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Rigid body concept for geometric nonlinear analysis of 3D frames, plates and shells based on the updated Lagrangian formulation

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Abstract

In the nonlinear analysis of elastic structures, the displacement increments generated at each incremental step can be decomposed into two components as the *rigid displacements* and *natural deformations*. Based on the updated Lagrangian (UL) formulation, the geometric stiffness matrix $[k_g]$ is derived for a 3D rigid beam element from the virtual work equation using a rigid displacement field. Further, by treating the three-node triangular plate element (TPE) as the composition of three rigid beams lying along the three sides, the $[k_g]$ matrix for the TPE can be assembled from those of the rigid beams. The idea for the UL-type incremental-iterative nonlinear analysis is that if the rigid rotation effects are fully taken into account at each stage of analysis, then the remaining effects of natural deformations can be treated using the small-deformation linearized theory. The present approach is featured by the fact that the formulation is *simple*, the expressions are *explicit*, and *all kinds* of actions are considered in the stiffness matrices. The robustness of the procedure is demonstrated in the solution of several benchmark problems involving the postbuckling response.

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1. Introduction

The nonlinear analysis of elastic structures is usually conducted in an incremental-iterative way based on the three configurations: the initial configuration C_0 , last calculated configuration C_1 , and current deformed configuration C_2 , as indicated in Fig. 1. In a step-by-step nonlinear analysis, we are interested in the behavior of the structure during the incremental step from C_1 to C_2 . The deformations occurring within each incremental step are assumed to be *small*, but the displacements accumulated for all incremental steps can be very large. The concept to be presented herein is based on the *updated Lagrangian* (UL) formulation, in that all quantities of the structure are expressed with reference to the last configuration C_1 .

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The displacement increments generated at each incremental step of an elastic nonlinear analysis can be composed into two components as the *rigid displacements* and *natural deformations* [1,2]. For most structures encountered in practice, the rigid component constitutes a much larger portion of the displacement increments at each incremental step with respect to the deformational component. For a UL-type incremental-iterative analysis, the idea is that if the rigid rotation effects for elements with *initial forces* (or stresses) are fully taken into account at each stage of analysis, then the remaining effects of natural deformations can be treated using the small-deformation linearized theory.

Concerning the incremental-iterative procedure, distinction should be made between the *predictor* and *corrector* stages [3,4]. The predictor relates to solution of the displacement increments $\{U\}$ of the structure for given load increments $\{P\}$ based on the structural equation $[K]\{U\} = \{P\}$. This stage determines the *trial* direction of iteration of the structure in the load-deflection space and

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Fig. 1. Motion of body in three-dimensional space.

thus affects the number of iterations or speed of convergence. For this reason, the stiffness matrix [K] used in the structural equation need *not* be exact, but should be kept *rigid-body qualified* to avoid convergence to incorrect directions. In the UL formulation, the corrector refers to recovery of the force increments $\{2f\}$ at C_2 from the displacement increments $\{u\}$ made available through the structural displacement increments $\{U\}$, and the superimposition of these force increments with the initial nodal forces $\{{}_1^1f\}$ following the rigid body rule [5,6] for obtaining the total element forces $\{{}_2^2f\}$ at C_2 .

In this paper, a rigid-body qualified geometric stiffness matrix $[k_g]$ will be derived for the 3D beam element from the virtual work equation by assuming the displacement field to be of the rigid type. Such an element is referred to as the *rigid element*. To the knowledge of the authors, no similar elements were presented by other scholars to explicitly accommodate the rigid behaviors of structures. For the 3D beam, the initial surface tractions may generate some moment terms upon 3D rotations during the incremental step from C_1 to C_2 , commonly known as the moments induced by the semitangential torques and quasi-tangential bending moments [1,7]. Naturally, all such terms should be included in the virtual work formulation for the 3D beam. The other issue to be considered for the space frames is the equilibrium of angled joints in the rotated configuration C_2 , rather than in C_1 . Based on such a consideration, only the symmetric part of the geometric stiffness matrix of each element has to be retained in the structural stiffness matrix, as the antisymmetric parts of all the elements meeting at the same joint cancel out with each other [6,7].

As for the analysis of plate/shell problems, a *triangular plate element* (TPE) with three translational and three rotational degrees of freedom (DOFs) at each of the three tip nodes will be considered, for its compatibility with the 12-DOF beam element derived above. Since the *rigid body behavior* of each finite element is solely determined by its external shape or nodal DOFs, the geometric stiffness matrix for the TPE is derived by treating the TPE as the composition of three *rigid beams* lying along the three sides. The geometric stiffness matrix so derived is explicit and capable of dealing with all kinds of in-plane and outplane actions.

For a review of related works on geometric nonlinear analysis of structures, Ref. [8] may be consulted, in which a total of 122 papers were cited. The purpose of this paper is not to review any related works. Rather, efforts will be focused on application of the rigid body concept and derivation of rigid-body qualified geometric stiffness matrices $[k_g]$ for the 3D beam element and TPE. The elastic stiffness matrices $[k_e]$ adopted are those readily available in the literature, namely, the elastic stiffness matrix $[k_e]$ adopted for the 3D beam element is the one commonly used [6.9], and the elastic stiffness matrix $[k_e]$ adopted for the TPE is constructed as the composition of Cook's plane hybrid element for membrane actions [10] and the hybrid stress model (HSM) of Batoz et al. for bending actions [11]. For the sake of brevity, repetition of relevant derivations is kept to the minimum.

2. Theory of three-dimensional beams

Before we proceed to derive the rigid element for the 3D beam, a summary of the theory for the 3D beam with bisymmetric solid cross-sections is first given. The beam element considered has a total of 12 DOFs as shown in Fig. 2, with x denoting the centroidal axis and (y, z) the two principal axes of the cross-section. Based on the UL formulation, the virtual work equation for a 3D beam at C_2 , but with reference to C_1 , can be expressed in a linearized form as [6]:

$$\int_{V} (E_{1}e_{xx}\delta_{1}e_{xx} + 4G_{1}e_{xy}\delta_{1}e_{xy} + 4G_{1}e_{xz}\delta_{1}e_{xz}) dV + \int_{V} ({}^{1}\tau_{xx}\delta_{1}\eta_{xx} + {}^{2}{}^{1}\tau_{xy}\delta_{1}\eta_{xy} + {}^{2}{}^{1}\tau_{xz}\delta_{1}\eta_{xz}) dV = {}^{2}_{1}R - {}^{1}_{1}R,$$
(1)

where *E* and *G* denote the elastic and shear modulus, respectively, *V* is the volume of the element, and the factors of 4 and 2 are added to account for the symmetry of shear strains, i.e., $_{1}e_{xy} = _{1}e_{yx}$, $_{1}e_{xz} = _{1}e_{zx}$, $_{1}\eta_{xy} = _{1}\eta_{yx}$, $_{1}\eta_{xz} = _{1}\eta_{zx}$, $(^{1}\tau_{xx}, ^{1}\tau_{xy}, ^{1}\tau_{xz})$ are the initial (Cauchy) axial and shear stresses, and δ denotes the variation of the quantity following. The linear and nonlinear components of the strain increments can be expressed with reference to C_{1} as



Fig. 2. Three-dimensional beam element.

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$${}_{1}e_{xx} = \frac{\partial u_{x}}{\partial x}, \quad {}_{1}e_{xy} = \frac{1}{2}\left(\frac{\partial u_{x}}{\partial y} + \frac{\partial u_{y}}{\partial x}\right), \quad {}_{1}e_{xz} = \frac{1}{2}\left(\frac{\partial u_{x}}{\partial z} + \frac{\partial u_{z}}{\partial x}\right), \quad (2)$$

$${}_{1}\eta_{xx} = \frac{1}{2} \left[\left(\frac{\partial u_{x}}{\partial x} \right)^{2} + \left(\frac{\partial u_{y}}{\partial x} \right)^{2} + \left(\frac{\partial u_{z}}{\partial x} \right)^{2} \right],$$

$${}_{1}\eta_{xy} = \frac{1}{2} \left[\left(\frac{\partial u_{x}}{\partial y} \right) \left(\frac{\partial u_{x}}{\partial x} \right) + \left(\frac{\partial u_{y}}{\partial y} \right) \left(\frac{\partial u_{y}}{\partial x} \right) + \left(\frac{\partial u_{z}}{\partial y} \right) \left(\frac{\partial u_{z}}{\partial x} \right) \right],$$

$${}_{1}\eta_{xz} = \frac{1}{2} \left[\left(\frac{\partial u_{x}}{\partial z} \right) \left(\frac{\partial u_{x}}{\partial x} \right) + \left(\frac{\partial u_{y}}{\partial z} \right) \left(\frac{\partial u_{y}}{\partial x} \right) + \left(\frac{\partial u_{z}}{\partial z} \right) \left(\frac{\partial u_{z}}{\partial x} \right) \right],$$
(3)

where (u_x, u_y, u_z) are the displacements of a generic point N(y, z) at cross-section x of the beam. By allowing the surface tractions ${}_{1}^{1}t_k$ and ${}_{1}^{2}t_k$ to exist only at the two end sections, the external virtual works ${}_{1}^{1}R$ and ${}_{1}^{2}R$ at C_1 and C_2 , respectively, can be written as

$${}_{1}^{1}R = \int_{S} \left({}_{1}^{1}t_{x}\delta u_{x} + {}_{1}^{1}t_{y}\delta u_{y} + {}_{1}^{1}t_{z}\delta u_{z} \right) dS,$$

$${}_{1}^{2}R = \int_{S} \left({}_{1}^{2}t_{x}\delta u_{x} + {}_{1}^{2}t_{y}\delta u_{y} + {}_{1}^{2}t_{z}\delta u_{z} \right) dS,$$
(4a, b)

where S is the surface area of the element at C_1 .

