Efficient quadratic approximation of floating strike Asian option values

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Abstract

We derive a new formula for Asian options with floating strike, which proves more accurate for both low and higher volatility values. Average Strike Options are less often considered in the literature because their valuation is more complex. Compared to a benchmark our analytical formula is very efficient in the sense of accuracy vs speed, whereas numerical methods: Monte-Carlo, numerical integration of the partial differential equation or numerical inversion of the Laplace transform all require considerable calculating time.

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1 Introduction and motivation

About 10 years ago financial markets gave rise to a new generation of options called "Asian." These options, first traded over the counter, met particular hedging needs of treasurers, corresponding to the entire budgetary period and not only that of the end of the period. Asian options payoff is based on the average price of a commodity or exchange rate observed during a given period of time. Such payoffs cannot be obtained by combining other hedging techniques and financial instruments such as standard options, futures or forwards. Moreover, most "Asians" are less expensive, which confers an additional advantage compared to traditional or some other exotic options. This is because the volatility of an average is lower than the volatility of the underlying asset.

The majority of traded Asian options are of European-style. As emphasized by Kemna and Vorst (1990), the reason is that a possibility of exercise before the expiration date would make the contract more vulnerable to price manipulation. Depending on the use of the average in the contract, one can distinguish two types of Asian options:

- *Average Price Options* (APO) are Asian options where the average relates to the underlying asset and the strike is fixed in advance.
- *Average Strike Options* (ASO) are Asian options where the average relates to the strike price. The payoff is determined by the difference between the underlying and its average. Average Strike Options are also known as *floating strike* options.

Explicit formula for all moments of an arithmetic continuous average A_τ of log-normal variates up to time τ has been obtained by Geman and Yor (1993). The fundamental Asian option valuation problem is that there are infinitely many moments of A_τ and, worse, knowledge of all moments is not equivalent to knowing the probability law of the average. This is why analytical approximations or numerical procedures are ultimately necessary.

The valuation of APOs has been considered by many authors. Geman and Yor (1993) also derive a closed form formula for an "in-the-money" APO call and a Laplace transform of the option premium for "at-the-money" and "out-of-the money" cases. However as emphasized by Fu, Madan and Wang (1998) and Temam (1998), numerical inversion of the Laplace transform is time-consuming and not numerically stable, especially for low values of volatility $\sigma < 20\%$. Levy (1992) employed Wilkinson approximation to obtain a formula for an APO call. Turnbull and Wakeman (1991) in their algorithm assumed that A_τ is log-normal. Recently, the limiting case of a perpetual average A_∞ has been considered by Milevsky and Posner (1998a) and (1998b) and an analytic formula obtained, but only for the case of *neg-*

ative cost of carry.[1] This is not surprising because A_∞ has a well known, reciprocal gamma distribution, which was first noticed by Yor (1993). Attempts has also been made, e.g. Hansen and Jorgensen (2000), to extend various analytical approximations to finite-lived, American-style Asian APOs. Finally, direct numerical methods for ASOs are abundant and fall in the range of either Monte-Carlo or quasi-Monte-Carlo simulations[2] or solving numerically the partial differential equation satisfied by an Asian APO.[3]

Analytical valuation methods for *Average Strike Options* are much less abundant because the problem is complex. Alziary *et al.* (1997) provide the put-call parity relations between APOs and ASOs but this is not sufficient to transform an ASO valuation into an APO valuation. This is because the relation ties simultaneously an ASO call *and* an ASO put. Indeed, to compute a premium of an ASO one must know the *joint* probability law of the couplet $\{A_\tau, S_\tau\}$, where S_τ is the *last* price of the underlying asset observed in the averaging period of length τ. This is in contrast to APOs, where knowing the law of the average A_τ is enough.

