on the foreign stock price level.

With regard to quanto derivatives, there are some articles on the valuation of quanto interest rate swaps (differential swaps) (Litzenberger, 1992; Jamshidian, 1993; Wei, 1994), and on the valuation of quanto options (Reiner, 1992; Wei, 1995; Chung, 2002). However, to our knowledge, no pricing formulae for general quanto equity swaps have ever been published. Hence, one purpose of this paper is to develop a pricing model for quanto equity swaps. As an illustration, we only price a quanto equity swap where a foreign equity index return is exchanged for a domestic floating interest rate with the notional principal denominated in the domestic currency. However, as discussed in the following section, our results can be directly applied to price general type of quanto equity swaps using the building block approach.

The difficulty on pricing quanto equity swaps is that their payoffs are contingent on many variables.[3] For example, consider a US investor enters into a two-year quanto equity swap with a local bank. The swap agreements dictate that the investor will pay the bank semiannually the US 6-month LIBOR rate plus (or minus) a spread, and in return will receive from the bank the FTSE 100 index return over six months for two years. The notional principal is denominated in US dollars. In a realistic case, one has to consider at least four factors that directly affect the value of this swap: the exchange rate ($/£), the US interest rate, the UK interest rate, and the FTSE 100 index price.

In this paper we extend Amin and Bodurtha's (1995) work to value quanto equity swaps. We consider a common type of quanto equity swap in which a foreign (domestic) equity index return is exchanged for a domestic (foreign) floating interest rate with the notional principal denominated in the domestic or foreign currency.

Using the building block approach, we show that our pricing formulae, combining with existing formulae for interest rate swaps, differential swaps and/or plain vanilla equity swaps, can be used to price general structures of quanto equity swaps.

Our valuation model is of considerable complexity, because it has to deal with at least four risk factors: the foreign asset price, the exchange rate, the domestic risk-free interest rate, and the foreign risk-free rate. The domestic and foreign interest rate dynamics are modelled here using a discrete version of the Heath *et al.* (1990, 1992, hereafter HJM) model. Thus, as in Jarrow and Turnbull (1996) and Kijima and Muromachi (2001), our valuation formulae are arbitrage-free. We first establish the domestic risk-neutral measure under which the relative price of any asset (including foreign assets) to the domestic money market account follows a martingale. The swap value is then determined by taking the expected present value of future cash flows under this measure. When the volatility and covariance functions are permitted to be arbitrary, our valuation model is 'path-dependent'. However, in the special case where all of the volatilities are constant or at most functions of time, we obtain a closed-form solution for a quanto equity swap price.

The rest of this paper is organized as follows. The following section covers the setting of notation and assumptions, the basic structures of quanto equity swaps, and the risk-neutral valuation relationship. The third section derives the pricing formula for a foreign equity/domestic floating rate swap with the notional principal denominated in the domestic currency. Numerical evaluations are presented in the fourth section and conclusions in the final section.

The Setting

Notation and Assumptions

The following notation is employed:[4]

- $I_f^*(t)$ foreign stock or index price at time t denominated in foreign currency;
- $S(t)$ spot exchange rate at time t (dollars per unit of foreign currency);
- T maturity date of the swap;
- $r_i(t)$ short rate at time t for borrowing and lending at the period from time t to $t+h$, $\forall i=d$ (domestic interest rate) and f (foreign interest rate);
- $f_i(t, T)$ forward rate at time t for borrowing and lending at the period from time T to $T+h$, $\forall i=d, f$;
- $P_d(t, T)$ price in domestic currency at time t of a domestic zero-coupon bond with maturity T and unit face value;
- $P_f^*(t, T)$ price in foreign currency at time t of a foreign zero-coupon bond with maturity T and unit face value.

There are four state variables relevant to the valuation of quanto equity swaps: the domestic interest rate, the foreign interest rate, the exchange rate, and the foreign stock price. We assume that the dynamics of the domestic and foreign forward rates follow the HJM model, and that the exchange rate and the foreign equity price are joint lognormally distributed. In other words, we assume that the four factors under the domestic risk neutral ($\tilde{Q}$) measure follow:[5]

$$f_d(t+h,\,T)-f_d(t,\,T)=\alpha_d(t,\,T,.)h+\sigma_d(t,\,T,.)X_d(t+h)\sqrt{h} \tag{1}$$

$$f_f(t+h,\,T)-f_f(t,\,T)=\alpha_f(t,\,T,.)h+\sigma_f(t,\,T,.)X_f(t+h)\sqrt{h} \tag{2}$$

$$\ln\frac{S(t+h)}{S(t)}=\left[\alpha_s(t,.)+r_d(t)-r_f(t)\right]h+\sigma_s(t,.)X_s(t+h)\sqrt{h} \tag{3}$$

$$\ln\frac{I_f^*(t+h)}{I_f^*(t)}=\alpha_{I_f^*}(t,.)h+\sigma_{I_f^*}(t,.)X_{I_f^*}(t+h)\sqrt{h} \tag{4}$$

where $\alpha_i(\cdot)$ and $\sigma_i(\cdot)$, $\forall i=d,f,s,I_f^*$, are respectively the drift and volatility of each variable, and the vectors of increments, $X=\left(X_d,\,X_f,\,X_s,\,X_{I_f^*}\right)$, are multivariate standard normal variables with the covariance matrix

$$Cov(X,\,X')=\begin{bmatrix} 1 & \rho_{df} & \rho_{ds} & \rho_{dI_f^*} \\ \rho_{df} & 1 & \rho_{fs} & \rho_{fI_f^*} \\ \rho_{ds} & \rho_{fs} & 1 & \rho_{I_f^*s} \\ \rho_{dI_f^*} & \rho_{fI_f^*} & \rho_{I_f^*s} & 1 \end{bmatrix}$$

where ρ_{ij} is the correlation coefficient between i and j.

We finally assume that the market is complete and there is no default risk for both parties of the swaps. Therefore, we can price quanto equity swaps from their expected present values of future cash flows.

Basic Structures of Quanto Equity Swaps

Depending on the payoff specifications, quanto equity swaps can be classified into three basic structures (Beder, 1992; Chance and Rich, 1998).

1. *Quanto equity/fixed rate swaps.* A foreign (domestic) equity index return is exchanged for a fixed interest rate with the notional principal denominated in the domestic (foreign) currency. For example, in a quanto foreign equity/fixed rate swap with the notional principal denominated in the domestic currency, the cashflow at time t_{i+1} for the domestic fixed-rate payer is

$$NP_d\times\left[\left(\frac{I_f^*(t_{i+1})}{I_f^*(t_i)}-1\right)-\bar{R}_d(t_{i+1}-t_i)\right]$$

where NP_d is the notional principal denominated in the domestic currency and Rd is the fixed rate per annual.

2. *Quanto equity/floating rate swaps.* A foreign (domestic) equity index return is exchanged for a domestic (foreign) floating interest rate with the notional principal denominated in the domestic or foreign currency. For example, in a quanto foreign equity/domestic floating rate swap with the notional principal denominated in the domestic currency, the cashflow at time t_{i+1} for the domestic floating-rate payer is

$$NP_d \times \left[\left(\frac{I_f^*(t_{i+1})}{I_f^*(t_i)} - 1 \right) - (R_d(q_i) + c_d)(t_{i+1} - t_i) \right]$$

where $q_i = t_{i+1} - t_i$, $R_d(q_i)$ is the domestic q_i-period LIBOR rate at time t_i and c_d is the constant margin rate to be specified when the swap is first initiated so that the net value of the swap is zero.