Based on the Bernoulli–Euler hypothesis of plane sections remaining plane and normal to the centroidal axis after deformation, the displacements (u_x, u_y, u_z) of the generic point N can be related to the displacements (u, v, w) of the centroid of the same cross-section as

$$u_x = u - yv' - zw', \quad u_y = v - z\theta_x, \quad u_z = w + y\theta_x, \tag{5}$$

where θ_x is the angle of twist. For the 3D beam, the displacement increments $\{u\}$ are

$$\{u\} = \langle u_a \ v_a \ w_a \ \theta_{xa} \ \theta_{ya} \ \theta_{za} \ u_b \ v_b \ w_b \ \theta_{xb} \ \theta_{yb} \ \theta_{zb} \rangle^{\mathrm{T}}.$$
(6)

Correspondingly, the nodal forces $\{{}_{1}^{1}f\}$ and $\{{}_{1}^{2}f\}$ acting on the element at C_{1} and C_{2} are

$$\{ {}_{1}^{1}f \} = \begin{cases} {}_{1}^{1}F_{xa} & {}^{1}F_{ya} & {}^{1}F_{za} & {}^{1}M_{xa} & {}^{1}M_{ya} & {}^{1}M_{za} \\ {}_{1}F_{xb} & {}^{1}F_{yb} & {}^{1}F_{zb} & {}^{1}M_{xb} & {}^{1}M_{yb} & {}^{1}M_{zb} \rangle^{\mathrm{T}}, \end{cases}$$
(7)

$$\{{}_{1}^{2}f\} = \frac{\langle {}^{2}F_{xa} \ {}^{2}F_{ya} \ {}^{2}F_{za} \ {}^{2}M_{xa} \ {}^{2}M_{ya} \ {}^{2}M_{za}}{{}^{2}F_{xb} \ {}^{2}F_{yb} \ {}^{2}F_{zb} \ {}^{2}M_{xb} \ {}^{2}M_{yb} \ {}^{2}M_{zb} \rangle^{\mathrm{T}},$$
(8)

where the left subscript "1" is omitted from each term on the right hand side, i.e., ${}^{1}F_{xa} \equiv {}^{1}_{1}F_{xa}$, ${}^{1}M_{xa} \equiv {}^{1}_{1}M_{xa}$, ..., ${}^{2}F_{xa} \equiv {}^{2}_{1}F_{xa}$, ${}^{2}M_{xa} \equiv {}^{2}_{1}M_{xa}$, ..., etc. All the nodal forces and moments should be interpreted as the stress resultants, as will be given below.

2.1. Stress resultant definitions

Let us cut a beam at section x and consider its left portion as in Fig. 3a. Also, let us attach two sets of coordinates to the centroid C of the cross-section. The first set is the $\eta-\zeta$ coordinates embedded at the cross-section, with η and ζ denoting the two principal axes. The second set is assumed to have an origin fixed at the centroid C, referred to as the ${}^1\bar{x}-{}^1\bar{y}-{}^1\bar{z}$ axes at C_1 , with $({}^1\bar{y}, {}^1\bar{z})$ coincident with (η, ζ) (see Fig. 3a), and as the ${}^2\bar{x}-{}^2\bar{y}-{}^2\bar{z}$ axes at C_2 , with $({}^2\bar{y}, {}^2\bar{z})$ parallel to $({}^1\bar{y}$ and ${}^1\bar{z})$ (see Fig. 3b).

Based on the conditions of equilibrium, the forces acting on section x at C_k , with k = 1, 2, can be interpreted as stress resultants over the cross-section of area A [6,7]:

$${}^{k}F_{x} = \int_{A}{}^{k}S_{xx} dA, \quad {}^{k}F_{y} = \int_{A}{}^{k}S_{xy} dA, \quad {}^{k}F_{z} = \int_{A}{}^{k}S_{xz} dA,$$
(9)
$${}^{k}M_{x} = \int_{A}{}^{k}({}^{k}S_{xz} \cdot {}^{k}\bar{y} - {}^{k}S_{xy} \cdot {}^{k}\bar{z}) dA,$$

$${}^{k}M_{y} = \int_{A}{}^{k}({}^{k}S_{xx} \cdot {}^{k}\bar{z} - {}^{k}S_{xz} \cdot {}^{k}\bar{x}) dA,$$
(10)
$${}^{k}M_{z} = \int_{A}{}^{k}({}^{k}S_{xy} \cdot {}^{k}\bar{x} - {}^{k}S_{xx} \cdot {}^{k}\bar{y}) dA,$$

where $\binom{k}{1}S_{xx}, \binom{k}{1}S_{xy}, \binom{k}{1}S_{xz}$ are the second Piola–Kirchhoff stresses acting at C_k with reference to C_1 . For the element at C_1 ,



Fig. 3. Coordinates of a generic point N at section x at (a) C_1 , (b) C_2 .

these stresses reduce to the Cauchy stresses $({}^{1}\tau_{xx}, {}^{1}\tau_{xy}, {}^{1}\tau_{xz})$, and the embedded coordinates (η, ζ) coincide with the $({}^{1}\bar{\nu}, {}^{1}\bar{z})$ axes, i.e.,

$${}^{1}\bar{x} = 0, \quad {}^{1}\bar{y} \equiv \eta = y, \quad {}^{1}\bar{z} \equiv \zeta = z.$$
(11)

It follows that the forces and moments in Eqs. (9) and (10) reduce to those commonly known for the beam at C_1 . In particular, the moments acting at C_1 are

$${}^{1}M_{x} = \int_{A} ({}^{1}\tau_{xz}y - {}^{1}\tau_{xy}z) dA,$$

$${}^{1}M_{y} = \int_{A} {}^{1}\tau_{xx}z dA, \quad {}^{1}M_{z} = -\int_{A} {}^{1}\tau_{xx}y dA.$$
(12)

For the element to deform from C_1 to C_2 , the displacements (u_x, u_y, u_z) of the generic point N at C_1 were given in Eq. (5). Consequently, the coordinates of point N at C_2 are (Fig. 3b)

$${}^{2}\bar{x} = -y\theta_{z} + z\theta_{y}, \quad {}^{2}\bar{y} = y - z\theta_{x}, \quad {}^{2}\bar{z} = z + y\theta_{x}.$$
(13)

Meanwhile, the second Piola-Kirchhoff stresses can be expressed in an incremental form,

$${}_{1}^{2}S_{i} = {}^{1}\tau_{i} + S_{i} \tag{14}$$

for i = xx, xy, xz, where S_i are the stress increments. Since the meanings for the forces $({}^2F_x, {}^2F_y, {}^2F_z)$ at C_2 are selfexplanatory, we shall focus on derivation of the moments $({}^2M_x, {}^2M_y, {}^2M_z)$ at C_2 . By substituting Eqs. (13) and (14) into Eq. (10), along with the stress resultant definitions in Eq. (12), the following can be obtained for the moments at C_2 :

$${}^{2}M_{x} = \int_{A} ({}^{2}_{1}S_{xz} \cdot y - {}^{2}_{1}S_{xy} \cdot z) \, \mathrm{d}A,$$

$${}^{2}M_{y} = \int_{A} {}^{2}_{1}S_{xx} \cdot z \, \mathrm{d}A - {}^{1}M_{z}\theta_{x} + \frac{1}{2}{}^{1}M_{x}\theta_{z},$$

$${}^{2}M_{z} = -\int_{A} {}^{2}_{1}S_{xx} \cdot y \, \mathrm{d}A + {}^{1}M_{y}\theta_{x} - \frac{1}{2}{}^{1}M_{x}\theta_{y},$$
(15)

where products of displacement and stress increments are neglected and the factor 1/2 is adopted for bisymmetric solid cross-sections. The expressions in Eq. (15) are derived based merely on the hypothesis of planar sections in Eq. (5) and the stress resultant definitions in Eq. (10). Evidently, the torque ${}^{1}M_{x}$ should be interpreted as the *semitangential moment*, as indicated by the terms $1/2{}^{1}M_{x}\theta_{z}$ and $-1/2{}^{1}M_{x}\theta_{y}$, and the bending moments ${}^{1}M_{y}$ and ${}^{1}M_{z}$ as the *quasi-tangential moments*, as indicated by the terms ${}^{1}M_{y} \theta_{x}$ and $-{}^{1}M_{z}\theta_{x}$ [1,6,7,12].

2.2. Strain energy term

With the linear strain components in Eq. (2) and the displacements in Eq. (5), one can obtain the strain energy δU in variational form from the first integral of Eq. (1) as follows [6]:

$$\delta U = \int_{V} (E_1 e_{xx} \delta_1 e_{xx} + 4G_1 e_{xy} \delta_1 e_{xy} + 4G_1 e_{xz} \delta_1 e_{xz}) dV$$

=
$$\int_{0}^{L} (EAu' \delta u' + EI_y w'' \delta w'' + EI_z v'' \delta v'' + GJ \theta'_x \delta \theta'_x) dx,$$

(16)

where A is the cross-sectional area, L is the length, I_y and I_z are the moments of inertia, and J is the torsional constant. From the strain energy term δU , a 12 × 12 elastic matrix $[k_e]$ can be derived:

$$\delta U = \{\delta u\}^{\mathrm{T}}[k_{\mathrm{e}}]\{u\} \tag{17}$$

as available elsewhere [6,9]. However, for the case of rigid displacements to be considered in the following, the strain energy term δU simply vanishes.