In their seminal paper Bouaziz, Briys and Crouhy (1994) used linear approximation technique approximating the true law of $\{A_\tau, S_\tau\}$ by a joint log-normal distribution. The only available closed-form formula for ASOs has been obtained by Conze and Visvanathan (1991), but only for geometric averages A^g_τ. Not surprisingly, the pair $\{A^g_\tau, S_\tau\}$ comprising the geometric mean A^g_τ *is* joint log-normally distributed. This is why the formula provided by Bouaziz *et al.* (1994) can be seen as relying to some extent on a geometric type of approximation. Good approximations are obtained for relatively small values of volatility. Not surprisingly, for higher volatility values, the same order of error can be observed if one uses exact Conze and Visvanathan (1991) geometric ASO pricing formula to value an *arithmetic* ASO Asian option.

In this paper we incorporate second order terms into the probability law approximation of the couplet $\{A_\tau, S_\tau\}$. Our closed-form formula is no longer based on geometric approximation and, compared to a benchmark, yields far more accurate results, even for higher values of volatility. As a benchmark we use accurate numerical values obtained via Monte-Carlo simulations. Our quadratic method can be extended to incorporate further expansion terms yielding a family of closed-form formulas converging to the true value. Finally, the approach can be applied to similar derivative pricing problems when no solution exists.

The paper is organized as follows: in section 2 we introduce the Asian

[1] Continuous-time adjusted *cost of carry* is $r - \frac{\sigma^2}{2} - \delta < 0$, where r, δ are risk-free and dividend rates respectively.

[2] See Boyle *et al.* (1997), Broadie and Glasserman (1996), Joy *et al.* (1996), Grant *et al.* (1997) and references therein.

[3] See Schreve and Večer (2000), Alziary *et al.* (1997) or Rogers and Shi (1995) and references therein.

option pricing setup and summarize existing results. In sections 3 and 4 we derive our quadratic pricing formula for floating strike Asian options. Section 5 focuses on computational details of our derivation. Section 6 stress tests the quadratic approximation against Monte-Carlo benchmark. Finally section 7 concludes.

2 Pricing average strike option

We assume that assumptions underlying Black-Scholes (1973) model hold. The market is perfect and complete, trades take place continuously. There exists risk-free asset paying continuous flow at rate $r > 0$ per unit of time. The price of the underlying asset (the "stock") evolves, under the risk-neutral martingale measure $\mathcal{Q}$, according to the stochastic differential equation

$$dS_t = S_t \left(r \ dt + \sigma \ dW_t \right) \qquad S_0 > 0 \tag{1}$$

where W_t is standard Brownian motion under $\mathcal{Q}$. The volatility $\sigma > 0$ is constant.

An Asian option contract is signed at time $t = 0$ and expires at maturity $T > 0$. The *arithmetic average* A is a claim, the value of which can be computed at maturity T, given the history of prices from time $t_0 < T$. In general, t_0 can also take negative values. Let $D > 0$ denote the duration of the averaging

$$D \triangleq T - t_0 \tag{2}$$

Three cases are of interest. If $T = D$, the computation of the average A corresponds to the whole life of the Asian option, in which case such a contract is termed a *plain vanilla* option. If $T > D$, the option is of *forward starting* type. Finally, if $T < D$, we have an *in progress averaging* Asian option and $t_0 < 0$.

Before maturity, $t < T$, the average A is not $\mathcal{F}_t$-measurable. This feature is taken into account defining an $\mathcal{F}_t$-measurable *partial average*A_t, the computation of which ends at time t such that $t \leq T$

$$A_t \triangleq \frac{1}{D} \int_{t_0}^{t} S_u du \tag{3}$$

The value of the arithmetic average A at maturity can now be defined as $A \triangleq A_T$.

An *average strike* Asian option (ASO) is defined by its payoff at maturity which is a non-negative function of the terminal value of the stock, S_T, and the terminal value of the arithmetic average A. Payoffs for an average strike Asian call we have

$$c_T^{aso} = \left(S_T - A_T \right)^+ \tag{4}$$

According to Ingersoll (1988), the price of an Asian option prior to maturity is a function of time t and two state variables: the partial average A_t and the stock price S_t

$$c_t^{aso} \triangleq c(A_t, S_t, t) \tag{5}$$

satisfying a partial differential equation[4] subject to boundary conditions. Equivalently, the price of an average strike option can be seen as a risk-neutral expectation of the terminal payoff (4) discounted using the risk-free rate