3. *Quanto equity/equity swaps.* A domestic equity index return is exchanged for a foreign equity index return with the notional principal denominated in the domestic or foreign currency. For instance, in a quanto foreign equity/domestic equity swap with the notional principal denominated in the domestic currency, the cashflow at time t_{i+1} for the domestic equity return payer is

$$NP_d \times \left[\frac{I_f^*(t_{i+1})}{I_f^*(t_i)} - \frac{I_d(t_{i+1})}{I_d(t_i)} + c'_d(t_{i+1} - t_i) \right]$$

where c'_d is the constant margin rate to be specified when the swap is first initiated so that the net value of the swap is zero.

Using the building block approach, it is not difficult to show that the above three basic structures of quanto equity swaps can be created from each other by combining with interest rate swaps, differential swaps, and/or plain vanilla equity swaps. For example, if the domestic fixed-rate payer in a quanto foreign equity/fixed rate swap enters into a domestic interest rate swap in which he pays the floating rate and receives a fixed rate (suppose the swap rate is $R'd$), then the cashflow from this interest rate swap at time t_{i+1} for him is

$$NP_d \times [(R'd - R_d(q_i))(t_{i+1} - t_i)]$$

If the market is arbitrage free, then the total cashflows from both swaps at time t_{i+1} for the investor must equal the cashflow at time t_{i+1} for the domestic floating-rate payer in a quanto foreign equity/domestic floating rate swap with the notional principal denominated in the domestic currency. Therefore, the following equality is sustained when the market is arbitrage free:

$$c_d = \bar{R}_d - \bar{R}'_d$$

In conclusion, using the no-arbitrage argument and the existing results in pricing interest rate swaps, differential swaps, and/or plain vanilla equity swaps, the valuation of the above three basic structures of quanto equity swaps can be simplified as just the valuation of any one of them. As an illustration, we will focus on pricing a quanto foreign equity/domestic floating rate swap with the notional principal denominated in the domestic currency.

Risk-neutral Valuations

In this article we apply the risk-neutral valuation technique developed by Amin and Bodurtha (1995) to identify a change of measure and work directly with the risk-neutral processes of the four state variables relevant to the pricing of a quanto equity swap. In the absence of arbitrage, there exists a probability measure, called the domestic risk-neutral measure (denoted as $\tilde{Q}$), under which all asset prices (denominated in the domestic currency), normalized by the domestic money market account, follow martingales. Since the original probability measure governing the evolution of state variables is irrelevant to the valuation, we derive our pricing model only under the $\tilde{Q}$ measure.

In a domestic risk-neutral world, all asset prices should be denominated in the domestic currency. Therefore, we multiply foreign currency values by the spot exchange rate, $S(t)$, to convert foreign asset prices to the domestic currency. For instance, the foreign money market account denominated in the domestic currency, $B_f(t)$, equals $B_f^*(t) \times S(t)$. We can hence define the relative prices of all the assets to the domestic money market account in the economy by

$$Z_f(t) = \frac{B_f(t)}{B_d(t)} = \frac{B_f^*(t) \times S(t)}{B_d(t)}$$

$$Z_d(t, T) = \frac{P_d(t, T)}{B_d(t)}$$

$$Z_f(t, T) = \frac{P_f(t, T)}{B_d(t)} = \frac{P_f^*(t, T) \times S(t)}{B_d(t)}$$

$$Z_{I_f^*}(t) = \frac{I_f^*(t)}{B_d(t)} = \frac{I_f^*(t) \times S(t)}{B_d(t)}$$

where $Z_f(t)$, $Z_d(t, T)$, $Z_f(t, T)$, and $Z_{I_f^*(t)}$ are the relatives prices (denominated in the domestic currency) at time t of the foreign money market account, the domestic zero-coupon bond (with maturity T), the foreign zero-coupon bond (with maturity T), and the foreign stock, respectively. By applying the martingale condition to $Z_f(t)$, $Z_d(t, T)$, $Z_f(t, T)$, and $Z_{I_f^*(t)}$, one can determine the values of the drifts $\alpha_d(.)$, $\alpha_f(.)$, $\alpha_s(.)$, and $\alpha_{I_f^*}(.)$ in Lemma 1.

Lemma 1. The prices in domestic currency of all assets relative to the domestic money market account are $\tilde{Q}-$martingales if and only if the following conditions are satisfied:

$$\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} \alpha_d(t, ih,.)h^2 = \ln\left\{\widetilde{E}_t\left[\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\left[\sigma_d(t, ih,.)X_d(t+h)\sqrt{h^3}\right]\right)\right]\right\}$$

$$\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} \alpha_f(t, ih,.)h^2 = \alpha_s(t,.)h + \ln\left\{\widetilde{E}_t\left[\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\left[\sigma_f(t, ih,.)X_f(t+h)\sqrt{h^3}\right] + \sigma_s(t,.))X_s(t+h)\sqrt{h}\right)\right]\right\} \tag{5}$$

$$\alpha_s(t,.)h = -\ln\left\{\widetilde{E}_t\left[\exp\left(\sigma_s(t,.)X_s(t+h)\sqrt{h}\right)\right]\right\}$$

$$\alpha_{I_f^*}(t,.)h = r_f(t)h + \ln\,\widetilde{E}_t\left\{\exp\left[\sigma_s(t,.)X_s(t+h)\sqrt{h}\right]\right\} - \ln\left\{\widetilde{E}_t\left(\exp\left[\sigma_{I_f^*}(t,.)X_{I_f^*}(t+h)\sqrt{h} + \sigma_s(t,.)X_s(t+h)\sqrt{h}\right]\right)\right\}$$

where $\widetilde{E}_t$ indicates the time t conditional expectation with respect to the $\widetilde{Q}$ measure.

Proof. See the Appendix.

Pricing Quanto Foreign Equity/Domestic Floating Rate Swaps

As discussed in the previous section, we will focus on pricing a quanto foreign equity/domestic floating rate swap with the notional principal denominated in the domestic currency in this paper. To the party paying the domestic floating rate and receiving a foreign equity return, the cashflow at time t_{i+1} ($CF_{t_{i+1}}$) is

$$CF_{t_{i+1}} = NP_d \times \left[\left(\frac{I_f^*(t_{i+1})}{I_f^*(t_i)} - 1\right) - (R_d(q_i) + c_d)(t_{i+1} - t_i)\right]$$

Note that the domestic q_i-period LIBOR rate at time t_i, $R_d(q_i)$, is defined by

$$R_d(q_i)(t_{i+1} - t_i) = \frac{1}{\widetilde{E}_{t_i}\left[\exp\left(-\sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} r_d(ih)h\right)\right]} - 1$$

To get the present value of the cash flow $CF_{t_{i+1}}$, we need to discount it back to time t at the domestic interest rate. Let $V_{t_{i+1}}$ denote the expected present value of

$CF_{t_{i+1}}$; then

$$V_{t_{i+1}} = \widetilde{E}_t \left\{ \exp\left[-\sum_{i=\frac{t}{h}}^{\frac{t_{i+1}}{h}-1} r_d(ih)h \right] CF_{t_{i+1}} \right\} \quad (6)$$

The term structure model (HJM model) used in this article is generally path dependent. Thus, one needs to apply numerical methods such as the Monte Carlo simulation method (Carr and Yang, 1998) or the lattice approach (Amin and Bodurtha, 1995) to evaluate equation (6). However, if the volatilities of all state variables are constants or at most are functions of time,[6] then there exists a closed-form solution for equation (6). To facilitate the derivation of closed-form solutions, it is assumed that[7]

$$\sigma_d(t, T,.) = \sigma_d \exp[-k_d(T-t)]$$
$$\sigma_f(t, T,.) = \sigma_d \exp\left[-k_f(T-t)\right]$$