2.3. Virtual work of initial stresses

By substituting the nonlinear strain components in Eq. (3) into the second integral of Eq. (1), along with the displacements in Eq. (5) and the definitions for forces in Eq. (9) (for k = 1) and moments in Eq. (12), the virtual work δV of the initial stresses can be derived as [6]

$$\begin{split} \delta V &= \int_{V} ({}^{1}\tau_{xx}\delta_{1}\eta_{xx} + 2{}^{1}\tau_{xy}\delta_{1}\eta_{xy} + 2{}^{1}\tau_{xz}\delta_{1}\eta_{xz}) \, dV \\ &= \frac{1}{2} \int_{0}^{L} \left[{}^{1}F_{x}\delta(u'^{2} + v'^{2} + w'^{2}) + {}^{1}F_{x} \left(\frac{I_{y}}{A}\delta w''^{2} + \frac{I_{z}}{A}\delta v''^{2} \right) + {}^{1}F_{x} \frac{I_{y} + I_{z}}{A}\delta \theta_{x}'^{2} \right] dx \\ &+ \int_{0}^{L} [-{}^{1}M_{z}\delta(w'\theta_{x}') - {}^{1}M_{y}\delta(v'\theta_{x}') - {}^{1}M_{y}\delta(u'w'') + {}^{1}M_{z}\delta(u'v'')] \, dx \\ &+ \int_{0}^{L} [{}^{1}F_{y}\delta(w'\theta_{x} - u'v') - {}^{1}F_{z}\delta(v'\theta_{x} + u'w')] \, dx \\ &+ \frac{1}{2} \int_{0}^{L} [{}^{1}M_{x}\delta(v''w') - {}^{1}M_{x}\delta(w'v')] \, dx, \end{split}$$
(18)

where higher order terms have been neglected.

2.4. External virtual work terms

By treating the surface tractions at C_1 as Cauchy stresses, i.e., ${}_1^1t_x \equiv {}^1\tau_{xx}$, ${}_1^1t_y \equiv {}^1\tau_{xy}$, ${}_1^1t_z \equiv {}^1\tau_{xz}$, and by using the displacements in Eq. (5) and the definitions for forces in Eq. (9) (for k = 1) and moments in Eq. (12), the external virtual work ${}_1^1R$ at C_1 , Eq. (4a), can be written as

$$R = \{\delta u\}^{\mathrm{T}}\{{}_{1}^{1}f\},\tag{19}$$

where the displacements $\{u\}$ and initial forces $\{{}_{1}^{1}f\}$ were defined in Eqs. (6) and (7).

Similarly, by treating the surface tractions at C_2 as the second Piola–Kirchhoff stresses, i.e., ${}_{1}^{2}t_{x} \equiv {}_{1}^{2}S_{xx}$, ${}_{1}^{2}t_{y} \equiv {}_{1}^{2}S_{xy}$, ${}_{1}^{2}t_{z} \equiv {}_{1}^{2}S_{xz}$, expressing the stresses in incremental form as in Eq. (14), and using the displacement field in Eq. (5) and the definitions for forces in Eq. (9) (for k = 2) and moments in Eq. (12), one derives from Eq. (4b) the external virtual work ${}_{1}^{2}R$ at C_{2} as

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$${}^{2}_{1}R = \{\delta u\}^{\mathrm{T}}\{{}^{2}_{1}f\} + \left[\left({}^{1}M_{z}\theta_{x} - \frac{1}{2}{}^{1}M_{x}\theta_{z}\right)\delta\theta_{y} + \left(-{}^{1}M_{y}\theta_{x} + \frac{1}{2}{}^{1}M_{x}\theta_{y}\right)\delta\theta_{z}\right]_{0}^{L},$$
(20)

where $\{{}_{1}^{2}f\}$ was defined in Eq. (8). Clearly, the *semi*- and *quasi-tangential* properties of the torque ${}^{1}M_{x}$ and bending moments ${}^{1}M_{y}$, ${}^{1}M_{z}$, respectively, are maintained in this expression.

3. Geometric stiffness matrix for the rigid beam element

Now, we shall derive the geometric stiffness matrix for the 3D rigid beam. For an element subjected only to nodal loads, the bending moments can be related to those at the two ends as

$${}^{1}M_{y} = -{}^{1}M_{ya}(1 - x/L) + {}^{1}M_{yb}(x/L),$$

$${}^{1}M_{z} = -{}^{1}M_{za}(1 - x/L) + {}^{1}M_{zb}(x/L).$$
 (21)

Further, based on the conditions of equilibrium, the following can be written:

$${}^{1}F_{x} = -{}^{1}F_{xa} = {}^{1}F_{xb}, \quad {}^{1}F_{y} = -{}^{1}F_{ya} = {}^{1}F_{yb} = -\frac{1}{L}({}^{1}M_{za} + {}^{1}M_{zb}),$$
(22)

$${}^{1}F_{z} = -{}^{1}F_{za} = {}^{1}F_{zb} = \frac{1}{L} ({}^{1}M_{ya} + {}^{1}M_{yb}), \quad {}^{1}M_{x} = -{}^{1}M_{xa} = {}^{1}M_{xb}.$$
(23)

For a *rigid rotation*, the axial and angular displacements can be expressed as

$$u = (1 - x/L)u_a + (x/L)u_b, \quad \theta_x = (1 - x/L)\theta_{xa} + (x/L)\theta_{xb}$$
(24)

subject to the constraints:

$$u_a = u_b, \quad \theta_{xa} = \theta_{xb}.$$
 (25a, b)

The lateral and rotational displacements can be expressed as

$$v = (1 - x/L)v_a + (x/L)v_b, \quad w = (1 - x/L)w_a + (x/L)w_b,$$
(26)

$$\theta_y = (1 - x/L)\theta_{ya} + (x/L)\theta_{yb}, \quad \theta_z = (1 - x/L)\theta_{za} + (x/L)\theta_{zb}$$
(27)

subject to the constraints:

$$\theta_{ya} = \theta_{yb}, \quad \theta_{za} = \theta_{zb}.$$
 (28)

For the rigid beam, the strain energy vanishes, i.e., $\delta U = 0$. Consequently, the virtual work equation in Eq. (1), along with the aid of Eqs. (16), (18)–(20), reduces to the following:

$$\frac{1}{2} \int_{0}^{L} \left[{}^{1}F_{x}\delta(u'^{2} + v'^{2} + w'^{2}) + {}^{1}F_{x}\left(\frac{I_{y}}{A}\delta w''^{2} + \frac{I_{z}}{A}\delta v''^{2}\right) + {}^{1}F_{x}\frac{I_{y} + I_{z}}{A}\delta \theta_{x}'^{2}} \right] dx$$

$$+ \int_{0}^{L} \left[{}^{-1}M_{z}\delta(w'\theta_{x}') - {}^{1}M_{y}\delta(v'\theta_{x}') - {}^{1}M_{y}\delta(u'w'') + {}^{1}M_{z}\delta(u'v'') \right] dx$$

$$+ \int_{0}^{L} \left[{}^{1}F_{y}\delta(w'\theta_{x} - u'v') - {}^{1}F_{z}\delta(v'\theta_{x} + u'w') \right] dx$$

$$+ \frac{1}{2} \int_{0}^{L} \left[{}^{1}M_{x}\delta(v''w') - {}^{1}M_{x}\delta(w''v') \right] dx$$

$$= \left\{ \delta u \right\}^{T} \left\{ {}^{2}_{1}f \right\} - \left\{ {}^{1}_{1}f \right\} \right)$$

$$+ \left[\left({}^{1}M_{z}\theta_{x} - \frac{1}{2}{}^{1}M_{x}\theta_{z} \right) \delta\theta_{y} + \left({}^{-1}M_{y}\theta_{x} + \frac{1}{2}{}^{1}M_{x}\theta_{y} \right) \delta\theta_{z} \right]_{0}^{L}.$$
(29)

Since this is a problem with constraints, the following points must be considered in deriving each of the terms involved in Eq. (29) using the rigid displacement field: (1) The following variational terms are equal to zero: $\delta u^{\prime 2} =$ 0, $\delta \theta_x'^2 = 0$, $\delta v''^2 = 0$, $\delta w''^2 = 0$, $\delta (u'w'') = 0$, $\delta (u'v'') = 0$, as a direct consequence of the rigid displacement field. (2) The variation of a quantity that is zero in the rigid displacement field need not be zero. For instance, $u' = (u_b - u_a)/(u_b - u_a)/(u_a$ L = 0 according to Eq. (25a), but $\delta u' = (\delta u_b - \delta u_a)/L$ is not equal to zero since $\delta u_b \neq \delta u_a$. (3) Even though some of the variational terms in Eq. (29) are known to be zero, such as $u' \,\delta v'$, $u' \,\delta w'$, $\theta'_{u} \,\delta v'$, $\theta'_{u} \,\delta w'$, they are kept in the derivation for the sake of symmetry of the related entries in the stiffness matrix. (4) The second derivative terms v''and w'' should be treated as θ'_{z} and $-\theta'_{y}$, respectively, and interpolated by Eq. (27) in order to capture the property of rotations.