$$c_t^{aso} = e^{-r(T-t)}\mathbb{E}\left[(S_T - A_T)^+ \middle| \mathcal{F}_t \right] \tag{6}$$

where $0 \le t \le T$ and $\mathbb{E}$ denotes risk-neutral expectation under the martingale measure $\mathcal{Q}$. Using the well known "tower law" it is then straightforward to rewrite (6) as

$$c_t^{aso} = e^{-r(T-t)}\mathbb{E}\left\{ \mathbb{E}\left[(S_T - A_T)^+ \middle| \mathcal{F}_{t_0} \right] \middle| \mathcal{F}_t \right\} \tag{7}$$

Finally, representing S_T in (7) as the solution to stochastic differential equation (1) with boundary condition $S_{t_0} > 0$, we obtain

$$c_t^{aso} = e^{-r(T-t)}\mathbb{E}\left\{ S_{t_0}\mathbb{E}\left[(\xi(t_0, T))^+ \middle| \mathcal{F}_{t_0} \right] \middle| \mathcal{F}_t \right\}$$

where

$$\xi(t_0, T) \triangleq e^{\tilde{r}D + \sigma\left(W_T - W_{t_0}\right)} - \frac{1}{D}\int_{t_0}^{T} e^{\tilde{r}(u-t_0) + \sigma\left(W_u - W_{t_0}\right)} du \tag{8}$$

and

$$\tilde{r} \triangleq r - \frac{\sigma^2}{2}$$

Bouaziz *et al.* (1994) linearize both exponential terms present in (8) according to Taylor series expansion

$$e^{\tilde{r}\Delta t + \sigma\Delta W} \approx 1 + \tilde{r}\Delta t + \sigma\Delta W \tag{9}$$

where Δt is either equal to D or $u - t_0$ and ΔW corresponds to either $(W_T - W_{t_0})$ or $(W_u - W_{t_0})$. Such approach assumes that *all* terms $\tilde{r}\Delta t + \sigma\Delta W$ are *small.* In order for this first order approximation to hold, this means that:

[4] See Ingersoll (1988). As compared to the Black-Scholes partial differential equation, it will involve one supplementary term $D^{-1}S_t \; \partial c/\partial A_t$. The partial average A_t plays a role of an extra state variable and *a fortiori* generates partial differentiation with respect to the partial average A_t.

1. *Interest rate* r should be comparable to the "speed" parameter $\frac{\sigma^2}{2}$ in order to make $\tilde{r} = r - \frac{1}{2}\sigma^2$ small;
2. The *averaging duration* D should be small;
3. The *volatility* σ should be small.

Using (9) in (8) it is then straightforward to see that an approximated linear expression for $\xi(t_0, T)$ will be *normally* distributed with volatility σ entering the resulting expression *linearly*, as a constant of proportionality. This linear approximation has *conditional* expectation and variance equal to

$$m = \frac{\tilde{r}D}{2} \quad (10)$$

$$\nu = \frac{\sigma^2 D}{3} \quad (11)$$

Once the conditional expectation m and variance ν are "matched," it is then straightforward to obtain the following pricing formula for *plain-vanilla* and *forward-starting* average strike Asian options[5]

$$c_t^{aso} = S_t e^{-r(T-t)} \left[m\mathcal{N}\left(\frac{m}{\sqrt{\nu}}\right) + \sqrt{\frac{\nu}{2\pi}} e^{-\frac{m^2}{2\nu}} \right] \quad (12)$$

into which appropriate values of m and ν must be inserted. The price does not depend here on the average A_t because for $t \le t_0$ its "calculation" has not started yet. For *plain-vanilla* option $t = t_0 = 0$ and $S_t = S_0$.

Linear approximation (9) introduces errors in the sense that the original distribution of $\xi(t_0, T)$ is not normal and, secondly, one will then in some sense rely only on the two first moments, m and ν, of the random variable $\xi(t_0, T)$. Moreover, the moments m and ν are themselves a first order approximation of the true moments. Our approach corrects this error of "second kind." This is done in the next two sections.