Letting $h \rightarrow 0$ and applying Lemma 1 and the property of lognormally-distributed variables,[8] we can evaluate (6) and obtain the following closed-form solution[9]

$$\begin{aligned} V_{t_{i+1}} &= NP_d \frac{P_d(t, t_{i+1}) P_f^*(t, t_i)}{P_f^*(t, t_{i+1})} \exp[-b_1 + b_2 - b_3 - b_4 + b_5 - b_6] \\ &\quad - NP_d\, P_d(t, t_i) - NP_d \frac{c_d(t_{i+1}-t_i)}{360} P_d(t, t_{i+1}) \end{aligned} \quad (7)$$

where,

$$\begin{aligned} b_1 = \frac{\sigma_d \sigma_f \rho_{df}}{k_d k_f} \Bigg[& \frac{1}{k_f}\left(e^{-k_f(t_{i+1}-t)} - e^{-k_f(t_i - t)}\right) + \frac{1}{k_d}\left(e^{-k_f(t_{i+1}-t_i)} - 1\right) + (t_{i+1} - t_i) \\ & + \frac{1}{k_d + k_f}\left(1 - e^{-\left(k_d + k_f\right)(t_{i+1}-t)} - e^{-k_d(t_{i+1}-t_i)} + e^{-k_d(t_{i+1}-t) - k_f(t_i - t)}\right) \Bigg] \end{aligned}$$

$$\begin{aligned} b_2 = \frac{\sigma_f^2}{k_f^3} \Bigg[& -\frac{1}{2} + k_f(t_{i+1}-t_i) + \frac{1}{2} e^{-k_f(t_{i+1}-t_i)} + e^{-k_f(t_{i+1}-t)} \\ & - e^{-k_f(t_i - t)} - \frac{1}{2} e^{-2k_f(t_{i+1}-t)} + \frac{1}{2} e^{-k_f(t_{i+1}+t_i - 2t)} \Bigg] \end{aligned}$$

$$b_3 = \frac{\sigma_f \sigma_s \rho_{fs}}{k_f} \times \left[(t_{i+1} - t_i) - \frac{1}{k_f}\left(e^{-k_f(t_i - t)} - e^{-k_f(t_{i+1}-t)}\right) \right]$$

$$b_4 = \frac{\sigma_{I_f^*} \sigma_d \rho_{I_f^* d}}{k_d} \times \left[(t_{i+1} - t_i) - \frac{1}{k_d}\left(1 - e^{-k_d(t_{i+1}-t_i)}\right) \right]$$

$$b_5 = \frac{\sigma_{I_f^*} \sigma_f \rho_{I_f^* f}}{k_f} \times \left[(t_{i+1} - t_i) - \frac{1}{k_f}\left(1 - e^{-k_f(t_{i+1}-t_i)}\right) \right]$$

$$b_6 = \sigma_{I_f^*} \sigma_s \rho_{I_f^* s} \times (t_{i+1} - t_i).$$

The terms b_1, b_2, b_3, b_4, b_5, and b_6 are meant to adjust for the effects of the correlation between domestic and foreign interest rates, the volatility of the foreign interest rate, the correlation between the foreign interest rate and exchange rate, the correlation between the foreign equity price and domestic interest rate, the correlation between the foreign equity price and foreign interest rate, and the correlation between the foreign equity price and exchange rate.

To the party paying the domestic floating rate and receiving a foreign equity return, the value of the swap is simply the summation of the present values of all future cash flows, i.e.

$$V=\sum_{i=\frac{t}{h}}^{n-1} V_{t_{i+1}} \tag{8}$$

where n is the number of payment dates after t (notice that $t_n = T$). Since the constant margin rate (c_d) is specified to let the value of the swap be zero at the initialized date, the constant margin rate at the initialized date must be

$$c_d=\frac{\sum_{i=\frac{t}{h}}^{n-1}\left\{\frac{P_d(t,\, t_{i+1})P_f^*(t,\, t_i)}{P_f^*(t,\, t_{i+1})}\exp[-b_1+b_2-b_3-b_4+b_5-b_6]-P_d(t,\, t_i)\right\}}{\frac{(t_{i+1}-t_i)}{360}\sum_{i=\frac{t}{h}}^{n-1} P_d(t,\, t_{i+1})} \tag{9}$$

So far we have derived the complete pricing formula for a quanto foreign equity/ domestic floating rate swap with the notional principal denominated in the domestic currency. There are several points worth noting. First, at the start date or immediately after the payment is made, the value of a quanto equity swap will mainly depend on the term structures of interest rates in both countries at each reset date. Hence, the level of the foreign equity price is irrelevant to the valuation of a quanto equity swap.[10] However, the subsequent value of the swap does depend on the level of the foreign equity price. The pricing formula of a quanto equity swap at the subsequent date is shown in the Appendix.

Chance and Rich (1998) also find that the foreign equity price level is irrelevant for pricing other equity swaps. However, unlike them, the quanto equity swap value does depend on the volatility of the equity price process, as well as its correlations with the exchange rate and interest rates in both countries (i.e. b_4 to b_6 in equation (7)). Kijima and Muromachi (2001) have similar findings to ours in the case of a plain vanilla equity swap with a *variable* notional principal in a stochastic interest rate economy.

Second, when the volatility of the foreign equity price is zero (i.e. $\sigma_{I_f^*}=0$), the foreign equity price will be equivalent to the foreign money market account. As a result, with $b_4=b_5=b_6=0$, our pricing formula is very similar to that for a differential swap[11] (Wei, 1994; Chang *et al.*, 2002). Moreover, if the foreign country is identical to the domestic country (i.e. $S=1$, $\sigma_s=0$, $k_d=k_f$, $\sigma_d=\sigma_f$, $\rho_{df}=1$), then the quanto equity swap reduces to be a single-currency equity swap and the constant margin rate will be zero in our pricing formula since $b_1=b_2$, $b_4=b_5$, and $b_3=b_6=0$. The result is consistent with the finding of Chance and Rich (1998) whereby the paying-floating, receive-equity swap is an equivalent exchange (i.e. the constant margin rate is zero). The above arguments also provide a simple way to check our complex formula.

Third, although the exchange rate itself does not appear in our pricing formula, the exchange rate volatility, the correlation with the foreign interest rate, and the correlation with the foreign equity price do have effects on the value of a quanto equity swap. Analogous to the discussion of Wei (1994) in the case of a differential swap, the correlation terms here express the 'fixed exchange rate effect'. That is, these terms capture the potential loss or gain due to future changes of the exchange rate. When the correlations relative to the exchange rate are zero, the swap value is completely independent of the exchange rate.

Fourth, it should be pointed out that the quanto equity swap described above is actually a 'quanto domestic equity/foreign floating rate swap with the notional principal denominated in the foreign currency' from the viewpoint of a 'foreign' investor. Therefore, there exists a similar pricing duality to that of Chang *et al.* (2002), and our pricing formula can be easily converted to the pricing formula of a quanto domestic equity/foreign floating rate swap with the notional principal denominated in the foreign currency.[12]

Finally, the valuation of a general quanto equity swap does not impose any extra difficulty. For instance, with a quanto equity swap whose periodic payments are based on the returns of the FTSE 100 and Nikkei 225, but are denominated in US dollars, we simply have to value each leg as a foreign equity-driven floater. In this case, the value of a general quanto equity swap will depend on some extra parameters.