With the above considerations in mind, we may substitute the force expressions in Eqs. (21)–(23) and the displacement fields in Eqs. (24), (26) and (27) into the virtual work equation for the rigid beam in Eq. (29). By taking into account the arbitrary nature of the variations, we can derive the stiffness equation for the rigid beam element as

$$[k_{\rm g}]^{\rm beam}\{u\} = \{{}^2_1f\} - \{{}^1_1f\},\tag{30}$$

where the geometric stiffness matrix is

$$[k_{g}]^{\text{beam}} = \begin{bmatrix} [g] & [h_{a}] & -[g] & [h_{b}] \\ [h_{a}]^{\text{T}} & [i_{a}] & -[h_{a}]^{\text{T}} & [0] \\ -[g] & -[h_{a}] & [g] & -[h_{b}] \\ [h_{b}]^{\text{T}} & [0] & -[h_{b}]^{\text{T}} & [i_{b}] \end{bmatrix}.$$
 (31)

Here, [0] is a 3×3 null matrix containing all zero entries, and

$$[g] = \begin{bmatrix} 0 & -{}^{1}F_{yb}/L & -{}^{1}F_{zb}/L \\ -{}^{1}F_{yb}/L & {}^{1}F_{xb}/L & 0 \\ -{}^{1}F_{zb}/L & 0 & {}^{1}F_{xb}/L \end{bmatrix},$$
(32)

$$[h_b] = \begin{bmatrix} 0 & 0 & 0 \\ {}^1M_{yb}/L & -{}^1M_{xb}/2L & 0 \\ {}^1M_{zb}/L & 0 & -{}^1M_{xb}/2L \end{bmatrix},$$
(33)
$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$[i_b] = \begin{bmatrix} -{}^1M_{zb} & 0 & {}^1M_{xb}/2 \\ {}^1M_{yb} & -{}^1M_{xb}/2 & 0 \end{bmatrix}.$$
 (34)

The submatrices $[h_a]$ and $[i_a]$ associated with end *a* can be obtained by replacing the terms $({}^{1}M_{xb}, {}^{1}M_{yb}, {}^{1}M_{zb})$ in the submatrices $[h_b]$ and $[i_b]$ by the terms $({}^{1}M_{xa}, {}^{1}M_{ya}, {}^{1}M_{za})$, respectively. The matrix as presented in Eq. (31) is asymmetric, due to the asymmetry of the submatrices $[i_a]$ and $[i_b]$ originating from the boundary terms in Eq. (29).

The geometric stiffness matrix derived in Eq. (31) is the same as that derived from the incrementally small-deformation theory [13]. It is *qualified by the rigid body rule* [5,6], in the sense that when an element with the initial nodal forces $\{{}_{1}^{1}f\}$ is subjected to a rigid rotation $\{u\}_r$, the total forces $\{{}_{1}^{2}f\}$ obtained from Eq. (30) will have a magnitude equal to that of the initial nodal forces $\{{}_{1}^{1}f\}$, but with the directions of the acting forces rotated by an angle equal to the rigid rotation. Of interest to note is that the effects of all kinds of actions, i.e., the axial forces, shear forces, bending moments, and torques, undergoing the rigid rotations are considered in the $[k_g]^{\text{beam}}$ matrix. In particular, this matrix reduces to the one for the truss element acted upon by a pair of axial forces undergoing the rigid rotations.

4. Geometric stiffness matrix for the triangular plate element (TPE)

Let us consider the *triangular plate element* (TPE) shown in Fig. 4, which has three translational and three rotational DOFs at each of the three tip nodes [2]. We choose this element simply because it is compatible with the 12-DOF beam element derived above.

In Section 3, the geometric stiffness matrix for the rigid beam has been explicitly given in terms of the nodal forces and element length. Thus, as far as the *rigid body behavior*



Fig. 4. Three-node triangular plate element.



Fig. 5. TPE treated as the composition of three rigid beams.



Fig. 6. The forces and moments acting on each beam element.

of an element is concerned, only the *initial forces* acting on the element and the *external shape* of the element need to be considered. The elastic properties that are essential to the deformation of the element, such as Young's modulus, cross-sectional area and moments of inertia, can be totally ignored. Based on such an idea, the rigid behavior of the TPE can be simulated as if it is composed of three rigid beams lying along the three sides of the element, as shown in Fig. 5. It is in this sense that the geometric stiffness matrix for the rigid TPE will be derived.

Fig. 6 shows the nodal forces acting on each of the three beam elements, named as beam 12, beam 23 and beam 31. In this figure, ${}^{1}F_{k}^{ij}$ and ${}^{1}M_{k}^{ij}$ denote the nodal force and nodal moment, respectively, acting on beam *ij* at the C_{1} configuration, with the right subscript *k* denotes the direction and the nodal point at which the force or moment is acting. In the following, we shall show how to determine the nodal forces acting on beam *ij*:

(1) Equations of equilibrium for each node of the TPE:

There are three forces and three moments acting at each node of the TPE. The following is the vector of forces $\{{}_{1}^{1}f\}$ for the element at the C_{1} configuration:

(37)

Considering the equilibrium of the forces and moments acting at each node, we can write

(a) Node 1

$${}^{1}F_{xa}^{12} + {}^{1}F_{xb}^{31} = {}^{1}F_{x1}, {}^{1}F_{ya}^{12} + {}^{1}F_{yb}^{31} = {}^{1}F_{y1}, {}^{1}F_{za}^{12} + {}^{1}F_{zb}^{31} = {}^{1}F_{z1}, {}^{1}M_{xa}^{12} + {}^{1}M_{xb}^{31} = {}^{1}M_{x1}, {}^{1}M_{ya}^{12} + {}^{1}M_{yb}^{31} = {}^{1}M_{y1}, {}^{1}M_{za}^{12} + {}^{1}M_{zb}^{31} = {}^{1}M_{z1}.$$
(36)

(b) Node 2

$${}^{1}F_{xa}^{23} + {}^{1}F_{xb}^{12} = {}^{1}F_{x2}, \quad {}^{1}F_{ya}^{23} + {}^{1}F_{yb}^{12} = {}^{1}F_{y2}, \quad {}^{1}F_{za}^{23} + {}^{1}F_{zb}^{12} = {}^{1}F_{z2}, \quad {}^{1}M_{xa}^{23} + {}^{1}M_{xb}^{12} = {}^{1}M_{x2}, \quad {}^{1}M_{ya}^{23} + {}^{1}M_{yb}^{12} = {}^{1}M_{y2}, \quad {}^{1}M_{za}^{23} + {}^{1}M_{zb}^{12} = {}^{1}M_{z2}.$$

(c) Node 3

$${}^{1}F_{xa}^{31} + {}^{1}F_{xb}^{23} = {}^{1}F_{x3}, \quad {}^{1}F_{ya}^{31} + {}^{1}F_{yb}^{23} = {}^{1}F_{y3}, \quad {}^{1}F_{za}^{31} + {}^{1}F_{zb}^{23} = {}^{1}F_{z3},$$

$${}^{1}M_{xa}^{31} + {}^{1}M_{xb}^{23} = {}^{1}M_{x3}, \quad {}^{1}M_{ya}^{31} + {}^{1}M_{yb}^{23} = {}^{1}M_{y3}, \quad {}^{1}M_{za}^{31} + {}^{1}M_{zb}^{23} = {}^{1}M_{z3}$$

(38)

(2) Equations of equilibrium of beam *ij*:

All the forces and moments acting on beam ij (with ij = 12, 23 or 31) must satisfy the conditions of equilibrium for the beam, as given below:

(a) Equilibrium of forces:

$${}^{1}F_{xa}^{ij} + {}^{1}F_{xb}^{ij} = 0, \quad {}^{1}F_{ya}^{ij} + {}^{1}F_{yb}^{ij} = 0, \quad {}^{1}F_{za}^{ij} + {}^{1}F_{zb}^{ij} = 0.$$
(39)

(b) Equilibrium of moments:

$$\begin{aligned} & \left(Y_{i}^{1}F_{za}^{ij}+Y_{j}^{1}F_{zb}^{ij}\right)+{}^{1}M_{xa}^{ij}+{}^{1}M_{xb}^{ij}=0, \\ & -\left(X_{i}^{1}F_{za}^{ij}+X_{j}^{1}F_{zb}^{ij}\right)+{}^{1}M_{ya}^{ij}+{}^{1}M_{yb}^{ij}=0, \\ & -\left(Y_{i}^{1}F_{xa}^{ij}+Y_{j}^{1}F_{xb}^{ij}\right)+\left(X_{i}^{1}F_{ya}^{ij}+X_{j}^{1}F_{yb}^{ij}\right)+{}^{1}M_{za}^{ij}+{}^{1}M_{zb}^{ij}=0. \end{aligned}$$

$$(40)$$

for ij = 12, 23 or 31, where (X_i, Y_i) are the coordinates of node *i*.

(3) Equations of equilibrium for the TPE:

The conditions of equilibrium must be obeyed by the TPE as a whole, as given below:

(a) Equilibrium of forces:

$$\sum_{k=1}^{3} {}^{1}F_{xk} = 0, \quad \sum_{k=1}^{3} {}^{1}F_{yk} = 0, \quad \sum_{k=1}^{3} {}^{1}F_{zk} = 0.$$
(41)

(b) Equilibrium of moments:

$$\sum_{k=1}^{3} \left(Y_k^{\ 1} F_{zk} + {}^{1} M_{xk} \right) = 0, \quad \sum_{k=1}^{3} \left(-X_k^{\ 1} F_{zk} + {}^{1} M_{yk} \right) = 0,$$
$$\sum_{k=1}^{3} \left(-Y_k^{\ 1} F_{xk} + X_k^{\ 1} F_{yk} + {}^{1} M_{zk} \right) = 0.$$
(42)

Now, we can solve Eqs. (36)–(42) to obtain the nodal forces for beam *ij* as

$${}^{1}F_{xa}^{ij} = \frac{1}{3}{}^{1}F_{xij} + f_{x}, \quad {}^{1}F_{ya}^{ij} = \frac{1}{3}{}^{1}F_{yij} + f_{y}, \quad {}^{1}F_{za}^{ij} = \frac{1}{3}{}^{1}F_{zij} + f_{z},$$

$${}^{1}F_{xb}^{ij} = -{}^{1}F_{xa}^{ij}, \quad {}^{1}F_{yb}^{ij} = -{}^{1}F_{ya}^{ij}, \quad {}^{1}F_{zb}^{ij} = -{}^{1}F_{za}^{ij}$$