3 Enhanced linear approximation

Linear approximations (10) and (11) for m and ν do very well when precision is sought for *very low* values of averaging duration D and, simultaneously, for interest rate r *comparable* to the speed parameter $\frac{\sigma^2}{2}$. In order to relax these two constraints we proceed with the following, enhanced linearization scheme

$$e^{\tilde{r}\Delta t + \sigma \Delta W} \approx e^{\tilde{r}\Delta t} (1 + \sigma \Delta W) \quad (13)$$

[5]Bouaziz *at al.* (1994) also provide a formula for *in progress averaging* Asian call.

where ΔW corresponds to either $(W_T - W_{t_0})$ or $(W_u - W_{t_0})$. Our approach preserves the exponential terms $\exp(\tilde{r}D)$ and $\exp[\tilde{r}(u - t_0)]$ from being developed, thus allowing the averaging duration D to be *large* and the interest rate r to be *different* from the speed parameter $\frac{\sigma^2}{2}$.

Notice that the conditional mean of $\xi(t_0, T)$ depends only on the length of the averaging period D. The same will hold for conditional variance of $\xi(t_0, T)$. Applying our enhanced linear approximation (13) to (8) we obtain

$$\xi_1 = e^{\tilde{r}D} - \frac{e^{\tilde{r}D} - 1}{D\tilde{r}} + \sigma\left[e^{\tilde{r}D}W_D - \frac{1}{D}\int_0^D e^{\tilde{r}u}W_u du\right]$$

Defining

$$x \triangleq \tilde{r}D$$

the mean of ξ_1 i.e. $m_1 \triangleq \mathbb{E}[\xi_1]$ can be written as

$$m_1 = \frac{e^x(x-1)+1}{x} \tag{14}$$

Further computations show that the variance $\nu_1 \triangleq \mathbb{E}\left[(\xi_1 - m_1)^2\right]$ is equal to

$$\nu_1 = \sigma^2 D\,\frac{e^{2x}[2x(x(x-2)+3)-3] - 4e^x(x-1) - 1}{2\,x^3} \tag{15}$$

As expected, it is now straightforward to verify that for *very small* $\tilde{r}D$, the *first order* Taylor series expansion of m_1 and the *zeroth order* Taylor series expansion of ν_1 in $x = \tilde{r}D$ around $x = 0$ will yield m and ν as given by the "straight" approximations (10) and (11)

$$\begin{aligned} m_1 &= \frac{1}{2}x + o\left(x^2\right) \approx \frac{\tilde{r}D}{2} \\ \nu_1 &= \sigma^2 D\left[\frac{1}{3} + o(x)\right] \approx \frac{\sigma^2 D}{3} \end{aligned}$$

Enhanced linear approximation does well for small values of volatility σ. As can be seen on Figure 1. For $\sigma < 45\%$ the enhanced parameters m_1 and ν_1 improve the approximation as compared to "straight" parameters m and ν. However, for higher values of volatility the enhanced linear approximation is no longer accurate. Straight approximation does slightly better for higher values of σ but the actual picture is that *both* straight and enhanced linear approximations largely *underestimate* benchmark values.

Approximation error is not only due to volatility σ taking higher values. Linearization methods assume that terms such as σW_u for $u \leq D$ and in particular σW_D are small. This means in turn that the volatility σ *and* the two Brownian terms W_u, W_D should be small. However, when the averaging period D becomes large this is no longer true as variances of these two terms, $\sigma^2 u$ and $\sigma^2 D$, are *linearly* increasing with time and *quadratically* increasing with volatility.

In order to incorporate these two effects — the first being linked to larger volatility and the second to the length of averaging — in the next section we will consider *quadratic* expansions.

