Laminar Water Wave and Current Passing Over Porous Bed

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Abstract: The problem of the dynamic interaction of water waves, current, and a hard poroelastic bed is dealt with in this study. Finite-depth homogeneous water with harmonic linear water waves passing over a semi-infinite poroelastic bed is investigated. In order to reveal the importance of viscous effect for different bed forms, viscosity of water is considered herein. In a boundary layer correction approach, the governing equations of the poroelastic material are decoupled without losing physical generality. The contribution of pressure effect and shear effect to the hard poroelastic bed, which is a valuable indication to the mechanism of ripple formation, is clarified in the present study. This approach will be helpful in saving time and storage capacity when it is applied to numerical computation.

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Introduction

For a realistic analysis of the problem of the dynamic interaction of water flow and seabed, the flow over the seabed is generally accompanied with waves and current. The water is actually viscous with a boundary layer near the interface between the homogeneous water and the porous bed. Thus it tends to include viscous flow, instead of potential flow, for water waves and current acting simultaneously on a deformable seabed.

The earliest investigations on linear water waves of an inviscid, incompressible, and irrotational fluid flow interacting with Darcy's flow within a rigid, isotropic porous skeleton were found in the works of Putnam (1949) and Reid and Kajiura (1957). Sleath (1970) and Moshagen and Torum (1975) further studied the similar problem but took the anisotropic permeability in horizontal and vertical directions into account. Liu (1973) proceeded with the order of magnitude analysis to simplify the boundary conditions by a boundary layer approach. In fact, fluid within a porous material interacting with a deforming solid skeleton, as Biot (1956) stated, is a more complicated two-phase problem for a realistic analysis. Biot (1956) developed the theory of poroelasticity to discuss elastic waves in a fluid saturated porous solid. For the problem of a low frequency wave acting on fluid saturated poroelastic media, Mei and Foda (1981) proposed a boundary layer correction to simplify the analysis; however, their approach was without systematic perturbation analysis. Huang and Song (1993) solved the problem of oscillatory linear water waves interacting with a deformable bed by using three decoupled Helmholtz equations derived by Huang and Chwang (1990) to treat the poroelastic bed. In their solution, five nondimensional parameters were derived. Chen et al. (1997) also applied Huang and Chwang's (1990) approach of poroelastic media flow together with conventional Stokes expansion of a deepwater wave to investigate the dynamic response of a hard permeable bed to nonlinear water waves.

As for the bed form formation problem, Darwin (1884) conducted experiments on sand-ripple caused by an oscillatory moving bed and concluded that the eddies induced by a series of vortices acting on the sandy bed would probably render an unstable ripple into a stable state. Exner (1925) established a differential erosion equation for two-dimensional flow to show the change in bed elevation due to a longitudinal variation of bottom velocity. Anderson (1953) applied Exner's (1925) erosion equation to explain the mechanism of the formation of ripple. Vittori and Blondeaux (1992) further considered the shear effect to discuss the formation of brick-pattern ripple under sea waves. On the other hand, Kennedy (1963) applied an empirical sediment transport formula to govern the continuity of the porous bed and used the instability analysis of potential flow theory to obtain his famous results of dune and antidune formations in alluvial channels. Unfortunately, owing to the constraint of instability analysis, Kennedy (1963) could only find the dominant wavelength of stable bed forms instead of the whole bed forms. Hsieh et al. (2001) proposed a boundary layer correction approach with a systematic two-parameter perturbation to obtain the bed forms of dune, antidune, and flat bed under nonlinear oscillatory water waves accompanied with a constant current. Although a clearer hydraulic mechanism of formation of bed forms was found in Hsieh et al. (2001) the potential flow theory was still adopted.

Based on the foregoing comments, we found that most studies did not consider the viscous effects of homogeneous fluid flow and porous media flow. We therefore try to solve the interaction problem of laminar water waves and a steady nonuniform current acting on a hard poroelastic bed in order to reveal the importance of fluid viscosity versus pressure gradient in this study.