Numerical Evaluations

Our pricing formulae are applied to price a range of quanto foreign equity/domestic floating rate swaps so as to investigate how much the swap values change when parameters change. All analyses in this section are based on the viewpoint of an investor who pays domestic floating rates and receives foreign equity returns. The sensitivity analyses are divided into three parts: term structure effect, volatility effect, and correlation effect. The benchmark parameters, mainly adopted from Wei (1994), are as follows: $\sigma_d=\sigma_f=0.02$, $\sigma_s=0.3$, $\sigma_{I_f^*}=0.3$, $\rho_{df}=0.3$, $\rho_{fs}=\rho_{fI_f^*}=-0.3$, $\rho_{dI_f^*}=\rho_{sI_f^*}=-0.2$, $k_d=k_f=0.15$, and $NP_d=\$100$.

We first examine the effects of different initial term structures in both countries on the swap values (assuming $c_d=0$) and constant margin rates (assuming $V=0$)) for three different tenures. The term structures of both countries are classified as three kinds: positive slope, negative slope, and flat. 'Flat' means the yield is fixed at 8% for all maturities of zero-coupon bonds. 'Positive slope' ('Negative slope') represents an upward (downward) yield curve with a slope of 0.4%/year (−0.4%/year) starting with an instantaneous rate of 8.0%.

From Table 1, one can verify that when the differences between the two term structures are the same, the swap values (and thus the constant margin rates) are very close. In the case that the tenure equals one year, the swap values are all close to \$1.849 (assuming $c_d=0$) when both term structures have the same slope. While the domestic term structure is 0.4%/year steeper than the foreign term structure (i.e. domestic positive foreign flat, or domestic flat foreign negative), the swap values are

Table 1. Effects of different term structures on the swap values and constant margin rates

Domestic term structure	Foreign term structure	V(\$)	c_d(%)
$T-t=1$ year			
Positive slope	Positive slope	1.8478	1.9666
Positive slope	Flat	1.4567	1.5503
Positive slope	Negative slope	1.0665	1.1350
Flat	Positive slope	2.2412	2.3793
Flat	Flat	1.8488	1.9627
Flat	Negative slope	1.4573	1.5471
Negative slope	Positive slope	2.6361	2.7917
Negative slope	Flat	2.2424	2.3747
Negative slope	Negative slope	1.8497	1.9588
$T-t=3$ years			
Positive slope	Positive slope	5.5458	2.1520
Positive slope	Flat	2.4286	0.9424
Positive slope	Negative slope	−0.6645	−0.2579
Flat	Positive slope	8.7811	3.3591
Flat	Flat	5.5969	2.1410
Flat	Negative slope	2.4373	0.9323
Negative slope	Positive slope	12.1299	4.5736
Negative slope	Flat	8.8767	3.3470
Negative slope	Negative slope	5.6489	2.1299
$T-t=5$ years			
Positive slope	Positive slope	9.0174	2.3107
Positive slope	Flat	1.4348	0.3677
Positive slope	Negative slope	−6.0521	−1.5509
Flat	Positive slope	17.2709	4.2759
Flat	Flat	9.2727	2.2957
Flat	Negative slope	1.3766	0.3408
Negative slope	Positive slope	26.3227	6.2901
Negative slope	Flat	17.8778	4.2721
Negative slope	Negative slope	9.5421	2.2802

Table shows the effects of different initial term structures in both countries on the swap values (assuming $c_d=0$) and constant margin rates (assuming $V=0$) for three different tenures. The benchmark parameters are as follows: $\sigma_d=\sigma_f=0.02$, $\sigma_s=0.3$, $\sigma_{I_f^*}=0.3$, $\rho_{df}=0.3$, $\rho_{fs}=\rho_{fI_f^*}=-0.3$, $\rho_{dI_f^*}=\rho_{sI_f^*}=-0.2$, $k_d=k_f=0.15$, and $NP_d=\$100$.

very close to \$1.457 in both cases. Thus, we can conclude that the difference between the two term structures, and not the absolute level of two interest rates, is important in valuing a quanto equity swap.

When the slope of the foreign (domestic) interest rate is higher (lower), the swap value is higher. The intuition behind this phenomenon is simple. Although the cash flows in a quanto equity swap are not based on the foreign interest rate, the value of the swap is affected by the foreign interest rate since the drift of the foreign equity price process comprises it. The higher the foreign interest rate is, the higher the drift of the foreign equity price will be. Therefore, when the foreign (domestic) term structure is higher (lower), the investor[13] will expect to receive more (pay less), which makes the value of a quanto equity swap higher. In these circumstances, when the tenure is longer, the price of the swap is higher. By contrast, if the slope of the domestic term structure is larger than the foreign one, then the quanto equity swap value decreases when the tenure increases.

We secondly study the volatility effects on the quanto equity swap values. Here we assume the swap maturity is three years and both term structures have a positive slope. We calculate swap values by varying one volatility while keeping other parameters constant. The results are presented in Table 2. We find that the swap

Table 2. Effects of volatilities on the swap values and constant margin rates

Varying σ_f while keeping other parameters unchanged		
σ_f	V($)	c_d(%)
0.010	5.2168	2.0243
0.015	5.3726	2.0848
0.02	5.5458	2.1520
0.025	5.7364	2.2259
0.030	5.9445	2.3067
Varying σ_d while keeping other parameters unchanged		
σ_d	V($)	c_d(%)
0.010	5.5267	2.1446
0.015	5.5362	2.1483
0.02	5.5458	2.1520
0.025	5.5553	2.1557
0.030	5.5648	2.1594
Varying σ_s while keeping other parameters unchanged		
σ_s	V($)	c_d(%)
0.10	1.8801	0.7296
0.15	2.7942	1.0843
0.2	3.7099	1.4395
0.25	4.6270	1.7954
0.3	5.5458	2.1520
0.35	6.4661	2.5091
0.40	7.3879	2.8668
Varying $\sigma_{I_f^*}$ while keeping other parameters unchanged		
$\sigma_{I_f^*}$	V($)	c_d(%)
0.10	2.3115	0.8969
0.15	3.1183	1.2100
0.2	3.9262	1.5235
0.25	4.7354	1.8375
0.3	5.5458	2.1520
0.35	6.3574	2.4669
0.40	7.1701	2.7823

Table shows the volatility effects on the swap values (assuming c_d=0) and constant margin rates (assuming V=0). The benchmark parameters are as follows: $\sigma_d=\sigma_f=0.02$, $\sigma_s=0.3$, $\sigma_{I_f^*}=0.3$, $\rho_{df}=0.3$, $\rho_{fs}=\rho_{fI_f^*}=-0.3$, $\rho_{dI_f^*}=\rho_{sI_f^*}=-0.2$, $k_d=k_f=0.15$, $T-t=3$ years, and NP_d=\$100.

value is more sensitive to the volatility of the foreign interest rate than to the volatility of the domestic interest rate. This may be explained from our pricing formula, whereby σ_f appears in b_1, b_2, b_3, and b_5, but σ_d only appears in b_1 and b_4. Since the interest rate volatility is generally far less than the exchange rate volatility and the foreign stock price volatility, the swap value and the constant margin rate are more sensitive to the latter. Moreover, the swap value increases when the volatilities of the four state variables increase.

We finally investigate how the correlation coefficients affect the swap values and the constant margin rates. The swap maturity here is also three years. We vary one correlation coefficient from -0.3 to 0.3 and keep others unchanged. All correlation coefficients are considered in our analyses except correlation between the domestic interest rate and the exchange rate since it does not appear in our pricing formula, i.e. equation (7). From Table 3, we find that the effect of the correlation between the domestic and foreign interest rate is extremely small. The price of a quanto equity swap is actually sensitive to the correlation between the foreign interest rate and the exchange rate and the correlation between the exchange rate and the foreign equity price, especially to the latter. When the correlation between exchange rate and foreign equity price is positive, the value of a quanto equity swap is negative and vice versa. The above results can be verified by examining our pricing formula. From equation (7), it is obvious that b_6 (reflecting the correlation between the exchange rate and the foreign equity price) dominates b_1 to b_5, because the exchange rate volatility and the foreign equity price volatility are generally far larger than both the domestic and the foreign interest rate volatilities.

