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Increase in Risk and Weaker Marginal-Payoff-Weighted Risk Dominance

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Abstract

This study investigates the comparative statics of an increase in risk for all risk-averse individuals with positive prudence. By extending the concept of risk dominance defined by Gollier (1995), this study provides the necessary and sufficient conditions—termed the weaker marginal-payoff-weighted risk dominance—of unambiguous comparative statics for all risk-averse individuals with positive prudence. The article further applies the concept of weaker marginal-payoff-weighted risk dominance to examine the relationship between an increase in risk and the demand for proportional insurance and to demonstrate the difference between Gollier's condition of risk dominance and weaker marginal-payoff-weighted risk dominance.

Introduction

In the early 1970s, Rothschild and Stiglitz (1970, 1971) were the first to study whether a risk-averse individual demands less risky assets when facing an increase in risk. Since then, other researchers (Dreze and Modigliani, 1972; Diamond and Stiglitz, 1974; Dionne and Eeckhoudt, 1987; and Briys, Dionne, and Eeckhoudt, 1989) have found conditions on the utility functions that can generate unambiguous comparative statics with a mean preserving increase in risk. Still others (Eeckhoudt and Hansen, 1980, 1983; Meyer and Ormiston, 1983, 1985; Black and Bulkley, 1989; and Hadar and Seo, 1990, 1992) have found the constraints on the increase in risk that can provide clear prediction. Gollier (1995) found the necessary and sufficient condition, a marginal-payoff-weighted risk dominance, for unambiguous comparative statics of risk increases. His theorem is very powerful because it provides the necessary and sufficient condition and, moreover, holds for all risk-averse individuals.

However, Gollier's condition may not be easily met in some cases. To make the conditions more applicable, this article intends to extend Gollier's results for risk-averse individuals with positive prudence. This article provides the necessary and sufficient

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conditions—termed the weaker marginal-payoff-weighted risk dominance—for unambiguous comparative statics of an increase in risk for all risk-averse individuals with positive prudence. Compared to Gollier, since the conditions in this study apply to fewer individuals, they are weaker than Gollier's condition.

This article further applies the concept of weaker marginal-payoff-weighted risk dominance to examine the relationship between an increase in risk and the demand for proportional insurance—demonstrating for the latter case the difference between Gollier's condition and weaker marginal-payoff-weighted risk dominance.

Model

We follow the notation of Gollier (1995). Let α be a decision variable and $x \in [a,b]$ be a random variable having a probability density function $f(x)$. Payoff of the individual is denoted by $z(x,\alpha)$, where $z_x(x,\alpha) > 0$. Let u denote the utility of the individual and be a function of $z(x,\alpha)$. The individual chooses α to maximize his or her expected utility. The problem can be written as:

$$\operatorname*{Max}_{\alpha} H(\alpha;u,f,z) = \int_a^b u(z(x,\alpha))f(x)dx. \tag{1}$$

The first-order condition of the problem is:

$$H'(\alpha;u,f,z) = \int_a^b z_\alpha(x,\alpha)u'(z(x,\alpha))f(x)dx = 0. \tag{2}$$

After an integration by parts, Equation (2) can be rewritten as:

$$\begin{aligned} H'(\alpha;u,f,z) &= u'(z(b,\alpha))T(b,\alpha,f,z) \\ &\quad - \int_a^b z_x(x,\alpha)u''(z(x,\alpha))T(x,\alpha,f,z)dx = 0, \end{aligned} \tag{3}$$

where

$$T(x,\alpha,f,z) = \int_a^x z_\alpha(t,\alpha)f(t)dt. \tag{4}$$

A marginal-payoff-weighted risk dominance defined by Gollier (1995) can be expressed as:

There exists a real scalar γ *such that* $T(x,\alpha,g,z) \le \gamma T(x,\alpha,f,z), \forall x \in [a,b]$.

Gollier (1995) showed that the above risk dominance is the sufficient and necessary condition for all risk-averse individuals to reduce α after the distribution of x changes from $f(x)$ to $g(x)$.

After an integration by parts again, Equation (3) can be rewritten as:

$$\begin{aligned} H'(\alpha;u,f,z) &= u'(z(b,\alpha))T(b,\alpha,f,z) \\ &\quad - u''(z(b,\alpha))S(b,\alpha,f,z) \\ &\quad + \int_a^b z_x(x,\alpha)u'''(z(x,\alpha))S(x,\alpha,f,z)dx = 0, \end{aligned} \tag{5}$$

where

$$S(x,\alpha,f,z)=\int_a^x z_v(v,\alpha)T(v,\alpha,f,z)dv. \tag{6}$$

Assume that the individual's utility has the following characteristics: $u'(z(x,\alpha))>0$, $u''(z(x,\alpha))<0$, and $u'''(z(x,\alpha))>0$. Two reasons exist that explain why the individual in this study is assumed to be risk-averse with positive prudence. First, positive prudence, as defined by Kimball (1990), is known to be a necessary condition for decreasing absolute risk aversion, as defined by Pratt (1964), which is one of the widely accepted assumptions in the literature and highly correlated with the comparative statics of an increase in wealth (Mossin, 1968). Second, Kimball (1990), Eeckhoudt and Kimball (1992), and Eeckhoudt, Gollier, and Schlesinger (1996) have shown that positive prudence is essential for analyzing precautionary saving and the impact of an increase in background risk. Thus, it is worth analyzing specifically the behavior of risk-averse individuals with positive prudence with respect to an increase in risk.