(43)

and the nodal moments as

$${}^{1}M_{xa}^{ij} = \frac{1}{3}{}^{1}M_{xij} + \frac{1}{3}Y_{ij}{}^{1}F_{zj} + (m_{x} - Y_{i}f_{z}),$$

$${}^{1}M_{ya}^{ij} = \frac{1}{3}{}^{1}M_{yij} - \frac{1}{3}X_{ij}{}^{1}F_{zj} + (m_{y} + X_{i}f_{z}),$$

$${}^{1}M_{za}^{ij} = \frac{1}{3}{}^{1}M_{zij} - \frac{1}{3}\left(Y_{ij}{}^{1}F_{xj} - X_{ij}{}^{1}F_{yj}\right) + [m_{z} + (Y_{i}f_{x} - X_{i}f_{y})],$$

$${}^{1}M_{xb}^{ij} = -{}^{1}M_{xa}^{ij} - Y_{ij}{}^{1}F_{za}^{ij}, \quad {}^{1}M_{yb}^{ij} = -{}^{1}M_{ya}^{ij} + X_{ij}{}^{1}F_{za}^{ij},$$

$${}^{1}M_{zb}^{ij} = -{}^{1}M_{za}^{ij} + Y_{ij}{}^{1}F_{xa}^{ij} - X_{ij}{}^{1}F_{ya}^{ij},$$

$$(44)$$

where a variable with right subscript *ij* denotes the difference of two quantities, e.g., ${}^{1}M_{xij} = {}^{1}M_{xi} - {}^{1}M_{xj}$, $X_{ij} = X_i - X_j$, and f_x , f_y , f_z , m_x , m_y and m_z are the constants to be determined. One feature with the *rigid* element is that there are more unknowns than the equations that can be utilized. For instance, the total number of equations given in Eqs. (36)–(42) is 42. By deducting the 12 dependent equations, the total number of equations that can be used is 30, but the total number of unknown forces in Eqs. (43) and (44) is 36, with 12 for each of the three beam elements. This means that the force distribution of a *rigid* element is not unique, unlike that of an *elastic* element, and that six more conditions should be made in order to obtain a solution. In this paper, we simply let the six variables f_x , f_y , f_z , m_x , m_y and m_z equal to zero.

All the nodal forces and moments solved and given in Eqs. (43) and (44) have been expressed with reference to the coordinates of the TPE, as indicated in Fig. 6. They have to be transformed into the local coordinates of each element in order to compute the geometric stiffness matrix $[k_g]^{\text{beam}}$ of the element. Let $\{{}^1_1f^{ij}\}$ denote the nodal forces of element *ij* in the global coordinates of the TPE and $\{{}^1_1\bar{f}^{ij}\}$ in the local coordinates of element *ij*,

$$\begin{cases} {}^{1}_{1}f^{ij} \\ {}^{1}_{1}f^{ij} \end{cases} = \begin{cases} {}^{1}F^{ij}_{xa} & {}^{1}F^{ij}_{ya} & {}^{1}F^{ij}_{za} & {}^{1}M^{ij}_{xa} & {}^{1}M^{ij}_{ya} & {}^{1}M^{ij}_{za} \\ {}^{1}F^{ij}_{xb} & {}^{1}F^{ij}_{yb} & {}^{1}F^{ij}_{zb} & {}^{1}M^{ij}_{xb} & {}^{1}M^{ij}_{yb} & {}^{1}M^{ij}_{zb} \end{cases}^{\mathrm{T}},$$
 (45)

$$\begin{cases} {}_{1}\bar{f}^{ij} \\ {}_{1}\bar{f}^{ij} \end{cases} = \begin{cases} {}_{xa}^{i} \bar{F}^{ij}_{xa} & {}^{1}\bar{F}^{ij}_{za} & {}^{1}\bar{M}^{ij}_{xa} & {}^{1}\bar{M}^{ij}_{ya} & {}^{1}\bar{M}^{ij}_{za} \\ {}_{1}\bar{F}^{ij}_{xb} & {}^{1}\bar{F}^{ij}_{yb} & {}^{1}\bar{F}^{ij}_{zb} & {}^{1}\bar{M}^{ij}_{xb} & {}^{1}\bar{M}^{ij}_{yb} & {}^{1}\bar{M}^{ij}_{yb} \end{cases}$$
(46)

The following is the transformation between the two sets of forces for element *ij*:

$$\begin{cases} {}_{1}\bar{f}^{ij} \\ {}_{1}\bar{f}^{ij} \end{cases} = [T] \begin{cases} {}_{1}^{1}f^{ij} \\ {}_{1}f^{ij} \end{cases}, \tag{47}$$

where the transformation matrix [T] is

$$[T] = \begin{bmatrix} [R] & [0] & [0] & [0] \\ [0] & [R] & [0] & [0] \\ [0] & [0] & [R] & [0] \\ [0] & [0] & [0] & [R] \end{bmatrix}, \quad [R] = \begin{bmatrix} -\frac{X_{ij}}{L_{ij}} & -\frac{Y_{ij}}{L_{ij}} & 0 \\ \frac{Y_{ij}}{L_{ij}} & -\frac{X_{ij}}{L_{ij}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(48)$$

with L_{ij} denoting the length of element *ij*. By substituting the nodal forces $\{ {}_{1}^{i} \bar{f}^{ij} \}$ into Eq. (31), the geometric stiffness matrix $[k_{g}]^{\text{beam_}ij}$ for element *ij* can be obtained. Finally, by assembling the geometric stiffness matrices for the three elements following the standard finite element procedure, the geometric stiffness matrix $[k_{g}]^{\text{TPE}}$ for the TPE can be obtained

$$[k_{g}]^{\text{TPE}} = \sum_{ij=12,23,31} [T]^{\text{T}} [k_{g}]^{\text{beam_}ij} [T],$$
(49)

which has been explicitly given in Appendix.

It is easy to verify that the geometric stiffness matrix $[k_g]^{\text{TPE}}$ derived above for the TPE passes the rigid body test [5,6], in that the resulting forces $\{{}^2_1f\}$ computed from Eq. (30) for the TPE undergoing a rigid rotation maintain a magnitude equal to that of the initial forces $\{{}^1_1f\}$, but with the directions of action rotated by an angle equal to the rigid rotation.

5. Joint equilibrium condition in the rotated position

Both the geometric stiffness matrix $[k_g]^{\text{beam}}$ for the rigid beam, as given in Eq. (31), and the $[k_g]^{\text{TPE}}$ matrix for the TPE, as given in the Appendix, are *asymmetric*, as indicated by the submatrices $[i_k]$ and $[I_k]$, respectively, which relate to the behavior of nodal moments undergoing 3D rotations [6,7]. Such a property of asymmetry is restricted to the *element level*, but not the structure level, which adds little extra effort to nonlinear analysis. To explain such a fact, let us decompose the geometric stiffness matrices for the rigid beam and TPE, as given in Eq. (31) and the Appendix, respectively, into the *symmetric* and *anti-symmetric* parts:

$$[k_g]^{\text{beam}} = [k_g]^{\text{beam}}_{\text{sym}} + [k_g]^{\text{beam}}_{\text{anti-sym}},$$
(50)

$$[k_g]^{\text{TPE}} = [k_g]^{\text{TPE}}_{\text{sym}} + [k_g]^{\text{TPE}}_{\text{anti-sym}},$$
(51)

where the anti-symmetric parts are

$$[k_{g}]_{anti-sym}^{beam} = \begin{bmatrix} [0] & [0] & [0] & [0] \\ [0] & [A_{a}] & [0] & [0] \\ [0] & [0] & [0] & [0] \\ [0] & [0] & [0] & [A_{b}] \end{bmatrix},$$
(52)

$$[k_{g}]_{anti-sym}^{TPE} = \begin{bmatrix} [0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [A_{1}] & [0] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [A_{2}] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [A_{3}] \end{bmatrix}$$
(53)

as indicated by the submatrix $[A]_k$, which is a function of the nodal moments,

$$[A]_{k} = \frac{1}{2} \begin{bmatrix} 0 & {}^{1}M_{zk} & -{}^{1}M_{yk} \\ -{}^{1}M_{zk} & 0 & {}^{1}M_{xk} \\ {}^{1}M_{yk} & -{}^{1}M_{xk} & 0 \end{bmatrix}$$
(54)

with the left subscript k = a, b for the beam element, and k = 1, 2, and 3 for the TPE.

In the literature [6,7], it has been demonstrated that the anti-symmetric parts of the geometric stiffness matrices of all *beam elements* meeting at the same joint will cancel out, once the conditions of equilibrium for the joint in the *rotated* configuration C_2 are enforced. As a result, the stiffness matrix assembled for the *structure* turns out to be symmetric. The same is also true for the *plates* and *shells* represented by the TPE, as will be proved below.