Conclusion

In this paper we extend the arbitrage-free pricing model of Amin and Bodurtha (1995) to derive pricing formulae for quanto equity swaps. We first determine the drift of all asset prices in a domestic risk-neutral world such that the relative price in domestic currency of any asset relative to the domestic money market account is a $\widetilde{Q}$ martingale. The pricing formulae are then obtained by discounting the cash flows of quanto equity swaps to current time under the $\widetilde{Q}$ measure. We find that the value of a quanto equity swap at the start date does not depend on the foreign stock price level, but rather on the term structures of both countries and other parameters.

The numerical results indicate that the difference between initial term structures of both countries will affect the swap value significantly. To the party paying a domestic interest rate and receiving a foreign equity return, the higher the domestic term structure is (compared to the foreign one), the smaller the swap value is. In this circumstance the swap value will decrease as the swap maturity increases. We also find that the swap value rises as the volatilities of all state variables increase. Moreover, among all correlation coefficients, the swap value is the most sensitive to the correlation between the exchange rate and the foreign equity price.

Table 3. Effects of correlations on the swap values and constant margin rates

Varying $\rho_s I_f^*$ while keeping other parameters unchanged		
$\rho_s I_f^*$	V($)	cd(%)
−0.30	5.6290	2.1843
−0.2	5.6151	2.1789
−0.1	5.6013	2.1735
0	5.5874	2.1681
0.1	5.5735	2.1627
0.2	5.5596	2.1573
0.30	5.5458	2.1520
Varying ρ_{fs} while keeping other parameters unchanged		
ρ_{fs}	V($)	cd(%)
−0.30	5.5458	2.1520
−0.2	5.3420	2.0729
−0.1	5.1384	1.9939
0	4.9349	1.9149
0.1	4.7314	1.8360
0.2	4.5281	1.7570
0.30	4.3248	1.6782
Varying $\rho_{fI_f^*}$ while keeping other parameters unchanged		
$\rho_{fI_f^*}$	V($)	cd(%)
−0.30	5.5458	2.1520
−0.2	5.5857	2.1674
−0.1	5.6255	2.1829
0	5.6654	2.1984
0.1	5.7053	2.2139
0.2	5.7452	2.2293
0.30	5.7851	2.2448
Varying $\rho_{dI_f^*}$ while keeping other parameters unchanged		
$\rho_{dI_f^*}$	V($)	cd(%)
−0.30	5.5857	2.1674
−0.2	5.5458	2.1520
−0.1	5.5059	2.1365
0	5.4660	2.1210
0.1	5.4262	2.1055
0.2	5.3863	2.0901
0.30	5.3464	2.0746

Table 3. (*Continued.*)

Varying $\rho_s I_f$* while keeping other parameters unchanged

$\rho_s I_f$*	V($)	cd(%)
−0.30	8.0042	3.1059
−0.2	5.5458	2.1520
−0.1	3.0984	1.2023
0	0.6620	0.2569
0.1	−1.7634	−0.6843
0.2	−4.1779	−1.6212
0.30	−6.5816	−2.5539

Table shows the correlation effects on the swap values (assuming c_d=0) and constant margin rates (assuming V=0). The benchmark parameters are as follows: $\sigma_d=\sigma_f=0.02$, $\sigma_s=0.3$, $\sigma_{I_f^*}=0.3$, $\rho_{df}=0.3$, $\rho_{fs}=\rho_{fI_f^*}=-0.3$, $\rho_{dI_f^*}=\rho_{sI_f^*}=-0.2$, $k_d=k_f=0.15$, $T-t=3$ years, and NP_d=$100.

In future research we can apply our valuation model to price financial derivatives, such as quanto options and quanto equity swaptions, whose payoffs depend on a foreign equity price and/or foreign risk-free rate. Moreover, our discrete time model can be implemented with lattice approaches to price American-style options.

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Notes

[1] In this article, the 'domestic' setting is where the investor lives and receives his/her payoff. For example, to a US investor, US dollar and S&P 500 are his/her domestic currency and equity index, respectively.
[2] In other words, the exchange rate between the US dollar and Japanese Yen during the whole life of the swap is set initially to be one.
[3] Chance and Rich (1998) price cross-currency equity swaps in which a party pays a domestic equity return and receives a return on a foreign index. The valuation problem considered in their paper is relatively simple, because the foreign index returns are converted into the domestic currency using a floating exchange rate. By contrast, we consider a fixed exchange rate (quanto) effect on the equity swap pricing in this paper.
[4] It should be noted that a superscript * indicates that the variable is denominated in the foreign currency.
[5] The values of the drifts $\alpha_d(t, T,.)$, $\alpha_f(t, T,.)$, $\alpha_s(t,.)$, and $\alpha_{I_f^*}(t,.)$ under the $\tilde{Q}$ measure are shown in Lemma 1.
[6] This assumption implies that both the domestic and foreign interest rates are normally distributed. Kijima and Muromachi (2001) make the same assumption so as to derive closed-form solutions for other types of equity swaps.

[7] Under this assumption, the HJM model corresponds to the spot rate (extended-Vasicek) model of Hull and White (1990) with mean-reversion coefficients k_d (domestic) and k_f (foreign).
[8] The properties of lognormally distributed variables used here are that the followings hold for normally distributed variables x and y:

$$E[e^x] = \exp\left[E(x) + \frac{Var(x)}{2}\right]$$
$$E[e^y] = \exp\left[E(y) + \frac{Var(y)}{2}\right]$$
$$E[e^{x+y}] = E[e^x]E[e^y]e^{Cov(x,\, y)}$$

[9] Please see the Appendix for the key steps of the derivations.
[10] As pointed out by the referee, this is a simple consequence of the fact that the payment of the equity leg is defined as a 'return' over two consecutive payment dates. If it is defined in the other ways, e.g. a 'return' from the beginning date to the payment date, the results will be different.
[11] The difference is due to the fact that the payment at time t_i on the foreign leg is predetermined at time t_{i-1} for a differential swap, while it is determined at time t_i for an equity swap.
[12] As pointed out by the referee, the pricing duality is useful only under one foreign equity case. For swaps or options written on baskets of foreign assets in many countries, the duality property becomes irrelevant.
[13] Remenber that our analyses are based on the viewpoint of an investor who pays domestic floating rates and receives foreign equity returns.
[14] It should be noted that cash flows other than the first one after time t can still be priced by equation (7) since the equity leg is defined as a 'return' over two consecutive payment dates.