4 Quadratic approximation formula

Intuitively — although far from being a perfect solution — incorporating higher order terms should once again improve the approximation, allowing for longer averaging periods D and higher volatilities σ. Instead of expansion (13) we will now use[6]

$$e^{\tilde{r}\Delta t+\sigma\Delta W} \approx e^{\tilde{r}\Delta t}\left[1+\sigma\Delta W+\frac{1}{2}\left(\sigma\Delta W\right)^2\right] \tag{16}$$

in (8). As before we are interested in the first two moments of a random variable ξ_2, which will depend on the length D of the averaging period. Variable ξ_2 has the same distribution as the right hand side of equation (8) for the original $\xi\left(t_0, T\right)$, in which exponential terms are expanded quadratically according to (16). We obtain

$$\xi_2 = \xi_1 + \frac{\sigma^2}{2}\left[e^{\tilde{r}D}W_D^2 - \frac{1}{D}\int_0^D e^{\tilde{r}u}W_u^2 du\right] \tag{17}$$

In equation (17) the right hand side contains non-random, normally distributed and chi-square distributed terms. Moreover, the last term integrates chi-square distributed variates. It is not at all obvious what the law of this sum of chi-square and Gaussian normal variates is. In other words the distribution of ξ_2 — as compared to the distribution of ξ_1 — is *not normal.* However, our goal is here to improve the accuracy of inputs m and ν for the general pricing formula (12) obtained under assumption of normality, rather than deriving a brand new formula for an auxiliary and peculiar distribution. Therefore in what follows we will focus on obtaining the conditional first two moments, the mean m_2 and the variance ν_2, necessary for our quadratic expansion.

The expected value of ξ_2 is equal to m_1 plus a quadratic correction, depending not only on $x = \tilde{r}D$ but also on volatility σ

$$m_2 \triangleq \mathbb{E}\left[\xi_2\right] = m_1 + D\,\frac{\sigma^2}{2}\,\frac{e^x\left(x\left(x-1\right)+1\right)-1}{x^2} \tag{18}$$

[6]Note that ΔW is a *finite* increment of Brownian motion and thus $\left(\Delta W\right)^2 \neq dt$ i.e. we cannot use Itô's lemma.

where m_1 is given by (14). The variance of ξ_2 is equal to

$$\begin{aligned} \nu_2 &\triangleq \mathbb{E}\left[(\xi_2 - m_2)^2\right] \qquad (19) \\ &= \nu_1 + D^2\sigma^4 \frac{e^{2x}(2x(x(x(x-2)+5)-7)+7)+8e^x(x-1)+1}{4\,x^4} \end{aligned}$$

where ν_1 is given by (15).

The quadratic approximation formula obtains by plugging the new conditional mean and variance m_2 and ν_2, given by (18) and (19) respectively, into the approximative formula (12). Our quadratic method yields far more accurate results than any of either the "crude" linear method initially obtained by Bouaziz *et al.* (1994) or our enhanced linear method. Accuracy is satisfactory for low and medium values of volatility up to 50%. This has been illustrated on Figure 1. The next section focuses on computations necessary to obtain m_2 and ν_2, after which we numerically compare our formulation with benchmark values.

5 Computational technique

Computation of means m_1 and m_2 as well as variances ν_1 and ν_2 is straightforward because it relies on basic properties of Wiener process W. However, simplifications may become quite tedious. Computer algebra packages such as *Mathematica* or *Maple* can be employed to check the derivation. In what follows we give a brief idea of computation complexity involved.

The higher the order of the approximation, the more complex the computational task becomes. For instance, once the quadratic approximation mean, m_2, was obtained, the last step involves the computation of variance ν_2 given by:

$$\nu_2 = \mathbb{E}\left[(\xi_2 - m_2)^2\right] \qquad (20)$$

Using quadratic expansion (16) and the quadratic mean m_2 given by (18) in the definition of quadratic variance (20), after cancelling m_1 terms we obtain

$$\begin{aligned} \nu_2 &= \sigma^4\mathbb{E}\Bigg[\Bigg(\frac{e^{\tilde{r}D}}{\sigma}W_D - \frac{1}{\sigma D}\int_0^D e^{\tilde{r}u}W_u du \\ &\qquad + \frac{e^{\tilde{r}D}}{2}W_D^2 - \frac{1}{2D}\int_0^D e^{\tilde{r}u}W_u^2 du - \gamma\Bigg)^2\Bigg] \qquad (21) \end{aligned}$$

where the free term γ is equal to

$$\gamma = \frac{D}{2}\left[e^{\tilde{r}D} - \frac{1+e^{\tilde{r}D}(\tilde{r}D-1)}{(\tilde{r}D)^2}\right]$$