Using the symbol e_{ijk} to denote a permutation symbol [14], the anti-symmetric [A] for each node of the TPE can be represented as follows:

$$A_{ij} = \frac{1}{2} e_{ijk}{}^{1} M_{k} \quad (i, j, k = 1, 2, 3),$$
(55)

which implies that ${}^{1}M_{1} = {}^{1}M_{x}$, ${}^{1}M_{2} = {}^{1}M_{y}$, etc. Let $[\Phi]$ denote the transformation matrix from the local coordinates of an element to the global coordinates of the structure. Then,

$$[\boldsymbol{\Phi}]^{\mathrm{T}}[\boldsymbol{\Phi}] = [I],\tag{56}$$

where [I] is an identity matrix, or

$$\Phi_{pi}\Phi_{pj} = \delta_{ij} \tag{57}$$

with δ_{ij} denoting the Kronecker delta. Based on the tensorial operations, the following is shown to be valid:

$$e_{ijk}\Phi_{pi}\Phi_{qj}\Phi_{rk} = e_{pqr} \quad (i, j, k, p, q, r = 1, 2, 3).$$
(58)

The nodal moment vector $\{M\}$ in the element coordinates can be transformed to the structure coordinates as

$$\{\overline{M}\} = [\Phi]\{M\},\tag{59}$$

where $\{\overline{M}\}$ denotes the nodal moments in the structure coordinates, or

$${}^{1}\overline{M}_{p} = \Phi_{pi}{}^{1}M_{i} \quad (i, p = 1, 2, 3).$$
 (60)

Similarly, the anti-symmetric matrix [A] can be transformed to the global coordinates as

$$[\overline{A}] = [\Phi]^{\mathrm{T}}[A][\Phi] \tag{61}$$

or

$$\overline{A}_{pq} = \Phi_{pi} A_{ij} \Phi_{qj} \quad (i, j, p, q = 1, 2, 3).$$
(62)

Substituting Eq. (55) into Eq. (62) yields

$$\overline{A}_{pq} = \Phi_{pi} \Phi_{qj} A_{ij} = \frac{1}{2} e_{ijk} \Phi_{pi} \Phi_{qj}{}^1 M_k \quad (i, j, k, p, q, r = 1, 2, 3).$$
(63)

By the following relations:

$$e_{ijk}\Phi_{pi}\Phi_{qj} = e_{ijt}\Phi_{pi}\Phi_{qj}\delta_{tk} = e_{ijt}\Phi_{pi}\Phi_{qj}\Phi_{rt}\Phi_{rk} = e_{pqr}\Phi_{rk}.$$
 (64)

Eq. (63) can be rewritten as follows:

$$\overline{A}_{pq} = \frac{1}{2} e_{pqr} \Phi_{rk}{}^{1} M_{k} = \frac{1}{2} e_{pqr}{}^{1} \overline{M}_{r} \quad (k, p, q, r = 1, 2, 3).$$
(65)

Next, let us consider a *nodal point* of the plate/shell structure where *n* TPE elements lying along different directions are connected. For the nodal point to be in equilibrium in the *rotated* configuration C_2 , the sum of moments exerted by all the elements meeting at the node along the three global axes must be equal to zero,

$$\sum_{s=1}^{n} {}^{1}\overline{M}_{r}^{(s)} = 0.$$
(66)

Consequently, we have

$$\sum_{s=1}^{n} \overline{A}_{pq}^{(s)} = \sum_{s=1}^{n} \frac{1}{2} e_{pqr}{}^{1} \overline{M}_{r}^{(s)} = \frac{1}{2} e_{pqr} \sum_{s=1}^{n} {}^{1} \overline{M}_{r}^{(s)} = 0$$

(p,q,r = 1,2,3). (67)

This implies that when the anti-symmetric matrices [A], which are expressed in the global coordinates, are summed over all the TPE elements sharing a common node, the resulting stiffness matrix associated with this node is *zero*. Thus, we have proved that the total stiffness matrix for the structure in the *rotated* configuration C_2 is *symmetric*, although the element stiffness matrices are known to be asymmetric.

6. Predictor and corrector for incremental-iterative analysis

Only incremental-iterative nonlinear analysis of the UL type is of concern in this study. The idea is that if the rigid rotation effects are fully taken into account at each stage of analysis, then the remaining effects of natural deformations can be treated using the small-deformation linearized theory. In this regard, two major stages can be identified [3,4]. The first is the *predictor* or *trial* stage, which relates to solution of the structural displacements $\{U\}$, given the load increments $\{^2P\} - \{^1P\}$, as indicated by the following equation:

$$[K]{U} = {^2P} - {^1P}, (68)$$

where $\{{}^{1}P\}$ and $\{{}^{2}P\}$ denote the applied loads acting on the structure at C_{1} and C_{2} , respectively. Once the structural displacement increments $\{U\}$ are made available, the displacements increments $\{u\}$ for each element can be com-

puted accordingly. It is known that the predictor affects only the number of iterations or the speed of convergence [3,4]. Therefore, the structural stiffness matrix [K] is allowed to be *approximate*, but to the limit that the direction of iterations is not misguided. In addition to the elastic stiffness matrix, this will require the use of geometric stiffness matrices that are capable of simulating the rigid rotational behavior of the initially stressed elements, such as those presented herein for the 3D beam and TPE elements.

For the 3D beam element, both the elastic stiffness matrix $[k_e]$ available in Refs. [6,9] and the geometric stiffness matrix $[k_g]$ in Eq. (31) will be used in constructing the structural stiffness matrix [K]. For the TPE, the elastic stiffness matrix $[k_e]$ is obtained as the composition of Cook's plane hybrid element for membrane actions [10] and the HSM element of Batoz et al. for bending actions [11]. The elastic stiffness matrices $[k_e]$ so obtained, plus the geometric stiffness matrices $[k_g]^{TPE}$ given in the Appendix, will be used to assemble the structural stiffness matrix [K] for the plates and shell considered.

The corrector stage is concerned with the recovery of the element forces $\left\{ {}_{2}^{2}f \right\}$ at C_{2} with reference to C_{2} . For incremental analysis of the UL type, two contributions need to be considered. The first is the initial nodal forces $\{ {}^{1}_{1}f \}$ existing at C_1 and expressed with reference to C_1 . According to the rigid body rule, the initial nodal forces $\{{}^{1}_{1}f\}$ will be rotated by an angle equal to the rigid rotation (with no *limit* in magnitude of rotation) generated during the incremental step from C_1 to C_2 , while their magnitude remains unchanged [5,6]. Thus, one can directly treat the initial nodal forces $\{{}^{1}_{1}f\}$ as the forces acting at C_{2} and expressed with respect to C_2 . As the rigid rotation effect is already taken into account, the force increments $\{2f\}$ can be computed from the displacement increments $\{u\}$ using the elastic stiffness matrix $[k_e]$ alone based on the small-deformation linearized theory, i.e.,

$$\{2f\} = [k_e]\{u\}. \tag{69}$$

Summing the above two effects yields the total element forces $\binom{2}{2}f$ at C_2 as

$${2 \choose 2}f = {1 \choose 1} + {2f}, \tag{70}$$

where the reference configuration is C_2 .

As the structural and element displacements, i.e., $\{U\}$ and $\{u\}$, are made available from the predictor stage in Eq. (68), one may compute the nodal coordinates and element orientations for the structure at the displaced configuration C_2 . Here, the nodal rotations generated at each incremental step are assumed to be so small that the law of commutativity applies. Then, by summing the element forces $\{\frac{2}{2}f\}$ computed from Eq. (70) for each node of the structure at C_2 and by comparing them with the total applied loads $\{^2P\}$, the unbalanced forces for the structure at C_2 can be obtained. Another iteration involving the predictor and corrector stages should be conducted to eliminate the unbalanced forces, should they be greater than preset tolerances.

In a UL-type incremental-iterative nonlinear analysis, the accuracy of the solution is governed primarily by the corrector: while the predictor can only affect the speed of convergence or the number of iterations [3,6]. For the purpose of investigating the capacity of the geometric stiffness matrix derived in this paper, the following combinations of predictor and corrector will be tested in the analysis of the space frames:

- *Predictor*: The structural stiffness matrix [K] in Eq. (68) is assembled using either of the following:
 - (P1) $[k_e] + [k_g]^{beam}$, where $[k_g]^{beam}$ is the one derived in this paper for the rigid beam.
 - (P2) $[k_e] + [k_g] + [k_i]$, where $[k_g]$ is the (conventional) geometric stiffness matrix and $[k_i]$ the *induced* moment matrix available on pp. 361-362 of Ref. [6].
 - (P3) $[k_e]$ matrix only.
- Corrector:
 - (C1) $\{2f\} = [k_e]\{u\}$ as proposed in Eq. (69).
 - (C2) $\{2f\} = ([k_e] + [k_g] + [k_i])\{u_n\}$, where $\{u_n\}$ denotes the natural deformations of the element.

As for the *plates* and *shells*, the TRIC element developed by Argyris et al. [15] will be used to yield solutions for comparison. The following are the combinations to be used:

- *Predictor*: The structural stiffness matrix [K] in Eq. (68) is assembled using either of the following: (P1) $[k_e]^{TPE} + [k_g]^{TPE}$, where $[k_e]^{TPE}$ is composed of the
 - elastic stiffness matrices available in Refs. [10,11]; (P2) $[k_e]^{\text{TRIC}} + [k_g]^{\text{TRIC}}$, where both the stiffness matri-
 - ces are given in Ref. [15].
 - (P3) $[k_e]^{\text{TPE}}$ matrix only.
- Corrector:
 - (C1) $\{2f\} = [k_e]^{\text{TPE}}\{u\}$, as proposed in Eq. (69) of this paper. (C2) $\{_{2}f\} = [k_{e}]^{\text{TRIC}}\{u\}.$

 M_{Za} (N-mm)

7. Numerical examples

The generalized displacement control (GDC) method [16] is adopted for tracing the nonlinear load-deflection response of the problems studied. With the aid of the generalized stiffness parameter (GSP), this method can automatically adjust the load increment sizes to reflect the variation in stiffness of the structure, while reversing the direction of loading when passing a *limit point*. It can easily deal with the various critical points encountered in the postbuckling response of a structure. Since all the analyses are allowed to self adjust their load increments by the GDC method, the total computation time consumed by each scheme may not truly reflect its efficiency of computation.

Example 1. Fig. 7 shows a single beam restrained against the rotations about the x and y axes at the left end and against the rotations about the y and z axes at the other end. The following data are adopted: length L=100 mm, cross-sectional area $A = 0.18 \text{ cm}^2$, torsional constant moments of inertia $I_v = 0.54 \text{ mm}^4$, $J = 2.16 \text{ mm}^4$, $I_z = 1350 \text{ mm}^4$, Young's modulus $E = 71,240 \text{ N/mm}^2$, and shear modulus $G = 27,190 \text{ N/mm}^2$. The beam is subjected to a moment of M_{za} at the left end, and a perturbation of $M_{xb} = 0.01 M_{za}$ at the right end (to trigger the lateral buckling). Ten elements are used for the beam. The theoretical critical moment for the beam under uniform moment is $M_{\rm cr} = \pm \pi \sqrt{EI_v GJ}/L = \pm 1493.3$ N mm [6,7].