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Appendix

Proof of Lemma 1

Lemma 1 can be obtained by applying the martingale condition to $Z_f(t)$, $Z_d(t, T)$, $Z_f(t, T)$, and $Z_{I_f^*}(t)$, respectively. That is,

$$\widetilde{E}_t\left[\frac{Z_f(t+h)}{Z_f(t)}\right] = \widetilde{E}_t\left[\frac{Z_d(t+h, T)}{Z_d(t, T)}\right] = \widetilde{E}_t\left[\frac{Z_f(t+h, T)}{Z_f(t, T)}\right] = \widetilde{E}_t\left[\frac{Z_{I_f^*}(t+h)}{Z_{I_f^*}(t)}\right] = 1 \tag{10}$$

Since Amin and Bodurtha (1995) provide formulae for $\alpha_d(t, ih,.)$, $\alpha_f(t, ih,.)$, and $\alpha_s(t,.)$ in their *Lemma 1*, we will derive the formula for $\alpha_{I_f^*}(t,.)$ only. Given the definition of $Z_{I_f^*}(.)$ and equation (10), we obtain

$$\widetilde{E}_t\left[\frac{Z_{I_f^*}(t+h)}{Z_{I_f^*}(t)}\right] = \widetilde{E}_t\left[\frac{I_f^*(t+h) * S(t+h)}{I_f^*(t) * S(t)} \frac{B_d(t)}{B_d(t+h)}\right] = 1 \tag{11}$$

In our discrete time model, the price in the domestic currency at time t of a domestic zero-coupon bond with maturity T is given by

$$P_d(t, T) = \exp\left[-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1} f_d(t, ih)h\right] \tag{12}$$

The domestic money market account, $B_d(t)$, is defined by

$$B_d(t) = \exp\left[\sum_{i=\frac{t}{h}}^{\frac{t}{h}-1} r_d(ih)h\right] = \exp\left[\sum_{i=\frac{t}{h}}^{\frac{t}{h}-1} f_d(ih, ih)h\right] \tag{13}$$

The foreign zero-coupon bond price, $P_f^*(t, T)$, and foreign money market account, $B_f^*(t)$, are defined in the same way as the domestic case.

Substituting equations (3), (4) and (13) into equation (11) yields

$$\widetilde{E}_t\left\{\exp\left[\alpha_{I_f^*}(t,.)h + \sigma_{I_f^*}(t,.)X_{I_f^*}(t+h)\sqrt{h}\right]\exp[-r_d(t)h]\right.$$
$$\left.\exp\left[\left[\alpha_s(t,.) + r_d(t) - r_f(t)\right]h + \sigma_s(t,.)X_s(t+h)\sqrt{h}\right]\right\} = 1$$

Hence, we obtain

$$\begin{aligned}\alpha_{I_f^*}(t,.)h = &-\ln\left\{\widetilde{E}_t\left(\exp\left[\sigma_{I_f^*}(t,.)X_{I_f^*}(t+h)\sqrt{h} + \sigma_s(t,.)X_s(t+h)\sqrt{h}\right]\right)\right\}\\ &-\left[\alpha_s(t,.)h - r_f(t)h\right]\end{aligned} \tag{14}$$

Substituting the formula of $\alpha_s(t,.)h$ into the above equation yields the desired result.

Deriving the Pricing Formula of a Quanto Equity Swap at the Start Date

From the definition of q_i-period LIBOR rate, the cash flow $CF_{t_{i+1}}$ can be rewritten as

$$CF_{t_{i+1}} = NP_d \times \left[\frac{I_f^*(t_{i+1})}{I_f^*(t_i)} - \frac{1}{\widetilde{E}_{t_i}\left[\exp\left(-\sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} r_d(ih)\right)\right]} - c_d(t_{i+1} - t_i)\right]$$

The expected present values of the second and third parts of $CF_{t_{i+1}}$ are shown in Chang *et al*. (2002). Thus we will focus on the valuation of the first part of $CF_{t_{i+1}}$.

Applying the lognormal property $E[e^{x+y}]=E[e^x]\,E[e^y]e^{Cov(x,y)}$ to Lemma 1, we can rewrite the drift rate of $I_f^*(t_i)$ under the $\widetilde{Q}$ measure as

$$\alpha_{I_f^*}(t,.)h = r_f(t)h - \ln \widetilde{E}_t\left(\exp\left[\sigma_{I_f^*}(t,.)X_{I_f^*}(t+h)\sqrt{h}\right]\right) - \sigma_{I_f^*}(t,.)\sigma_s(t,.)\rho_{sI_f^*}h$$

From the dynamic process of $I_f^*(t_i)$ and the above equation, we can express the expected present value of the first part of $CF_{t_{i+1}}$ as (ignoring NP_d for brevity):

$$\begin{aligned}
&\widetilde{E}_t\left\{\exp\left[-\sum_{i=\frac{t}{h}}^{\frac{t_{i+1}}{h}-1} r_d(ih)h\right]\frac{I_f^*(t_{i+1})}{I_f^*(t_i)}\right\}\\
&= \widetilde{E}_t\left\{\exp\left[-\sum_{i=\frac{t}{h}}^{\frac{t_{i+1}}{h}-1} r_d(ih)h\right]\exp\left\{\sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} r_f(ih)h\right.\right.\\
&\quad - \sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} \ln \widetilde{E}_{ih}\left(\exp\left[\sigma_{I_f^*}(ih,.)X_{I_f^*}(ih+h)\sqrt{h}\right]\right)\\
&\quad \left.\left. - \sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} \sigma_{I_f^*}(ih,.)\sigma_s(ih,.)\rho_{sI_f^*}h + \sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} \sigma_{I_f^*}(ih,.)X_{I_f^*}(ih+h)\sqrt{h}\right\}\right\}
\end{aligned} \tag{15}$$

By definition, the expectation of the first term is the domestic discount bond price $P_d(t,\ T)$, i.e.

$$P_d(t,\ T) = \widetilde{E}_t\left[\exp\left(-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1} r_d(ih)h\right)\right] \tag{16}$$

The expectation of the second term can be rearranged as

$$
\begin{aligned}
&\widetilde{E}_t\left[\exp\left(\sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} r_f(ih)h\right)\right] \\
&=\widetilde{E}_t\left[\frac{\exp\left(-\sum_{i=t}^{\frac{t_i}{h}-1} r_f(ih)h\right)}{\exp\left(-\sum_{i=t}^{\frac{t_{i+1}}{h}-1} r_f(ih)h\right)}\right] \\
&=\frac{\widetilde{E}_t\left[\exp\left(-\sum_{i=t}^{\frac{t_i}{h}-1} r_f(ih)h\right)\right]}{\widetilde{E}_t\left[\exp\left(-\sum_{i=t}^{\frac{t_{i+1}}{h}-1} r_f(ih)h\right)\right]}\exp\left[cov\left(\sum_{i=t}^{\frac{t_{i+1}}{h}-1} r_f(ih)h,\ \sum_{\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} r_f(ih)h\right)\right]
\end{aligned}
\tag{17}
$$

where the second equality is derived by applying lognormal properties. Since $r_f(ih)=f_f(ih, ih)$, we obtain the foreign spot rate dynamics as follows:

$$
\begin{aligned}
r_f(ih)h &= f_f(ih, ih)h \\
&= \sum_{j=t}^{i-1}\left[f_f(jh+h, ih)h - f_f(jh, ih)h\right] + f_f(t, ih)h \\
&= \sum_{j=t}^{i-1}\alpha_f(jh, ih)h^2 + \sum_{j=t}^{i-1}\sigma_f(jh, ih)X_f(jh+h)\sqrt{h}h + f_f(t, ih)h
\end{aligned}
\tag{18}
$$

The drift rate of the foreign forward interest rate can be rewritten as

$$
\alpha_f(jh, ih)h^2 = \sum_{k=j+1}^{i}\alpha_f(jh, kh)h^2 - \sum_{k=j+1}^{i-1}\alpha_f(jh, kh)h^2
$$