Formulation

Fig. 1 indicates plane waves propagating over a horizontal, infinitely thick, and homogeneous poroelastic bed accompanied with

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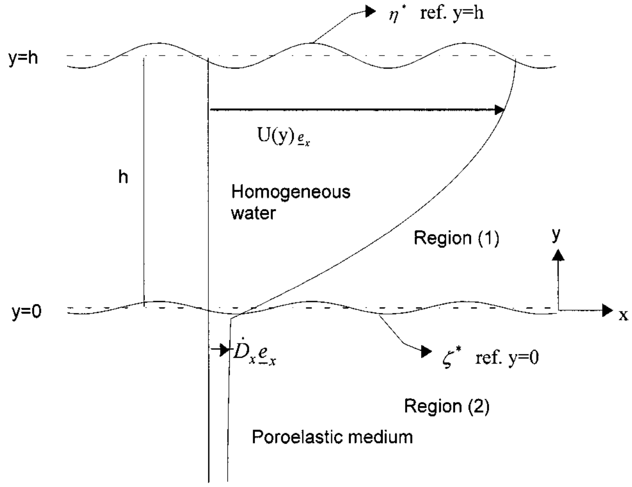

Fig. 1. Definition sketch

a steady nonuniform current. Region 1 is homogeneous water governed by laminar flow while region 2 is a semi-infinite porous medium saturated with water governed by Biot's (1962) theory of poroelasticity. The coordinates of region 1 range from $y = \zeta^*(x,t)$ to $y = h + \eta^*(x,t)$ and region 2 from $y = \zeta^*(x,t)$ to $y \to -\infty$. The symbols η^* and ζ^* represent the displacements of waves from the mean free surface $(y = h)$ and mean bed surface $(y = 0)$, respectively. Note that all symbols with an asterisk mean in the periodic motion with the time factor $e^{-i\omega t}$.

Steady Nonuniform Current

Considering a fully developed flow $(\partial/\partial x = 0)$ along an inclined plane with an angle θ, i.e., Poiseuille flow, the velocity can be found from the equation of momentum

$$\mu \frac{d^2 U(y)}{dy^2} + \rho g \sin\theta = 0 \tag{1}$$

where $U(y)$ = fluid velocity in the x direction; ρ = density of fluid; g = gravitational acceleration; and μ = dynamic viscosity of fluid.

Referring to Eq. (21) of the work of Song and Huang (2000), we can obtain

$$n_0 \mu \frac{d^2 \dot{D}_x}{dy^2} + n_0 \rho g \sin\theta = b \dot{D}_x \tag{2}$$

as the velocity distribution inside the poroelastic bed. Here, n_0 = porosity; $\dot{D}_x = x$ component of pore velocity of fluid; and

$$b = F(\kappa) \mu n_0^2 / k_p \tag{3}$$

of which $F(\kappa)$ = correction factor of frequency, whose value is unity for low frequency and k_p = coefficient of specific permeability.

The boundary conditions are

1. At the free surface:

$$\frac{dU(y)}{dy} = 0 \tag{4}$$

2. At the poroelastic bed surface: continuity of flux in the x direction

$$U(0) = n_0 \dot{D}_x \tag{5}$$

continuity of fluid stress in the x direction

$$n_0 \sigma_{xy}^{(1)} = \sigma_{xy}^{(2)}$$

where σ_{xy} is defined as Eqs. (33) and (35); the superscripts (1) and (2) denote region (1) and region (2), respectively. And the above equation could be simplified to

$$\frac{\partial U}{\partial y} = \frac{\partial \dot{D}_x}{\partial y} \tag{6}$$

after considering that the y components of homogeneous water and pore fluid are vanishing.

3. At the far field:

$$\dot{D}_x \to 0 \tag{7}$$

Based on the boundary value problem above, $U(y)$ and $\dot{D}_x$ can be solved as follows:

$$U(y) = \frac{-\rho g \sin\theta}{2\mu} y^2 + \frac{\rho g h \sin\theta}{\mu} y + n_0 \left(n_0 \sqrt{\frac{n_0 \mu}{b}} \frac{\rho g h \sin\theta}{\mu} + \frac{n_0 \rho g \sin\theta}{b} \right) \quad \text{for } y \geqslant 0 \tag{8}$$

$$\dot{D}_x(y) = n_0 \sqrt{\frac{n_0 \mu}{b}} \frac{\rho g h \sin\theta}{\mu} e^{\sqrt{(b/n_0 \mu)} y} + \frac{n_0 \rho g \sin\theta}{b} \quad \text{for } y \leqslant 0 \tag{9}$$

with h = depth of homogeneous water.