We can see that expression (21) involves raising to the second power a sum of 5 terms, which in turn will generate a sum of 15 terms. Next steps involve taking $\mathcal{Q}$-expectations of each of 15 terms, computing the remaining exponential integrals, collecting terms and simplifying so as to obtain (19). Some of cross terms will cancel i.e. in particular those involving powers of Wiener process W of *odd* orders. For example the expectation of the cross term

$$-2\left(\frac{1}{\sigma D}\int_0^D e^{\tilde{r}u}W_u du\right)\left(\frac{e^{\tilde{r}D}}{2}W_D^2\right)$$

is zero because $\mathbb{E}\left[W_u W_D^2\right] = 0$ which is a basic consequence of the *independence* property for non-overlapping increments of Brownian motion. As $D > u$ we have

$$\begin{aligned}
\mathbb{E}\left[W_u W_D^2\right] &= \mathbb{E}\left[W_u\left(W_D - W_u + W_u\right)^2\right] \\
&= \mathbb{E}\left[W_u\left(\left(W_D - W_u\right)^2 + 2\left(W_D - W_u\right)W_u + W_u^2\right)\right] \\
&= \mathbb{E}\left[W_u\right]\mathbb{E}\left[\left(W_D - W_u\right)^2\right] \\
&\quad +2\mathbb{E}\left[W_D - W_u\right]\mathbb{E}\left[W_u^2\right] + \mathbb{E}\left[W_u^3\right] \\
&= 0
\end{aligned}$$

because $\mathbb{E}\left[W_u\right] = 0$, $\mathbb{E}\left[W_D - W_u\right] = 0$ and $\mathbb{E}\left[W_u^3\right] = 0$.

Terms of *even* order in powers of Wiener process W will in general not vanish. For example the correlation term involving the integral of "discounted" square of Wiener process W can be written as

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{2D}\int_0^D e^{\tilde{r}u}W_u^2 du\right)^2\right] \\
&\quad = \frac{1}{4D^2}\mathbb{E}\left[\left(\int_0^D e^{\tilde{r}u}W_u^2 du\right)\left(\int_0^D e^{\tilde{r}t}W_t^2 dt\right)\right] \\
&\quad = \frac{1}{4D^2}\left(\int_0^D e^{\tilde{r}u}\left(\int_0^D e^{\tilde{r}t}\mathbb{E}\left[W_u^2 W_t^2\right] dt\right) du\right) \qquad (22)
\end{aligned}$$

Where now the expectation $\mathbb{E}\left[W_u^2 W_t^2\right]$ involves powers of Wiener process of even order. For $t > u > 0$ we have

$$\begin{aligned}
\mathbb{E}\left[W_u^2 W_t^2\right] &= \mathbb{E}\left[W_u^2\left(\left(W_t - W_u\right) + W_u\right)^2\right] \\
&= \mathbb{E}\left[W_u^2\left(\left(W_t - W_u\right)^2 + 2W_u\left(W_t - W_u\right) + W_u^2\right)\right] \\
&= \mathbb{E}\left[W_u^2\right]\mathbb{E}\left[\left(W_t - W_u\right)^2\right] \\
&\quad +2\mathbb{E}\left[W_u^3\right]\mathbb{E}\left[W_t - W_u\right] + \mathbb{E}\left[W_u^4\right] \\
&= u\left(t - u\right) + 3u^2 \\
&= ut + 2u^2
\end{aligned}$$

We conclude that, in general case of any $u > 0$ and $t > 0$

$$\mathbb{E}\left[W_u^2 W_t^2\right] = ut + 2\left(u \wedge t\right)^2$$

where $u \wedge t = \min\{u, t\}$. We can now easily compute the internal integral in (22) as

$$\begin{aligned}
&\int_0^D e^{\tilde{r}t}\mathbb{E}\left[W_u^2 W_t^2\right] dt \\
&\quad = \int_0^D e^{\tilde{r}t}\left[ut + 2\left(u \wedge t\right)^2\right] dt \\
&\quad = u\int_0^D te^{\tilde{r}t}dt + 2\left[\int_0^u t^2 e^{\tilde{r}t}dt + u^2\int_u^D e^{\tilde{r}t}dt\right] \\
&\quad = \frac{\tilde{r}u\left(e^{\tilde{r}D}\left(\tilde{r}\left(2u + D\right) - 1\right) - 4e^{\tilde{r}u} + 1\right) + 4\left(e^{\tilde{r}u} - 1\right)}{\tilde{r}^3}
\end{aligned}$$