The numerical results obtained are plotted in Fig. 8, along with the computation time of each analysis listed in Table 1. The peak load in Fig. 8 is 1463.0 N mm. As can be seen, the proposed scheme P1C1 can be used to yield solutions that are as accurate as the conventional one, though at the cost of longer computation time. It is interesting that the same problem can be solved using the elastic stiffness matrix $[k_e]$ alone in all stages of analysis, as indicated by P3C1 in Table 1. Of course, this is achieved at the cost of much longer computation time.



Fig. 7. Single beam subjected to moment M_{za}



Fig. 8. Moment-displacement curves for single beam: (a) U_{xa} , (b) U_{zb} .

Table 1 Running time for analysis of Example 1

Combination	P1C1	P2C2	P3C1
CPU time (s)	55.99	38.02	474.33



Fig. 9. Hinged angled frame in pure bending.

Example 2. Fig. 9 shows an angled frame under uniform bending (i.e., with $M_{za} = M_{zc}$) and a shear load F_{zb} (N) of magnitude $5 \times 10^{-5} M_{za}$ [1,6,7]. The length *L* is 240 mm and other properties are the same as in Example 1. Due to symmetry of the frame, only the left half modeled by 10 elements is analyzed. The member *ab* is restrained against rotations about the *x*, *y* axes and translations along the *y*, *z* axes. The translation along the *x*-axis at node *b* is also restrained. The theoretical critical loads for the frame are: $M_{cr} = \pm \pi \sqrt{EI_VGJ}/L = \pm 622.2$ N mm.

From the results plotted in Fig. 10, one observes that the proposed scheme P1C1 can be reliably used to trace the postbuckling behavior of the frame, though at the expense of slightly longer computation time, compared with the conventional one (P2C2). Of interest is that using the $[k_e]$ matrix alone in both the predictor and corrector, as indicated by P3C1, can arrive exactly at the same solution, but with much longer computation time (see Table 2).

Example 3. The right-angled frame subjected to an inplane load F_{yb} at the free end in Fig. 11(a) was studied [1], for which the critical load is $F_{yb,cr} = 1.088$ N. The same material and geometry data as those used in Example 2 are adopted. Each member of the frame is modeled by 10 beam elements. The entire frame was also modeled by 38 triangular elements as in Fig. 11b. An imperfection load F_{zb} equal to one thousandth of F_{yb} is applied at the free end.

From the results plotted in Fig. 12, it is seen that the proposed scheme P1C1 using either the beam or TPE approach is capable of tracing the postbuckling response of the angled frame. Table 3 indicates that the time consumed by the present beam approach is slightly longer than the conventional approach. The same is also true for the TPE approach, compared with the TRIC approach. It should be noted that the number of iterations at each incremental step is about the same for both the TPE and TRIC



Fig. 10. Moment–displacement curves for hinged angled frame: (a) U_{xa} , (b) θ_{za} .

P3C1

1081.13

Table 2Running time for analysis of Example 2			
Combination	P1C1	P2C2	

116.91

CPU time (s)

approaches. However, less time is consumed by the TRIC approach in computing the stiffness matrices, since it considers only the in-plane actions, while the TPE considers all kinds of in-plane and out-of-plane actions.

109.25

No convergent solution was obtained by the P3C1 scheme using only the $[k_e]$ matrix, due to the fact that the trial directions produced by the predictor based on the $[k_e]$ matrix alone deviates too much from the equilibrium path in the region near the bifurcation point.

Example 4. Fig. 13 shows a spherical shell subjected to the central point load *P* with all edges hinged and immovable [17]. The data adopted herein are: half of side length a = 784.9 mm, thickness t = 99.45 mm, Young's modulus radius of curvature R = 2540 mm, E = 68.95 N/mm², and Poisson's ratio v = 0.3. Because of symmetry, only one quarter of the shell is modeled as an 8×8 mesh and analyzed.

The results obtained for the load-deflection response of the central point using different combinations are plotted in Fig. 14, along with the running time listed in Table 4. Basically no distinction can be made between the results obtained by different approaches. Compared with the



Fig. 11. Right angled frame with fixed support: (a) beam elements, (b) plate elements.



Fig. 12. Load-displacement curves for angled frame: (a) beam approach, (b) plate approach.

Table 3	
Running time for analysis of Example 3	

Approach	Combination		
	P1C1	P2C2	P3C1
Beam	37.59	36.16	Divergent
Plate	1492.22	1211.25	Divergent

TRIC approach, slightly longer computation time is required by the TPE approach due to inclusion of all kinds of in-plane and out-of-plane actions in the stiffness matrices by the TPE approach, though the number of iterations at each incremental step is about the same. Of interest is



Fig. 13. Hinged spherical shell subjected to a central load.



Fig. 14. Load-deflection curve for hinged spherical shell.

Table 4		
Running time (s) for analysis of Example 4		
Mesh	Combination	

Wiesh	Comomation			
	P1C1	P2C2	P3C1	
8×8	114.39	100.01	184.92	

that using the $[k_e]^{\text{TPE}}$ stiffness matrix alone, as indicated by P3C1, is good enough for obtaining the solution, though at the cost of extra computation time.

Example 5. A cylindrical shell subjected to a central load *P* on the top surface is shown in Fig. 15 [18]. The longitudinal boundaries of the shell are hinged and immovable, whereas the curved edges are completely free. The following data are adopted: $E = 3.10275 \text{ kN/mm}^2$, R = 2540 mm, L = 254 mm, h = 12.7 mm, $\theta = 0.1 \text{ rad}$, v = 0.3. Because of symmetry, only one quarter of the shell is considered and modeled by an 8×8 mesh.

The results obtained for the central deflection of the shell are plotted in Fig. 16, along with the computation time of each analysis in Table 5. Again, no distinction can be made between the results obtained by different approaches. The computation time consumed by the present TPE approach is slightly longer than the TRIC approach for the reasons stated in Example 3. The P3C1 approach with the $[k_e]$ matrix alone is able to trace the entire load-deflection response, though at the cost of extra computation time.



Fig. 15. Hinged cylindrical shell.



Fig. 16. Central deflection of hinged cylindrical shell with h = 12.7 mm.

Table 5Running time (s) for analysis of Example 5

Combination		
P3C1		
668.48		



Fig. 17. Central deflection of hinged cylindrical shell with h = 6.35 mm.

Example 6. This example is identical to Example 5 except that the thickness of the shell is halved to h = 6.35 mm [18]. The results obtained using an 8×8 mesh have been shown in Fig. 17, together with the computation time in Table 6. A comparison of Fig. 17 with Fig. 16 indicates that reducing the thickness of the shell by half has resulted not only in a drastic decline of the limit load capacity, but also in a shift of the response curve pattern to embrace a snap-back region. As can be seen from Table 6, slightly longer computation time is consumed by the present TPE approach, compared with the TRIC approach, for which the reason was given in Example 3. Again, this problem can be solved using exclusively the $[k_e]$ matrix in each stage of analysis, as

Table 6 Running time (s) for analysis of Example 6

Mesh	Combination		
	P1C1	P2C2	P3C1
8×8	197.08	177.78	559.73

indicated by P3C1, though much longer computation time is required.

8. Concluding remarks

The procedure presented in this paper is based on the idea that if the rigid rotation effects are fully taken into account at each stage of the incremental-iterative nonlinear analysis, then the remaining effects of natural deformations can be treated using the small-deformation linearized theory. Based on the UL formulation, two stages are considered essential, i.e., the *predictor stage* for computing the structural displacements given the load increments, and the *corrector stage* for recovering the element forces. The former affects the number of iterations, but the latter determines the accuracy of solution.

As for use in the predictor, the geometric stiffness matrix $[k_g]$ is derived for a 3D *rigid beam element* from the virtual work equation by using a rigid displacement field, and the geometric stiffness matrix $[k_g]$ for the *rigid triangular plate element* (TPE) is assembled from those for the three rigid beams lying along the three sides of the element. One advantage with the present approach is that the geometric stiffness matrices $[k_g]$ for both the 3D rigid beam and rigid TPE are *explicitly* given and *all kinds of in-plane and out-of-plane actions* are taken into account. Using the rigid element concept, the work of derivation is greatly *reduced* for both elements, as only rigid displacement fields are required.

As for the corrector, the *rigid body rule* is used to update the initial nodal forces $\{{}_{1}^{1}f\}$ existing at C_{1} with no limit on the magnitude of rigid rotations. With this, the force increments generated at each incremental step are computed using only the elastic stiffness matrix $[k_{e}]$ based on the small-deformation linearized theory.

The robustness of the proposed combination of predictor and corrector has been demonstrated in the solution of a number of frame and shell problems involving postbuckling responses. A slightly longer computation time is required for the TPE approach due to consideration of all kinds of actions, compared with the TRIC approach that considers only in-plane actions, based on the framework of the GDC method for self adjustment of load increments. For problems with no abrupt change in the slope of the load–deflection curves, the entire postbuckling response can be traced by using only the elastic stiffness matrix $[k_e]$ in each stage of analysis, though at the cost of extra computation time.