Governing Equations of Homogeneous Water

Assuming that the homogeneous water in region 1 of Fig. 1 is incompressible, and the flow is laminar flow, then the equations of continuity and momentum can be expressed, respectively, as

$$\nabla \cdot \underset{\sim}{V}^{*(1)} = 0 \tag{10}$$

$$\frac{\partial \underset{\sim}{V}^{*(1)}}{\partial t} + (\underset{\sim}{V}^{*(1)} \cdot \nabla) \underset{\sim}{V}^{*(1)} = -\frac{1}{\rho} \nabla P^{*(1)} + \nu \nabla^2 \underset{\sim}{V}^{*(1)} \tag{11}$$

where $\underset{\sim}{V}^{*(1)}$ = velocity vector of flow; $P^{*(1)}$ = perturbed pressure; and ν = kinematic viscosity of fluid. The fluid stress is

$$\sigma_{ij}^{*(1)} = -P^{*(1)} \delta_{ij} + \mu (V_{i,j}^{*(1)} + V_{j,i}^{*(1)}) \tag{12}$$

Referring to Mei (1989) or Morse and Feshbach (1978), any vector can be taken as the sum of an irrotational and a solenoidal vector, so that the velocity can be decomposed as the sum of the steady current, the irrotational part, and the rotational (but solenoidal) part, i.e.

$$\underset{\sim}{V}^{*(1)} = U(y) \underset{\sim}{e}_x + \nabla \Phi_1^{*(1)} + \nabla \times \underset{\sim}{U}^{*(1)} \tag{13}$$

Substituting Eq. (13) into Eqs. (10) and (11) and after simplification by linearization, we will have

$$\nabla^2 \Phi_1^{*(1)} = 0 \tag{14}$$

$$\frac{\partial}{\partial t} (\nabla \times \underset{\sim}{U}^{*(1)}) + U (\nabla \times \underset{\sim}{U}_{,x}^{*(1)}) = \nu \nabla^2 (\nabla \times \underset{\sim}{U}^{*(1)}) \tag{15}$$

$$-\frac{1}{\rho}P^{*(1)}=-i\omega\Phi_1^{*(1)}+U\Phi_{1,x}^{*(1)}+\frac{U_{,y}}{ik_0}\Phi_{1,y}^{*(1)}+\frac{U_{,yy}}{ik_0}(\Phi_1^{*(1)}|_{y=h}-\Phi_1^{*(1)}) \tag{16}$$

where $\Phi_1^{*(1)}$ = perturbed irrotational velocity potential function. If the motion is periodical, Eq. (15) could be simplified to

$$\nabla^2(\nabla\times\underline{U}^{(1)})+k_w^2(\nabla\times\underline{U}^{(1)})=\frac{U}{\nu}(\nabla\times\underline{U}_{,x}^{(1)}) \tag{17}$$

with

$$k_w^2=\frac{i\omega}{\nu} \tag{18}$$

Referring to Morse and Feshbach (1978), $\underline{U}^{(1)}$ can be expressed as

$$\underline{U}^{(1)}=\Phi_2^{(1)}\underline{e}_z \tag{19}$$

then Eq. (17) can be rewritten as

$$\nabla^2\Phi_2^{(1)}+\left(k_w^2-\frac{ik_0U}{\nu}\right)\Phi_2^{(1)}=0 \tag{20}$$

where k_0 = wave number of an incoming water wave, which will be found as complex.

Governing Equations of Laminar Poroelastic Media Flow

The poroelastic material in region (2) of Fig. 1 is assumed to be water saturated. The skeleton density is denoted as ρ_s. Let $\underline{\dot{d}}^*$ and $\underline{\dot{D}}^*$ = the velocity vectors of solid and water in region (2), respectively. Then the continuity equations of solid and water are

$$\frac{\partial}{\partial t}[(1-n_0)\rho_s]+\nabla\cdot[(1-n_0)\rho_s\underline{\dot{d}}^*]=0 \tag{21}$$

$$\frac{\partial}{\partial t}[n_0\rho]+\nabla\cdot[n_0\rho\underline{\dot{D}}^*]=0 \tag{22}$$

Referring to the work of Verruijt (1969), the storage equation can be obtained as

$$\frac{\partial P^{*(2)}}{\partial t}=-\frac{K}{n_0}\left[(1-n_0)\nabla\cdot\left(\frac{\partial\underline{\dot{d}}^*}{\partial t}\right)+n_0\nabla\cdot\left(\frac{\partial\underline{\dot{D}}^*}{\partial t}\right)\right] \tag{23}$$

for perturbed pressure $P^{*(2)}$. In Eq. (23), K is the bulk modulus of compressibility of fluid inside the porous bed. Referring to the work of Song and Huang (2000), the linear momentum equations of solid skeleton and fluid for the porous bed based on the theory of poroelasticity may be written as

$$\nabla\cdot\underline{\underline{\sigma}}_s^*+(1-n_0)\rho_s\underline{g}=(1-n_0)\rho_s\underline{\ddot{d}}^*-F(\kappa)\frac{\mu n_0^2}{k_p}(\underline{\dot{D}}^*-\underline{\dot{d}}^*) \tag{24}$$

$$\nabla\cdot\underline{\underline{\sigma}}^*+n_0\rho\underline{g}=n_0\rho\underline{\ddot{D}}^*+F(\kappa)\frac{\mu n_0^2}{k_p}(\underline{\dot{D}}^*-\underline{\dot{d}}^*) \tag{25}$$

where $\underline{\underline{\sigma}}_s^*$ = solid stress tensor and $\underline{\underline{\sigma}}^*$ = normal stress tensor of fluid.