Finally, computing the external integral in (22) yields

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{2D}\int_0^D e^{\tilde{r}u}W_u^2 du\right)^2\right] \\
&\quad = \frac{1}{4D^2}\frac{e^{2\tilde{r}D}\left(\tilde{r}D\left(3\tilde{r}D - 8\right) + 8\right) + 2e^{\tilde{r}D}\left(\tilde{r}D - 5\right) + 2}{\tilde{r}^4}
\end{aligned}$$

Other computations are very similar in character to those presented above, sequentially yielding m_1, ν_1, m_2 and ν_2.

6 Quadratic formula vs benchmark

The price estimates of floating strike options are obtained using Monte Carlo simulation with antithetic variables and control variate reduction techniques. Optimal parameter $\alpha^\star$ has been chosen so as to decrease the standard deviation of the price estimates (comparing with using $\alpha = 1$ in the control variate technique). The simulation is very precise as the highest standard deviation of the price estimates was equal to $\delta = 0.08\%$ of the estimated option price for volatility $\sigma = 1$.

Since our standard deviation is very small (due to the fact that we use two variance reduction techniques to reduce the standard deviation of price estimates), the approximation error is very likely larger than 4δ. Instead, we compute the percentage error (in comparison with the mean) of approximation value. For low and medium volatility cases (volatility less than 50%), the errors are within $\pm 2\%$ of the option price, as compared to an undervaluation ranging from -6.66% to -27.84% yielded by crude linear approximation formula within the same volatility domain. Our values approximate Asian floating strike option reasonably well as compared to typical bid-ask spread plus transaction cost of 0.5%. Comparison with benchmark is presented in Table 1.

7 Concluding remarks

In this paper we derived an analytical quadratic approximation formula for Asian options with floating strike. Our results extend those previously obtained by Bouaziz, Briys and Crouhy (1994).

Our technique illustrates the principle that adding additional Taylor series expansion terms can significantly increase the accuracy of an option pricing method, where the probability distribution of the underlying variable is unknown. Numerical comparison with benchmark values suggests that our approximation is accurate for low and medium volatility values up to 50%.

Future research could investigate the possibility of applying quadratic expansion techniques to derivative's pricing problems where the distribution of the underlying asset is derived from empirical observations rather than from theoretical assumptions, as in recent studies of risk-neutral densities.

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Vol.	Approximation Error [%]		Vol.	Approximation Error [%]	
σ	Quadratic	Linear	σ	Quadratic	Linear
0.02	0.000767858	−6.66153	0.32	1.3593	−20.0705
0.04	0.0906345	−7.21022	0.34	1.1062	−20.9645
0.06	0.453462	−8.00031	0.36	0.814977	−21.8511
0.08	0.883893	−8.89112	0.38	0.486226	−22.7299
0.1	1.25898	−9.82092	0.4	0.119586	−23.6012
0.12	1.55355	−10.7663	0.42	−0.285203	−24.4649
0.14	1.77181	−11.7163	0.44	−0.728197	−25.3208
0.16	1.92239	−12.666	0.46	−1.20981	−26.1693
0.18	2.01297	−13.6129	0.48	−1.73028	−27.0103
0.2	2.05028	−14.5553	0.5	−2.29001	−27.8438
0.22	2.03958	−15.4918	0.52	−2.88905	−28.6698
0.24	1.9847	−16.4217	0.54	−3.52786	−29.4886
0.26	1.88775	−17.3447	0.56	−4.20658	−30.3001
0.28	1.75032	−18.2608	0.58	−4.92482	−31.104
0.3	1.57422	−19.1692	0.6	−5.68298	−31.9007

Table 1: Comparison of relative errors [%] as compared to benchmark values (Monte-Carlo simulation with two variance reduction techniques) yielded by *quadratic* approximation formula and *linear* approximation formula as given in Bouaziz *et al.* (1994).