Let us further define weaker marginal-payoff-weighted risk dominance as used in this study.

> Definition of weaker marginal-weighted risk dominance $g(x)$ . . . is called weaker marginal-payoff-weighted risk dominance on $f(x)$ if and only if there exists a real scalar γ such that $T(b,\alpha,g,z)\le\gamma T(b,\alpha,f,z)$ and $S(x,\alpha,g,z)\le\gamma S(x,\alpha,f,z),\forall x\in[a,b]$.

It is obvious that risk-averse individuals with positive prudence are a subset of risk-averse individuals. Moreover, risk dominance, as defined by Gollier (1995), is a sufficient condition of weaker marginal-payoff-weighted risk dominance.

Theorem 1:

> *After a change from $f(x)$ to $g(x)$, for all risk-averse individuals with positive prudence, α is reduced if and only if $g(x)$ is a weaker marginal-payoff-weighted risk dominated by $f(x)$.*

Proof of Theorem 1:

The sufficiency of Theorem 1 is shown in the following section and the necessity of Theorem 1, in the Appendix.

From Equation (5),

$$\begin{aligned}H'(\alpha_f;u,f,z)&=u'(z(b,\alpha_f))T(b,\alpha_f,f,z)\\&\quad-u''(z(b,\alpha_f))S(b,\alpha_f,f,z)\\&\quad+\int_a^b z_x(x,\alpha_f)u'''(z(x,\alpha_f))S(x,\alpha_f,f,z)dx=0,\end{aligned}$$

where α_f is the optimal solution of the above equation.

Thus,

$$H'(\alpha_f;u,g,z) = u'(z(b,\alpha_f))T(b,\alpha_f,g,z) \\ -u''(z(b,\alpha_f))S(b,\alpha_f,g,z) \\ +\int_a^b z_x(x,\alpha_f)u'''(z(x,\alpha_f))S(x,\alpha_f,g,z)dx.$$

Therefore,

$$\begin{aligned} &H'(\alpha_f;u,g,z) - \gamma H'(\alpha_f;u,f,z) \\ &\quad = u'(z(b,\alpha_f))[T(b,\alpha_f,g,z) - \gamma T(b,\alpha_f,f,z)] \\ &\quad\quad -u''(z(b,\alpha_f))[S(b,\alpha_f,g,z) - \gamma S(b,\alpha_f,f,z)] \\ &\quad\quad +\int_a^b z_x(x,\alpha_f)u'''(z(x,\alpha_f))[S(x,\alpha_f,g,z) - \gamma S(x,\alpha_f,f,z)]dx. \end{aligned} \tag{7}$$

It is obvious that if weaker marginal-payoff-weighted risk dominance is satisfied, then, for all risk-averse individuals with positive prudence,

$$H'(\alpha_f;u,g,z) - \gamma\ H'(\alpha_f;u,f,z) \le 0\ .$$

Because $H'(\alpha_f;u,f,z) = 0$,

$$H'(\alpha_f;u,g,z) \le 0\ .$$

Thus,

$$\alpha_g \le \alpha_f\ ,$$

where α_g is the optimal solution under distribution $g(x)$. Q. E. D.

An Application to Proportional Insurance

Proportional insurance is commonly used in primary property-liability insurance to control the problem of moral hazard. Over the past decade, the property-liability insurance industry has suffered a great fluctuation in loss distributions. Thus, the effect of increases in risks and the demand for proportional insurance have attracted a great deal of attention both in the industry and in the literature. In the following section, using proportional insurance as an example, the author demonstrates the application of weaker marginal-payoff-weighted risk dominance and analyzes the comparison between the condition defined by Gollier (1995)[1] and weaker marginal-payoff-weighted risk dominance.

[1] For convenience of comparison, the author will call the risk dominance defined by Gollier (1995) the marginal-payoff-weighted risk dominance.