If there is a steady streaming flow over the porous media, the solid displacement and fluid displacement may be separated as

$$\underline{d}^*(\underline{x},t)=\underline{d}^0(\underline{x})+\underline{d}'(\underline{x},t) \tag{26}$$

$$\underline{D}^*(\underline{x},t)=\underline{D}^0(\underline{x})+\underline{D}'(\underline{x},t) \tag{27}$$

respectively, with $|\underline{d}'|\ll|\underline{d}^0|$, $|\underline{D}'|\ll|\underline{D}^0|$. Then, from Eqs. (24) and (25), the equations for stationary deformed solid skeleton and steady streaming viscous flow are

$$\nabla\cdot\underline{\underline{\sigma}}_s^0+(1-n_0)\rho_s\underline{g}=-F(\kappa)\frac{\mu n_0^2}{k_p}\underline{\dot{D}}^0 \tag{28}$$

$$\nabla\cdot\underline{\underline{\sigma}}^0+n_0\rho\underline{g}=n_0\rho(\underline{\dot{D}}^0\cdot\nabla)\underline{\dot{D}}^0+F(\kappa)\frac{\mu n_0^2}{k_p}\underline{\dot{D}}^0 \tag{29}$$

on the other hand, the linear equations of motion of the solid and fluid for the remaining disturbance are

$$\nabla\cdot\underline{\underline{\sigma}}_s'=(1-n_0)\rho_s\underline{\ddot{d}}'-F(\kappa)\frac{\mu n_0^2}{k_p}(\underline{\dot{D}}'-\underline{\dot{d}}') \tag{30}$$

$$\nabla\cdot\underline{\underline{\sigma}}'=n_0\rho\underline{\ddot{D}}'+F(\kappa)\frac{\mu n_0^2}{k_p}(\underline{\dot{D}}'-\underline{\dot{d}}') \tag{31}$$

Assuming that the generalized Hooke's law for the solid skeleton and Newton's law for fluids are valid, stress tensors can be separated into a steady part and a perturbed part

$$\underline{\underline{\sigma}}_s^0=\underline{\underline{\tau}}_s^0-(1-n_0)P^0\underline{\underline{I}};\quad \underline{\underline{\sigma}}_s'=\underline{\underline{\tau}}_s'-(1-n_0)P'\underline{\underline{I}} \tag{32}$$

$$\underline{\underline{\sigma}}^0=n_0\underline{\underline{\tau}}^0-n_0P^0\underline{\underline{I}};\quad \underline{\underline{\sigma}}'=n_0\underline{\underline{\tau}}'-n_0P'\underline{\underline{I}} \tag{33}$$

where P' = perturbed pressure and $\underline{\underline{I}}$ = identity matrix. While the effective stresses for the solid skeleton and the shear stress of the fluid are separated into a steady part and a perturbed part

$$\underline{\underline{\tau}}_s^0=G[\nabla\underline{d}^0+(\nabla\underline{d}^0)^T]+\lambda(\nabla\cdot\underline{d}^0)\underline{\underline{I}};\quad \underline{\underline{\tau}}_s'=G[\nabla\underline{d}'+(\nabla\underline{d}')^T]+\lambda(\nabla\cdot\underline{d}')\underline{\underline{I}} \tag{34}$$

$$\underline{\underline{\tau}}^0=\mu[\nabla\underline{D}^0+(\nabla\underline{D}^0)^T]+\mu'(\nabla\cdot\underline{D}^0)\underline{\underline{I}};\quad \underline{\underline{\tau}}'=\mu[\nabla\underline{D}'+(\nabla\underline{D}')^T]+\mu'(\nabla\cdot\underline{D}')\underline{\underline{I}} \tag{35}$$

with G, λ = Lame's constants of elasticity and μ' = second fluid viscosity; superscript T denotes the transpose of matrix.