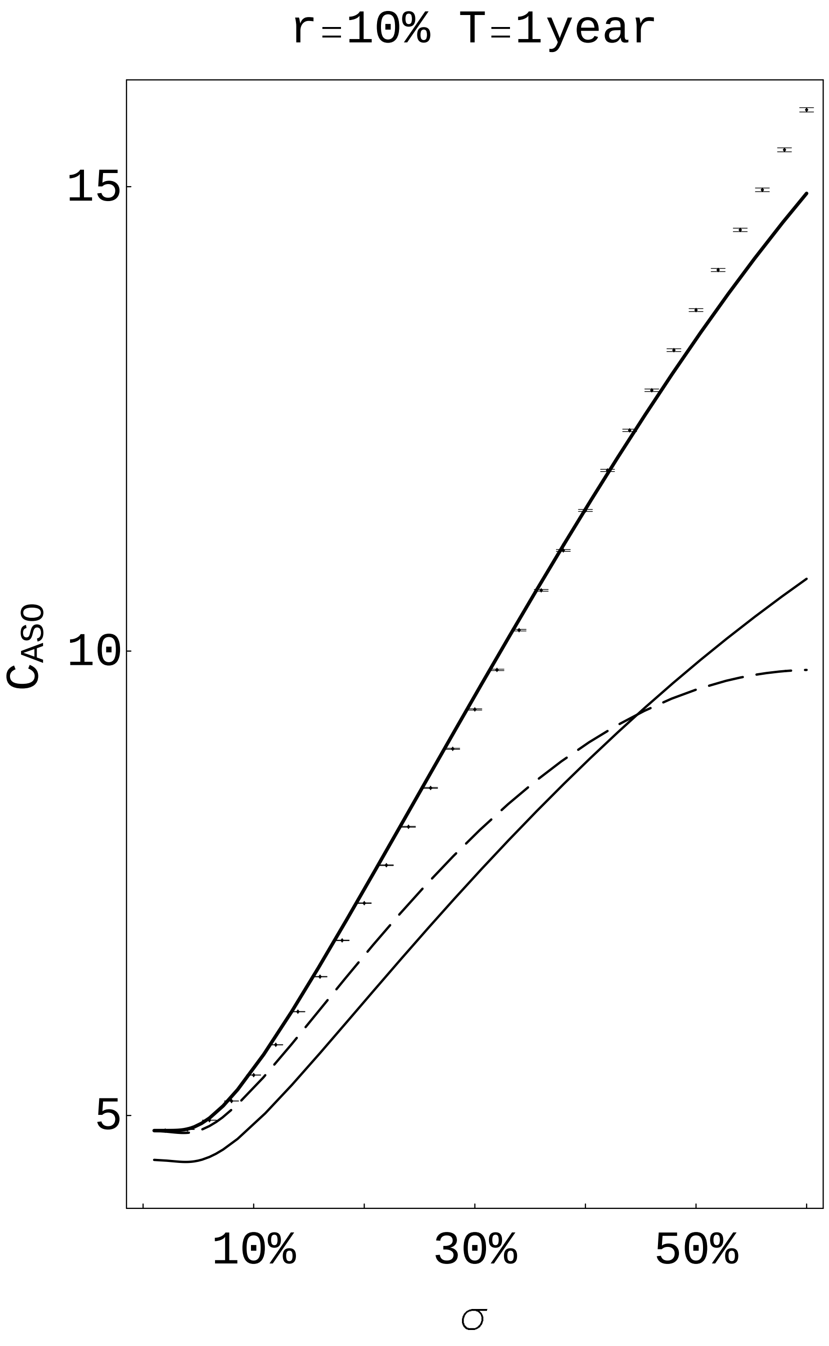

Figure 1: *Quadratic* approximation formula: bold line. *Linear* approximation: thin line. *Enhanced linear* approximation: thin dashed line. $S_0 = 100$, $T = 1$, $r = 10\%$.