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Appendix. Geometric stiffness matrix for the TPE

The geometric stiffness matrix derived in this paper for the TPE is

$$[k_g]^{\text{TPE}} = \begin{bmatrix} [E_1] & [F_1] & -[E_{12}] & [G_{12}] & -[E_{31}] & -[H_{31}] \\ [F_1]^{\text{T}} & [I_1] & -[H_{12}]^{\text{T}} & [0] & [G_{31}]^{\text{T}} & [0] \\ -[E_{12}]^{\text{T}} & -[H_{12}] & [E_2] & [F_2] & -[E_{23}] & [G_{23}] \\ [G_{12}]^{\text{T}} & [0] & [F_2]^{\text{T}} & [I_2] & -[H_{23}]^{\text{T}} & [0] \\ -[E_{31}]^{\text{T}} & [G_{31}] & -[E_{23}]^{\text{T}} & -[H_{23}] & [E_3] \\ -[H_{31}]^{\text{T}} & [0] & [G_{23}]^{\text{T}} & [0] & [F_3]^{\text{T}} & [I_3] \end{bmatrix} ,$$

where

$$\begin{split} & [E_{ij}] = \begin{bmatrix} a^{ij} & c^{ij} & d^{ij} \\ c^{ij} & b^{ij} & e^{ij} \\ d^{ij} & e^{ij} & f^{ij} \end{bmatrix}, \quad [G_{ij}] = \begin{bmatrix} n^{ij} & p^{ij} & 0 \\ q^{ij} & o^{ij} & 0 \\ r^{ij} & s^{ij} & -m^{ij} \end{bmatrix} \\ & [H_{ij}] = \begin{bmatrix} g^{ij} & i^{ij} & 0 \\ j^{ij} & h^{ij} & 0 \\ k^{ij} & l^{ij} & m^{ij} \end{bmatrix}, \\ & [I_1] = \begin{bmatrix} t_1^{12} + t_2^{31} & u_1^{12} + u_2^{31} & w_1^{12} + w_2^{31} \\ v_1^{12} + v_2^{31} & -(t_1^{12} + t_2^{31}) & y_1^{12} + y_2^{31} \\ x_1^{12} + x_2^{31} & z_1^{12} + z_2^{31} & 0 \end{bmatrix}, \\ & [I_2] = \begin{bmatrix} t_1^{23} + t_2^{12} & u_1^{23} + u_2^{12} & w_1^{23} + w_2^{12} \\ v_1^{23} + v_2^{12} & -(t_1^{23} + t_2^{12}) & y_1^{23} + y_2^{12} \\ x_1^{23} + x_2^{12} & z_1^{23} + z_2^{12} & 0 \end{bmatrix}, \\ & [I_3] = \begin{bmatrix} t_1^{31} + t_2^{23} & u_1^{31} + u_2^{23} & w_1^{31} + w_2^{23} \\ v_1^{31} + v_2^{23} & -(t_1^{31} + t_2^{23}) & y_1^{31} + y_2^{23} \\ x_1^{31} + x_2^{23} & z_1^{31} + z_2^{23} & 0 \end{bmatrix}, \\ & [E_1] = [E_{12}] + [E_{31}], \quad [E_2] = [E_{23}] + [E_{12}], \\ & [E_3] = [E_{31}] + [E_{23}], \\ & [F_1] = [H_{12}] - [G_{31}], \quad [F_2] = [H_{23}] - [G_{12}], \\ & [F_3] = [H_{31}] - [G_{23}]. \end{aligned}$$

In the above expressions, the parameters are given as follows:

$$\begin{split} a^{ij} &= -\frac{1}{L_{ij}^4} \Big(X_{ij} Y_{ij}^2 F_{xa}^{ij} - X_{ij}^2 Y_{ij} F_{ya}^{ij} - L_{ij}^2 Y_{ij} F_{ya}^{ij} \Big), \\ b^{ij} &= \frac{1}{L_{ij}^4} \Big(X_{ij} Y_{ij}^2 F_{xa}^{ij} - X_{ij}^2 Y_{ij} F_{ya}^{ij} + L_{ij}^2 X_{ij} F_{xa}^{ij} \Big), \\ c^{ij} &= -\frac{1}{L_{ij}^4} \Big(Y_{ij}^3 F_{xa}^{ij} + X_{ij}^3 F_{ya}^{ij} \Big), \quad d^{ij} &= -\frac{1}{L_{ij}^2} X_{ij} F_{za}^{ij}, \\ e^{ij} &= -\frac{1}{L_{ij}^2} Y_{ij} F_{za}^{ij}, \\ f^{ij} &= \frac{1}{L_{ij}^2} \Big(X_{ij} F_{xa}^{ij} + Y_{ij} F_{ya}^{ij} \Big), \\ g^{ij} &= -\frac{1}{2L_{ij}^4} \Big(X_{ij} Y_{ij}^2 M_{xa}^{ij} - X_{ij}^2 Y_{ij} M_{ya}^{ij} - L_{ij}^2 Y_{ij} M_{ya}^{ij} \Big) \end{split}$$

$$\begin{split} h^{ij} &= \frac{1}{2L_{ij}^4} \Big(X_{ij} Y_{ij}^2 M_{xa}^{ij} - X_{ij}^2 Y_{ij} M_{ya}^{ij} + L_{ij}^2 X_{ij} M_{xa}^{ij} \Big), \\ t^{ij} &= -\frac{1}{2L_{ij}^4} \Big(X_{ij}^2 Y_{ij} M_{xa}^{ij} - X_{ij}^2 Y_{ij}^2 M_{ya}^{ij} + L_{ij}^2 Y_{ij} M_{xa}^{ij} \Big), \\ s^{ij} &= \frac{1}{2L_{ij}^4} \Big(X_{ij}^2 Y_{ij} M_{xa}^{ij} - X_{ij}^3 M_{ya}^{ij} - L_{ij}^2 X_{ij} M_{ya}^{ij} - L_{ij}^2 Y_{ij} M_{xa}^{ij} - X_{ij}^2 Y_{ij} M_{xa}^{ij} - X_{ij}^2 Y_{ij} M_{ya}^{ij} - X_{ij}^2 Y_{ij} M_{ya}^{ij} - X_{ij}^2 Y_{ij} M_{ya}^{ij} - L_{ij}^2 Y_{ij} M_{xa}^{ij} + Y_{ij} M_{ya}^{ij} \Big), \\ s^{ij} &= -\frac{1}{L_{ij}^2} X_{ij} Y_{ij} F_{za}^{ij} + \frac{1}{2L_{ij}^4} \Big(X_{ij} Y_{ij}^2 M_{xa}^{ij} - X_{ij}^2 Y_{ij} M_{ya}^{ij} - L_{ij}^2 Y_{ij} M_{ya}^{ij} \Big), \\ o^{ij} &= -\frac{1}{L_{ij}^2} Y_{ij} F_{za}^{ij} + \frac{1}{2L_{ij}^4} \Big(Y_{ij}^3 M_{xa}^{ij} - X_{ij}^2 Y_{ij} M_{ya}^{ij} + L_{ij}^2 Y_{ij} M_{ya}^{ij} \Big), \\ s^{ij} &= \frac{1}{L_{ij}^2} Y_{ij}^2 F_{za}^{ij} + \frac{1}{2L_{ij}^4} \Big(Y_{ij}^3 M_{xa}^{ij} - X_{ij}^2 Y_{ij} M_{ya}^{ij} + L_{ij}^2 X_{ij} M_{ya}^{ij} \Big), \\ q^{ij} &= -\frac{1}{L_{ij}^2} Y_{ij}^2 F_{za}^{ij} - \frac{1}{2L_{ij}^2} \Big(X_{ij}^2 Y_{ij} M_{xa}^{ij} - X_{ij}^2 Y_{ij} M_{ya}^{ij} \Big), \\ s^{ij} &= \frac{1}{L_{ij}^2} Y_{ij} M_{za}^{ij} - \frac{1}{L_{ij}^2} \Big(Y_{ij}^2 F_{xa}^{ij} - X_{ij}^2 Y_{ij} Y_{jj}^{ij} \Big), \\ s^{ij} &= \frac{1}{L_{ij}^2} Y_{ij} M_{za}^{ij} - \frac{1}{L_{ij}^2} \Big(Y_{ij}^2 F_{xa}^{ij} - X_{ij}^2 Y_{ij} F_{ya}^{ij} \Big), \\ s^{ij} &= -\frac{1}{L_{ij}^2} (X_{ij} Y_{ij} M_{xa}^{ij} + Y_{ij}^2 M_{ja}^{ij} \Big), \\ s^{ij} &= -\frac{1}{2L_{ij}^2} \Big(X_{ij} Y_{ij} M_{xa}^{ij} - X_{ij}^2 M_{ja}^{ij} \Big), \\ s^{ij} &= \frac{1}{2L_{ij}^2} \Big(X_{ij}^2 Y_{ij} M_{xa}^{ij} + X_{ij}^2 Y_{ij} M_{ya}^{ij} \Big), \\ s^{ij} &= -\frac{1}{2L_{ij}^2} \Big(X_{ij}^2 M_{ya}^{ij} + X_{ij}^2 Y_{ij} M_{ya}^{ij} \Big), \\ s^{ij} &= -\frac{1}{2L_{ij}^2} \Big(X_{ij}^2 M_{xa}^{ij} + X_{ij}^2 Y_{ij} M_{ya}^{ij} \Big), \\ s^{ij} &= -\frac{1}{2L_{ij}^2} \Big(X_{ij}^2 M_{xa}^{ij} + X_{ij}^2 Y_{ij} M_{ya}^{ij} \Big), \\ s^{ij} &= -\frac{1}{2L_{ij}^2} \Big(X_{ij}^2 M_{xa}^{ij} + X_{ij}^2 Y_{ij} M_{ya}^{ij} \Big), \\ s^{ij} &= -\frac{1}{2L_{$$

In the above, the nodal forces for element *ij*, that is, $F_{xa}^{ij}, F_{ya}^{ij}, \ldots, M_{xa}^{ij}, M_{ya}^{ij}, \ldots$, with the left superscript "1" omitted, have been given in Eqs. (43) and (44), $X_{ij} = X_i - X_j$, $Y_{ij} = Y_i - Y_j$, and L_{ij} is the length of element *ij*.

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