Substituting Lemma 1 into the above equation and using the lognormal properties yields

$$\begin{aligned}\alpha_f(jh, ih)h^2 = &\ln\left\{\widetilde{E}_{jh}\left[\exp\left(-\sigma_f(jh, ih)X_f(jh+h)\sqrt{h}h\right)\right]\right\}\\ &-cov_{jh}\left(\sigma_f(jh, ih)X_f(jh+h)\sqrt{h}h, \sigma_s(jh)X_s(jh+h)\sqrt{h}\right)\\ &+\sigma_f(jh, ih)\sum_{k=j+1}^{i-1}\sigma_f(jh, kh)h^3\end{aligned} \tag{19}$$

Substituting (19) into (18), we obtain the dynamic process of the foreign spot interest rate as

$$\begin{aligned}r_f(ih)h = &\sum_{j=t}^{i-1}\ln\left\{\widetilde{E}_{jh}\left[\exp\left(-\sigma_f(jh, ih)X_f(jh+h)\sqrt{h}h\right)\right]\right\}\\ &+\sum_{j=t}^{i-1}\left[\sigma_f(jh, ih)\sum_{k=j+1}^{i-1}\sigma_f(jh, kh)h^3\right.\\ &\left.-cov_{jh}\left(\sigma_f(jh, ih)X_f(jh+h)\sqrt{h}h, \sigma_s(jh)X_s(jh+h)\sqrt{h}\right)\right]\\ &+\sum_{j=t}^{i-1}\sigma_f(jh, ih)X_f(jh+h)\sqrt{h}h + f_f(t, ih)h\end{aligned} \tag{20}$$

Hence, we obtain

$$\begin{aligned}&\widetilde{E}_t\left[\exp\left(-\sum_{i=t}^{\frac{T}{h}-1} r_f(ih)h\right)\right] = \exp\left(-\sum_{i=t}^{\frac{T}{h}-1} f_f(t, ih)h\right)\times\\ &\exp\left[\sum_{i=t}^{\frac{T}{h}-1}\sum_{j=t}^{i-1} cov_{jh}\left(\sigma_f(jh, ih)X_f(jh+h)\sqrt{h}h, \sigma_s(jh)X_s(jh+h)\sqrt{h}\right)\right]\\ &= P_f^*(t, T)\exp\left[\sum_{i=t}^{\frac{T}{h}-1}\sum_{j=t}^{i-1} cov_{jh}\left(\sigma_f(jh, ih)X_f(jh+h)\sqrt{h}h, \sigma_s(jh)X_s(jh+h)\sqrt{h}\right)\right]\end{aligned} \tag{21}$$

Applying the lognormal properties and substituting equations (16), (17), and (21) into equation (15) yields

$$P_d(0,\, t_{i+1})\frac{P_f^*(0,\, t_i)}{P_f^*(0,\, t_{i+1})}\exp[-b_1+b_2-b_3-b_4+b_5-b_6]$$

where

$$b_1 = cov\left(\sum_{i=\frac{t}{h}}^{\frac{t_{i+1}}{h}-1} r_d(ih)h,\ \sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} r_f(ih)h\right)$$

$$b_2 = cov\left(\sum_{i=\frac{t}{h}}^{\frac{t_{i+1}}{h}-1} r_f(ih)h,\ \sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} r_f(ih)h\right)$$

$$b_3 = \sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1}\sum_{j=\frac{t}{h}}^{i-1} cov_{jh}\left(\sigma_f(jh,\, ih,.)X_f(jh+h)\sqrt{h}h,\ \sigma_s(ih,.)X_s(jh+h)\sqrt{h}\right)$$

$$b_4 = cov\left(\sum_{i=\frac{t}{h}}^{\frac{t_{i+1}}{h}-1} r_d(ih)h,\ \sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} \sigma_{I_f^*}(ih,.)X_{I_f^*}(ih+h)\sqrt{h}\right)$$

$$b_5 = cov\left(\sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} r_f(ih)h,\ \sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} \sigma_{I_f^*}(ih,.)X_{I_f^*}(ih+h)\sqrt{h}\right)$$

$$b_6 = \sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} \sigma_{I_f^*}(ih,.)\sigma_s(ih,.)\rho_{sI_f^*}h$$

As an illustration, we derive the formula of b_1 when $\sigma_d(t,\, T,.)=\sigma_d\exp[-k_d(T-t)]$, $\sigma_f(t,\, T,.)=\sigma_f\exp[-k_f(T-t)]$, and $h\to 0$. Because $r_d(ih)=f_d(ih,\, ih)$, we obtain the follow equation:

$$\begin{aligned} r_d(ih)h &= f_d(ih,\, ih)h \\ &= \sum_{j=\frac{t}{h}}^{i-1} \alpha_d(jh,\, ih)h^2 + \left(\sum_{j=\frac{t}{h}}^{i-1} \sigma_d(jh,\, ih,.) X_d(jh+h)\sqrt{h}h \right) + f_d(t,\, ih)h \end{aligned} \tag{22}$$

The drift rate of domestic forward interest rate can be rewritten as

$$\alpha_d(jh,\, ih,.)h^2 = \sum_{k=j+1}^{i} \alpha_d(jh,\, kh,.)h^2 - \sum_{k=j+1}^{i-1} \alpha_d(jh,\, kh,.)h^2$$

Substituting Lemma 1 into the above equation and using the properties of lognormally distributed variables yields

$$\begin{aligned} \alpha_d(jh,\, ih,.)h^2 &= \ln\left\{ \widetilde{E}_{jh}\left[\exp\left(-\sigma_d(jh,\, ih,.) X_d(jh+h)\sqrt{h}h \right) \right] \right\} \\ &\quad + \sigma_d(jh,\, ih,.) \sum_{k=j+1}^{i-1} \sigma_d(jh,\, kh,.)h^3 \end{aligned} \tag{23}$$

Substituting equation (23) into equation (22) yields

$$\begin{aligned} r_d(ih)h &= \sum_{j=\frac{t}{h}}^{i-1} \ln\left\{ \widetilde{E}_{jh}\left[\exp\left(-\sigma_d(jh,\, ih,.) X_d(jh+h)\sqrt{h}h \right) \right] \right\} \\ &\quad + \sum_{j=\frac{t}{h}}^{i-1} \left(\sigma_d(jh,\, ih,.) \sum_{k=j+1}^{i-1} \sigma_d(jh,\, kh,.)h^3 \right) \\ &\quad + \sum_{j=\frac{t}{h}}^{i-1} \sigma_d(jh,\, ih,.) X_d(jh+h)\sqrt{h}h + f_d(t,\, ih)h \end{aligned} \tag{24}$$

Therefore we have

$$b_1 = cov\left(\sum_{i=\frac{t}{h}}^{\frac{t_{i+1}}{h}-1} r_d(ih)h,\ \sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1} r_f(ih)h\right)$$
$$= h^3 cov\left(\sum_{i=\frac{t}{h}}^{\frac{t_{i+1}}{h}-1}\sum_{j=\frac{t}{h}}^{i-1}\sigma_d(jh, ih,.)X_d(jh+h),\ \sum_{i=\frac{t_i}{h}}^{\frac{t_{i+1}}{h}-1}\sum_{j=\frac{t}{h}}^{i-1}\sigma_f(jh, ih,.)X_f(jh+h)\right)$$

When $h \to 0$,

$$b_1 = cov\left(\int_{s=t}^{t_{i+1}}\int_{u=t}^{s}\sigma_d(u, s)\mathrm{d}W_d(u)\mathrm{d}s,\ \int_{s=t_i}^{t_{i+1}}\int_{u=t}^{s}\sigma_f(u, s)\mathrm{d}W_f(u)\mathrm{d}s\right).$$