The displacement vectors of solid and fluid can be expressed as

$$\underline{d}^*=\nabla\Phi_1^{*(2)}+\nabla\Phi_2^{*(2)}+\nabla\times\underline{H}_3^{*(2)}+\nabla\times\underline{H}_4^{*(2)} \tag{36}$$

$$\underline{D}^*=\alpha_1\nabla\Phi_1^{*(2)}+\alpha_2\nabla\Phi_2^{*(2)}+\alpha_3\nabla\times\underline{H}_3^{*(2)}+\alpha_4\nabla\times\underline{H}_4^{*(2)} \tag{37}$$

In Eq. (36), $\Phi_1^{*(2)}$ and $\Phi_2^{*(2)}$ = displacement potentials of the first and the second longitudinal waves, respectively; while $\underline{H}_3^{*(2)}$ and $\underline{H}_4^{*(2)}$ = displacement vectors of the first and the second transverse waves and α_1 and α_2 = dilatational wave-induced coefficients of displacement between solid and fluid, while α_3 and α_4 are induced by transverse waves. Referring to the work of Morse and Feshbach (1978), Eqs. (36) and (37) can be rewritten as

$$\underline{H}_j^{*(2)}=\Phi_j^{*(2)}\underline{e}_z;\quad j=3,4. \tag{38}$$

Song and Huang (2000) used Eqs. (36) to (38) to simplify Eqs. (30) and (31) into four decoupled Helmholtz equations

$$\nabla^2\Phi_j^{*(2)}+k_j^2\Phi_j^{*(2)}=0;\quad j=1,2,3,4 \tag{39}$$

where the wave numbers k_j are given as Eqs. (40) and (41) and (43) and (44) in the work of Song and Huang (2000); where k_1 and k_2 = wave numbers of the first and second dilatational waves, and k_3 and k_4 = wave numbers of the first and second transverse waves. Moreover, the perturbed pore water pressure $P^{*(2)}$, the perturbed effective solid stresses $\underline{\underline{\tau}}_s^{*(2)}$, and the perturbed effective fluid stresses $\underline{\underline{\tau}}^{*(2)}$ can be, respectively, expressed as

$$P^{*(2)} = -\frac{K}{n_0}[(1-n_0+\alpha_1 n_0)k_1^2\Phi_1^{*(2)} + (1-n_0+\alpha_2 n_0)k_2^2\Phi_2^{*(2)}] \tag{40}$$

$$\underline{\underline{\tau}}_s^{*(2)} = G[2\nabla\nabla\Phi_1^{*(2)} + 2\nabla\nabla\Phi_2^{*(2)} + \nabla(\nabla\times\underline{H}_3^{*(2)}) + \{\nabla(\nabla\times\underline{H}_3^{*(2)})\}^T + \nabla(\nabla\times\underline{H}_4^{*(2)}) + \{\nabla(\nabla\times\underline{H}_4^{*(2)})\}^T] \tag{41}$$

$$\underline{\underline{\tau}}^{*(2)} = \mu[2\alpha_1\nabla\nabla\Phi_1^{*(2)} + 2\alpha_2\nabla\nabla\Phi_2^{*(2)} + \alpha_3(\nabla(\nabla\times\underline{H}_3^{*(2)}) + \{\nabla(\nabla\times\underline{H}_3^{*(2)})\}^T) + \alpha_4(\nabla(\nabla\times\underline{H}_4^{*(2)}) + \{\nabla(\nabla\times\underline{H}_4^{*(2)})\}^T)] \tag{42}$$

Compared with Biot's potential poroelastic theory, the laminar model possesses one more transverse wave due to the viscous effect.

Boundary Conditions

The following three boundaries: (1) free surface $\lfloor y=h+\eta^*(x,t)\rfloor$, (2) channel bed interface $\lfloor y=\zeta^*(x,t)\rfloor$, and (3) deep far field of porous bed $[y\rightarrow-\infty]$ must satisfy boundary conditions.

1. At the free surface
   - Dynamic boundary condition

$$\frac{\partial\Phi_1^{*(1)}}{\partial t} + U_{\text{ref}}\Phi_{1,x}^{*(1)} + g\eta^* = 0 \tag{43}$$

   where U_{ref}, i.e., $U(h)$, = the reference velocity.
   - Kinematic boundary condition

$$\frac{\partial\Phi_1^{*(1)}}{\partial y} = \frac{\partial\eta^*}{\partial t} + U_{\text{ref}}\frac{\partial\eta^*}{\partial x} \tag{44}$$