Assume that a risk-averse insured with positive prudence has initial wealth W. The insured faces a random loss $y \in [0, L]$, where y follows a distribution $M(y)$. Let $M(y)$ and $m(y)$ denote the cumulated and probability density functions, respectively. Let P denote the premium for insurance and $\alpha \in [0,1]$ the coverage of proportional insurance. Assume that the insurance premium is proportional to the insurance, i.e., $P = \alpha\mu$.[2] It then follows that the final wealth of the insured can be expressed as $z(y, \alpha) = W - y + \alpha y - P$. It is very important to recognize that $Z_y(y, \alpha) = -1 + \alpha \leq 0$, which does not satisfy the assumption in this study. A simple variable transformation is employed to cope with this problem. Let $x = W - y$, which represents the net wealth of the insured after a loss. Thus, the final wealth of the insured with insurance can be rewritten as $z(x, \alpha) = x + \alpha(W - x) - P$.[3] Assume that the cumulated and probability density distributions of the net wealth are $F(x)$ and $f(x)$, respectively, which are transformed directly from loss distributions. Also assume that the insured chooses optimal proportional insurance to maximize his or her expected utility $\int_{W-L}^{W} u(z(x, \alpha)) f(x) dx$.

Let us recall Equation (4) and follow Gollier (1995). After an integration by parts, a distribution with cumulated density function $G(x)$ is satisfied for marginal-payoff-weighted risk dominance to $F(x)$ if and only if there exists a real scalar γ such that $\forall x \in [W - L, W]$,

$$(W - \mu - x)[G(x) - \gamma F(x)] + \int_{W-L}^{x} [G(t) - \gamma F(t)] dt \leq 0. \qquad (8)$$

If $\gamma = 1$, the first (second) term in Equation (8) is related to first-(second-)order stochastic dominance. However, Gollier showed that marginal-payoff-weighted risk dominance is in general not equivalent either to first-order stochastic dominance or to second-order stochastic dominance. An interesting finding in the case of proportional insurance is that marginal-payoff-weighted risk dominance with $\gamma = 1$ is obviously a necessary condition of first-order stochastic dominance, which transforms only portions of the distribution under $W - \mu$.[4] Thus, marginal-payoff-weighted risk dominance is, indeed, a necessary condition of first-order stochastic dominance, which transforms only portions of the distribution under $W - \mu$.

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2 The figure μ is usually assumed to be the sum of one plus expense loading ratio times the mean of the loss distribution. It is important to recognize that μ is assumed to be a constant; the shift of the loss distribution has no effect on μ. In reality, it may not be true if the insurer can also observe the shift of the loss distribution. The assumption is made to simplify the problem to generate a clearer comparison between marginal-payoff-weighted risk dominance and weaker marginal-payoff-weighted risk dominance. The details of this issue are discussed in Powers and Tzeng (1999).

3 The results can be extended to other cases in which the payoff is a linear function of the random variable, such as a portfolio problem.

4 This can be stated as $G(t) \leq F(t), \forall t \in [W - L, W - \mu]$ and $G(t) = F(t), \forall t \in [W - \mu, W]$. The risk transformation, which affects only a portion of the distribution, has also been studied by Fishburn and Porter (1976) and Eeckhoudt, Gollier, and Schlesinger (1991).

Let us recall the definition of weaker marginal-payoff-weighted risk dominance. By the same token, a distribution is called weaker marginal-payoff-weighted risk dominance for $F(x)$ if and only if there exists a real scalar γ such that

$$(-\mu)[1-\gamma]+\int_{W-L}^{W}[G(t)-\gamma F(t)]dt \le 0\,,$$

and

$$(W-\mu-x)[\Gamma_G(x)-\gamma\Gamma_F(x)]+2\int_{W-L}^{W}[\Gamma_G(t)-\gamma\Gamma_F(t)]dt \le 0\,,\ \forall x \in [W-L,W], \quad (9)$$

where $\Gamma_F(v)=\int_{W-L}^{v}F(t)dt$.[5]

Similar to Gollier's finding (1995), weaker marginal-payoff-weighted risk dominance is generally neither second-order nor third-order stochastic dominance, although the first (second) term in Equation (9) with $\gamma = 1$ is related to second- (third-)order stochastic dominance. It is easy to find that weaker marginal-payoff-weighted risk dominance is a necessary condition of second-order stochastic dominance, which transforms only portions of the distribution under $W-\mu$.[6] It is worth noting that first-order stochastic dominance, which transforms only portions of the distribution under $W-\mu$, implies both marginal-payoff-weighted risk dominance and second-order stochastic dominance, which transforms only portions of the distribution under $W-\mu$. However, marginal-payoff-weighted risk dominance and second-order stochastic dominance, which transforms only portions of the distribution under $W-\mu$, do not imply each other. Furthermore, weaker marginal-payoff-weighted risk dominance is implied by both first- and second-order stochastic dominance, which transforms only portions of the distribution under $W-\mu$. Thus, an increase in risk with second-order stochastic dominance, which transforms only portions of the distribution under $W-\mu$, satisfies weaker marginal-payoff-weighted risk dominance but may not satisfy marginal-payoff-weighted risk dominance. Therefore, we can conclude that an increase in risk with second-order stochastic dominance, which transforms only portions of the distribution under $W-\mu$, makes all risk-averse individuals with positive prudence demand more proportional insurance, but it may not make all risk-averse individuals demand more.