Let $A = \int_{s=\tau_1}^{T}\int_{u=\tau_2}^{s}\sigma_f(u, s)\mathrm{d}W_f(u)\mathrm{d}s$, where $\tau_1 \geqslant \tau_2$. Substituting $\sigma_f(t, T) = \sigma_f \exp[-k_f(T-t)]$ into A yields

$$A = \sigma_f\int_{s=\tau_1}^{T}\int_{u=\tau_2}^{s}\exp\left[-k_f(s-u)\right]\mathrm{d}W_f(u)\mathrm{d}s$$
$$= \sigma_f\int_{s=\tau_1}^{T}\exp\left(-k_f s\right)\int_{u=\tau_2}^{s}\exp\left[k_f u\right]\mathrm{d}W_f(u)\mathrm{d}s$$
$$= -\frac{\sigma_f}{k_f}\int_{s=\tau_1}^{T}\int_{u=\tau_2}^{s}\exp\left[k_f u\right]\mathrm{d}W_f(u)\mathrm{d}\exp\left(-k_f s\right)$$

Using the formula of integration by parts, we obtain

$$A = -\frac{\sigma_f}{k_f}\left[\exp\left(-k_f s\right)\int_{u=\tau_2}^{s}\exp\left(k_f u\right)\mathrm{d}W_f(u)\Big|_{s=\tau_1}^{T} - \int_{s=\tau_1}^{T}\exp\left(-k_f s\right)\mathrm{d}\int_{u=\tau_2}^{s}\exp\left[k_f u\right]\mathrm{d}W_f(u)\right]$$
$$= -\frac{\sigma_f}{k_f}\left[\exp\left(-k_f T\right)\int_{u=\tau_2}^{T}\exp\left(k_f u\right)\mathrm{d}W_f(u) - \exp\left(-k_f\tau_1\right)\int_{u=\tau_2}^{\tau_1}\exp\left(k_f u\right)\mathrm{d}W_f(u) - \int_{s=\tau_1}^{T}\mathrm{d}W_f(s)\right] \tag{25}$$

and

$$\widetilde{E}_{\tau_2}(A)=0 \tag{26}$$

Note that if we let $B=\int_{s=\tau_1}^{T}\int_{u=\tau_2}^{s}\sigma_d(u,s)\mathrm{d}W_d(u)\mathrm{d}s$, then B also follows formulae similar to equations (25) and (26). Therefore we can rewrite b_1 as

$$\begin{aligned}
b_1 &= cov\left(\int_{s=t}^{t_{i+1}}\int_{u=t}^{s}\sigma_d(u,s)\mathrm{d}W_d(u)\mathrm{d}s,\ \int_{s=t_i}^{t_{i+1}}\int_{u=t}^{s}\sigma_f(u,s)\mathrm{d}W_f(u)\mathrm{d}s\right)\\
&=\frac{\sigma_f\sigma_d}{k_fk_d}\widetilde{E}_t\left\{\left[\exp(-k_dt_{i+1})\int_{u=t}^{t_{i+1}}\exp(k_du)\mathrm{d}W_d(u)-\int_{s=t}^{t_{i+1}}\mathrm{d}W_d(s)\right]\right.\\
&\quad\times\left[\exp\left(-k_ft_{i+1}\right)\int_{u=t}^{t_{i+1}}\exp\left(k_fu\right)\mathrm{d}W_f(u)\right.\\
&\quad\left.\left.-\exp\left(-k_ft_i\right)\int_{u=t}^{t_i}\exp\left(k_fu\right)\mathrm{d}W_f(u)-\int_{s=t_i}^{t_{i+1}}\mathrm{d}W_f(s)\right]\right\}\\
&=\frac{\sigma_d\sigma_f\rho_{df}}{k_dk_f}\left[\frac{1}{k_f}\left(e^{-k_f(t_{i+1}-t)}-e^{-k_f(t_i-t)}\right)+\frac{1}{k_d}\left(e^{-k_f(t_{i+1}-t_i)}-1\right)+(t_{i+1}-t_i)\right.\\
&\quad\left.+\frac{1}{k_d+k_f}\left(1-e^{-\left(k_d+k_f\right)(t_{i+1}-t)}-e^{-k_d(t_{i+1}-t_i)}+e^{-k_d(t_{i+1}-t)-k_f(t_i-t)}\right)\right]
\end{aligned}$$

where the second equality follows the definition of covariance and equations (25) and (26).

Pricing Formula of a Quanto Equity Swap at the Subsequent Date

The valuation formula in the earlier section (i.e. equation (7)) is applicable when current time t is at the starting date or immediately after the payment date of the swap. However, when current time t is between two payment dates, the first payment after time t cannot be evaluated by equation (7) since we have known the domestic and foreign interest rates and some information of stock level from the previous payment date to time t.[14] Let t_0 denote the previous payment date and t_1 denote the next payment date. Current time t is between t_0 and t_1. The cash flow received at t_1 becomes

$$\begin{aligned}
CF_{t_1}&=NP_d\times\left[\left(\frac{I_f^*(t_1)}{I_f^*(t_0)}-1\right)-(R_d(q_0)+c_d)(t_1-t_0)\right]\\
&=NP_d\times\left[\frac{I_f^*(t)}{I_f^*(t_0)}\frac{I_f^*(t_1)}{I_f^*(t)}-\left(\frac{1}{P_d(t_0,t_1)}+c_d\right)(t_1-t_0)\right]
\end{aligned}$$

Similar to the previous proof, one can obtain a simplified closed-form solution for the expected present value of CF_{t_1} at time t as

$$V_{t_1} = NP_d \frac{I_f^*(t)}{I_f^*(t_0)} \frac{P_d(t, t_1)}{P_f^*(t, t_1)} \exp\left[-b_1' + b_2' - b_3' - b_4' + b_5' - b_6'\right] - NP_d \frac{P_d(t, t_1)}{P_d(t_0, t_1)} - NP_d \frac{c_d(t_{i+1} - t_i)}{360} P_d(t, t_1) \tag{27}$$

where

$$b_1' = \frac{\sigma_d \sigma_f \rho_{df}}{k_d k_f} \left[\frac{1}{k_f}\left(e^{-k_f(t_1-t)} - 1\right) + \frac{1}{k_d}\left(e^{-k_f(t_1-t)} - 1\right) + (t_1 - t) + \frac{1}{k_d + k_f}\left(1 - e^{-\left(k_d + k_f\right)(t_1 - t)}\right)\right]$$

$$b_2' = \frac{\sigma_f^2}{k_f^3}\left[-\frac{3}{2} + k_f(t_1 - t) + 2e^{-k_f(t_1-t)} - \frac{1}{2}e^{-2k_f(t_1-t)}\right]$$

$$b_3' = \frac{\sigma_f \sigma_s \rho_{fs}}{k_f} \times \left[(t_1 - t) - \frac{1}{k_f}\left(1 - e^{-k_f(t_1-t)}\right)\right]$$

$$b_4' = \frac{\sigma_{I_f^*} \sigma_d \rho_{I_f^* d}}{k_d} \times \left[(t_1 - t) - \frac{1}{k_d}\left(1 - e^{-k_d(t_1-t)}\right)\right]$$

$$b_5' = \frac{\sigma_{I_f^*} \sigma_f \rho_{I_f^* f}}{k_f} \times \left[(t_1 - t) - \frac{1}{k_f}\left(1 - e^{-k_f(t_1-t)}\right)\right]$$

$$b_6' = \sigma_{I_f^*} \sigma_s \rho_{I_f^* s} \times (t_1 - t)$$

Obviously, equations (22) and (7) are equivalent when both t and t_i equal t_0. From equation (22), we find that between two payment dates, the value of quanto equity swaps does depend on the stock level at time t because the value of the first payoff involves the stock price.