2. At the bed surface
   - Continuity of vertical component of flow velocity

$$n_0(\dot{D}_y^* - \dot{d}_y^*) = (V_y^{*(1)} - \dot{d}_y^*) \tag{45}$$

   - Continuity of vertical component of fluid stresses

$$n_0\sigma_{yy}^{*(1)} = \sigma_{yy}^{*(2)} \tag{46}$$

   - Continuity of horizontal component of total stresses

$$(1-n_0)\sigma_{xy}^{*(1)} = \sigma_{sxy}^{*(2)} \tag{47}$$

   - Continuity of vertical component of total stresses

$$(1-n_0)\sigma_{yy}^{*(1)} = \sigma_{syy}^{*(2)} \tag{48}$$

   - Continuity of horizontal component of flow velocity

$$n_0(\dot{D}_x^* - \dot{d}_x^*) = (V_x^{*(1)} - \dot{d}_x^*) \tag{49}$$

   - Continuity of horizontal component of fluid stresses

$$n_0\sigma_{xy}^{*(1)} = \sigma_{xy}^{*(2)} \tag{50}$$

3. Far field $(y\rightarrow-\infty)$
   As $y\rightarrow-\infty$, no perturbed displacements exist, i.e.

$$\underline{d}^*,\ \underline{D}^* \rightarrow \underline{0} \tag{51}$$

Considering the kinematics of the porous bed interface, we have

$$\frac{\partial\zeta^*}{\partial t} = \frac{\partial\underline{d}^*}{\partial t}\cdot\left(-\frac{\partial\zeta^*}{\partial x},1\right) \tag{52}$$

and Eq. (52) will be used to solve ζ^*.

Note that governing equations (14) and (39), pressure and effective stresses equations (16), (40), and (41), together with boundary conditions (43)–(51) form the complete boundary value problem of the present study.

Nondimensionalizations and Multiparameter Perturbation Expansion

Chen et al. (1997) found that because the wavelength of the second longitudinal wave in the porous bed is much shorter than that of the water wave when the bed material is soft, i.e., $m \ll 1$, according to Huang and Song's (1993) definition of $m=(2G+\lambda)n_0/K$, stiffness ratio of solid and fluid, the conventional Stokes expansion based on $k_0 a$ is invalid for the soft bed material problem since there exists a boundary layer inside the soft bed material. That is why Hsieh et al. (2001) needed to propose a boundary layer correction approach for a soft bed material problem. On the contrary, the wavelength of the second longitudinal wave in the porous bed is longer than that of the water wave when the bed material is hard, i.e., $m \gg 1$, and there is no boundary layer inside the porous bed material. But if the viscosity of water is considered, the Stokes boundary layer exists in the region of homogeneous water while a boundary layer exists inside the hard bed material. After a tedious but straightforward analysis of order of magnitude for each dependent variable, we herewith define

$$\varepsilon_1 = k_0 a;\quad \varepsilon_2 = k_0/k_\omega;\quad \varepsilon_4 = k_0/k_4 \tag{53}$$

as the small-valued parameters for a hard bed material problem, and the dimensionless variables are selected as

$$\hat{x} = k_0 x;\quad \hat{y} = k_0 y \tag{54}$$

$$\bar{Y} = \hat{y}/\varepsilon_2$$

(only inside the boundary layer of homogeneous water) (55)

$\bar{\bar{Y}} = \hat{y}/\varepsilon_4$ (only inside the boundary layer of poroelastic bed) (56)

$$\hat{t} = \sqrt{gk_0}\,t;\quad \hat{\omega} = \omega/\sqrt{gk_0} \tag{57}$$

$$\eta_{10}^* = k_0\eta^*;\quad \hat{\zeta}^* = (K/\rho g a^2)\zeta^* \tag{58}$$

$$\phi_{10}^{*\mathrm{I}} = (k_0^2/\sqrt{gk_0})\Phi_1^{*(1)} \tag{59}$$

$$\phi_{10}^{*\mathrm{II}} = (k_0^2/\varepsilon_2\sqrt{gk_0})\Phi_2^{*(1)} \tag{60}$$

$$\phi_{10}^{*[1]} = \mathrm{Re}(e^{k_0 h})k_0^2\Phi_1^{*(2)} \tag{61}$$

$$\phi_{10}^{*[2]} = \mathrm{Re}(e^{k_0 h})k_0^2\Phi_2^{*(2)} \tag{62}$$

$$\phi_{10}^{*[3]} = \mathrm{Re}(e^{k_0 h})k_0^2\Phi_3^{*(2)} \tag{63}$$

$$\phi_{10}^{*[4]} = [\mathrm{Re}(e^{k_0 h})k_0^2\varepsilon_2/\varepsilon_4^2]\Phi_4^{*(2)} \tag{64}$$