CONCLUSION

This study provides the necessary and sufficient conditions—termed weaker marginal-payoff-weighted risk dominance—of unambiguous comparative statics for all risk-averse individuals with positive prudence. Compared to Gollier (1995), since the conditions in this study apply to a subset of risk-averse individuals, weaker marginal-payoff-weighted risk dominance is shown to be a necessary condition of Gollier's condition on risk dominance and is, therefore, easier to meet. This article further uses

[5] Third-order stochastic dominance, as defined by Whitmore (1970), is shown to be the necessary and sufficient condition for all risk-averse individuals with positive prudence to prefer one risk over another.

[6] This can be stated as
$\Gamma_G(t) \le \Gamma_F(t), \forall t \in [W-L, W-\mu]$ and $\Gamma_G(t) = \Gamma_F(t), \forall t \in [W-\mu, W]$.

the demand for proportional insurance to demonstrate the application of weaker marginal-payoff-weighted risk dominance. In the case of proportional insurance, the author finds that Gollier's condition on risk dominance and weaker marginal-payoff-weighted risk dominance are, respectively, necessary conditions for first-order stochastic dominance and second-order stochastic dominance, which transforms only portions of the distribution under $W-\mu$. Moreover, the author finds that second-order stochastic dominance, which transforms only portions of the distribution under $W-\mu$, implies weaker marginal-payoff-weighted risk dominance but may not imply Gollier's condition on risk dominance.

APPENDIX

Because a linear transformation of an individual's utility function has no effect on the nature of the problem, the problem can be simplified by assuming that $u'(z(b,\alpha_f))=0$. Thus, Equation (7) can be rewritten as

$$\begin{aligned} &H'(\alpha_f;u,g,z)-\gamma\ H'(\alpha_f;u,f,z)\\ &\quad=-u''(z(b,\alpha_f))[S(b,\alpha_f,g,z)-\gamma\ S(b,\alpha_f,f,z)]\\ &\quad+\int_a^b z_x(x,\alpha_f)u'''(z(x,\alpha_f))[S(x,\alpha_f,g,z)-\gamma S(x,\alpha_f,f,z)]dx. \end{aligned} \tag{10}$$

Therefore, proving the necessity of Theorem 1 is the same as proving the following statement:

> If $\forall\gamma$ such that $\exists x_0\in[a,b]\ni S(x_0,\alpha,g,z)>\gamma\ S(x_0,\alpha,f,z)$, then there exists a decisionmaker with $u''(\bullet)<0$, and $u'''(\bullet)>0$ who increases α.

Let us partition $[a,b]$ into three parts, Ω_0, Ω_-, and Ω_+, such that $S(x,\alpha,g,z)>(=,<)\gamma S(x,\alpha,f,z)$, for all $x\in\Omega_+(\Omega_0,\Omega_-)$, respectively. Without losing any generality, assume that $b\in\Omega_-$. Let us consider that the utility function of the individual can be expressed as:

$$u''(z(x,\alpha))=-\varepsilon_1+\theta z(x,\alpha)<0 \text{ for } x\in\Omega_- \text{ or } x\in\Omega_0,$$

$$u''(z(x,\alpha))=-\varepsilon_2+\kappa z(x,\alpha)<0 \text{ for } x\in\Omega_+.$$

Thus,

$$u'''(z(x,\alpha))=\theta>0 \text{ for } x\in\Omega_- \text{ or } x\in\Omega_0,$$

$$u'''(z(x,\alpha))=\kappa>0 \text{ for } x\in\Omega_+.$$

Therefore, Equation (10) can be rewritten as

$$\begin{aligned} &H'(\alpha_f;u,g,z)-\gamma\ H'(\alpha_f;u,f,z)\\ &\quad=(\varepsilon_1-\theta z(x,b))[S(b,\alpha_f,g,z)-\gamma\ S(b,\alpha_f,f,z)]\\ &\quad+\kappa\int_{\Omega_-} z_x(x,\alpha_f)[S(x,\alpha_f,g,z)-\gamma\ S(x,\alpha_f,f,z)]dx\\ &\quad+\theta\int_{\Omega_+} z_x(x,\alpha_f)[S(x,\alpha_f,g,z)-\gamma\ S(x,\alpha_f,f,z)]dx. \end{aligned} \tag{11}$$

In this equation, the first and the third terms are positive, and the second term is negative. Moreover, $\kappa > 0$ and is related only to ε_2. By holding ε_1 and θ constant and making κ sufficiently large, Equation (11) becomes positive. Thus, the individual increases α. Q. E. D.

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