$$\hat{P}^{*(1)} = (k_0/\rho g)P^{*(1)};\quad \hat{P}^{*(2)} = (k_0/\rho g)P^{*(2)} \tag{65}$$

$$\hat{U}^* = (k_0/\sqrt{gk_0})U^* \tag{66}$$

All the symbols of variables on the left-hand side of Eqs. (54)–(66) are dimensionless, but those on the right-hand side are dimensional. Note that since multiple length scales in vertical direction are needed (see the discussion in the work of Hsieh et al. 2001), $\hat{y}$, $\overline{Y}$, and $\bar{\bar{Y}}$ are proposed for the boundary layer correction approach.

Applying the multiparameter perturbation expansion, in order to grasp the least order of physical characteristics, we can write all the velocity potentials and the displacement potentials as

$$\Phi^{*(1)} = \varepsilon_1 \phi_{10}^{*\mathrm{I}} + \cdots + \varepsilon_1 \varepsilon_2 \phi_{10}^{*\mathrm{II}} \tag{67}$$

$$\Phi^{*(2)} = \varepsilon_1 (\phi_{10}^{*[1]} + \phi_{10}^{*[2]} + \phi_{10}^{*[3]}) + \cdots + (\varepsilon_1 \varepsilon_4^2/\varepsilon_2) \phi_{10}^{*[4]} \tag{68}$$

if $\|\varepsilon_1\|$, $\|\varepsilon_2\|$, and $\|\varepsilon_4\|$ are smaller than unity.

For a periodic motion with the frequency ω accompanied with a steady nonuniform flow, the aforementioned variables $[\ ]^*(\underset{\sim}{R},t)$ can be written as $[\ ](\underset{\sim}{R})e^{-i\omega t}$, where $\underset{\sim}{R}$ is the position vector. Let the given incoming-wave amplitude before being distributed by the porous bed be a (i.e., $\hat{\eta}_{10} = e^{i\hat{x}}$) with a wave number k_0. If both $|\eta^*|$ and $|\zeta^*|$ are much smaller than the relative wavelengths, it is more convenient to shift the boundary conditions at free surface, $y = h + \eta^*(x,t)$, and porous bed interface, $y = \zeta^*(x,t)$, to $y = h$ and $y = 0$ first before solving the boundary value problem. Thus after performing Taylor series expansions at the free surface and at the channel bed interface, respectively, the boundary value problem of leading order without the time factor, $e^{-i\omega t}$, is obtained in the following.

Without Boundary Layer

The governing equation of homogeneous water is

$$\hat{\nabla}^2 \phi_{10}^1 = 0 \tag{69}$$

The governing equation of poroelastic material is

$$\hat{\nabla}^2 \phi_{10}^{[j]} + \frac{k_j^2}{k_0^2} \phi_{10}^{[j]} = 0; \quad j = 1,2,3 \tag{70}$$

The boundary conditions are

1. At the mean free surface
   - Dynamic boundary condition

$$-i\hat{\omega}\phi_{10}^{\mathrm{I}} + \hat{U}_{\mathrm{ref}} \phi_{10,\hat{x}}^{\mathrm{I}} + \eta_{10} = 0 \tag{71}$$

   - Kinematic boundary condition

$$\phi_{10,\hat{y}}^{I} = -i\hat{\omega}\eta_{10} + \hat{U}_{\mathrm{ref}} \eta_{10,\hat{x}} \tag{72}$$

2. At the mean bed surface
   - Continuity of vertical component of flow velocity

$$\phi_{10,\hat{y}}^{\mathrm{I}} = -\frac{i\hat{\omega}}{\mathrm{Re}(e^{k_0 h})} [(1 - n_0 + \alpha_1 n_0) \phi_{10,\hat{y}}^{[2]} + (1 - n_0 + \alpha_2 n_0) \phi_{20,\hat{y}}^{[2]} - (1 - n_0 + \alpha_3 n_0) \phi_{30,\hat{x}}^{[2]}] \tag{73}$$

   - Continuity of vertical component of fluid stresses

$$i\hat{\omega}\phi_{10}^{\mathrm{I}} = \frac{K\,\mathrm{Re}(k_0)}{n_0 \rho g\,\mathrm{Re}(e^{k_0 h})} \left[(1 - n_0 + \alpha_1 n_0) \frac{k_1^2}{k_0^2} \phi_{10}^{[2]} + (1 - n_0 + \alpha_2 n_0) \frac{k_2^2}{k_0^2} \phi_{20}^{[2]} \right] \tag{74}$$

   - Continuity of horizontal component of total stresses

$$2\phi_{10,\hat{x}\hat{y}}^{[2]} + 2\phi_{20,\hat{x}\hat{y}}^{[2]} + \phi_{30,\hat{y}\hat{y}}^{[2]} - \phi_{30,\hat{x}\hat{x}}^{[2]} = 0 \tag{75}$$

   - Continuity of vertical component of total stresses

$$(\phi_{10,\hat{y}\hat{y}}^{[2]} + \phi_{20,\hat{y}\hat{y}}^{[2]} - \phi_{30,\hat{x}\hat{y}}^{[2]}) - \frac{\lambda}{2G} \left(\frac{k_1^2}{k_0^2} \phi_{10}^{[2]} + \frac{k_2^2}{k_0^2} \phi_{20}^{[2]} \right) = 0 \tag{76}$$

3. Far field ($y \to -\infty$)

$$\phi_j^{[2]} \to 0; \quad j = 1,2,3 \tag{77}$$

Inside Boundary Layer

The governing equations inside the boundary layer are

$$\phi_{10,\overline{YY}}^{\mathrm{II}} + \left(1 - \frac{\hat{U}}{\hat{\omega}}\right) \phi_{10}^{\mathrm{II}} = 0 \tag{78}$$

$$\phi_{10,\bar{\bar{Y}}\bar{\bar{Y}}}^{[4]} + \phi_{10}^{[4]} = 0 \tag{79}$$

where ϕ_{10}^{II} and $\phi_{10}^{[4]}$ = nondimensional rotational potential function of flow and the displacement function of the second rotational inside the poroelastic bed.

The boundary conditions inside the boundary layer are

1. Continuity of horizontal component of flow velocity

$$\phi_{10,\hat{x}}^{\mathrm{I}} + \phi_{10,\overline{Y}}^{\mathrm{II}} = -\frac{i\hat{\omega}}{\mathrm{Re}(e^{k_0 h})} [(1 - n_0 + \alpha_1 n_0) \phi_{10,\hat{x}}^{[1]} + (1 - n_0 + \alpha_2 n_0) \phi_{10,\hat{x}}^{[2]} + (1 - n_0 + \alpha_3 n_0) \phi_{10,\hat{y}}^{[3]}] \tag{80}$$

2. Continuity of horizontal component of fluid stresses

$$\phi_{10,\overline{YY}}^{\mathrm{II}} \approx -i\hat{\omega} \frac{\alpha_4}{\mathrm{Re}(e^{k_0 h})} \phi_{10,\bar{\bar{Y}}\bar{\bar{Y}}}^{[4]} \tag{81}$$

General Solutions

After the time factor $e^{-i\omega t}$ is omitted, the given incoming water wave profile with magnitude a is

$$\eta_{10}(x) = a e^{i k_0 x} \quad (0 < x < \infty) \tag{82}$$

With the input of incoming water wave, the leading order of the aforementioned boundary value problem can be solved by the method of separation of variables. The dimensional solutions of the velocity potentials and the displacement potentials are

$$\Phi_1^{(1)} = \{A_0 \cosh[k_0(h - y)] + B_0 \sinh[k_0(h - y)]\} e^{i k_0 x} \tag{83}$$

$$\Phi_2^{(1)} = \varepsilon_2 \frac{a_0}{k_0^2} e^{i\sqrt{1 - \hat{U}/\hat{\omega}}(k_w/k_0) y} e^{i k_0 x} \tag{84}$$

with $k_w^2 = i\omega/\nu$ and $\mathrm{Im}(\sqrt{1 - \hat{U}/\hat{\omega}} k_w/k_0) > 0$;

$$\Phi_j^{(2)} = \frac{a_j}{k_0^2} e^{K_j y} e^{i k_0 x}; \quad j = 1,2,3 \tag{85}$$

with $K_j^2 = k_0^2 - k_j^2$, $\mathrm{Re}(K_j) > 0$; $\quad j = 1,2,3$

$$\Phi_4^{(2)} = \frac{\varepsilon_4^2}{\varepsilon_2} \frac{a_4}{k_0^2} e^{-i(k_4/k_0) y} e^{i k_0 x} \tag{86}$$

with $\mathrm{Im}(k_4/k_0) > 0$.

Based on Eqs. (83)–(86), there are seven unknown coefficients. The coefficients A_0, B_0, a_1, a_2, and a_3 can be solved straightforwardly outside the boundary layer, while a_0 and a_4

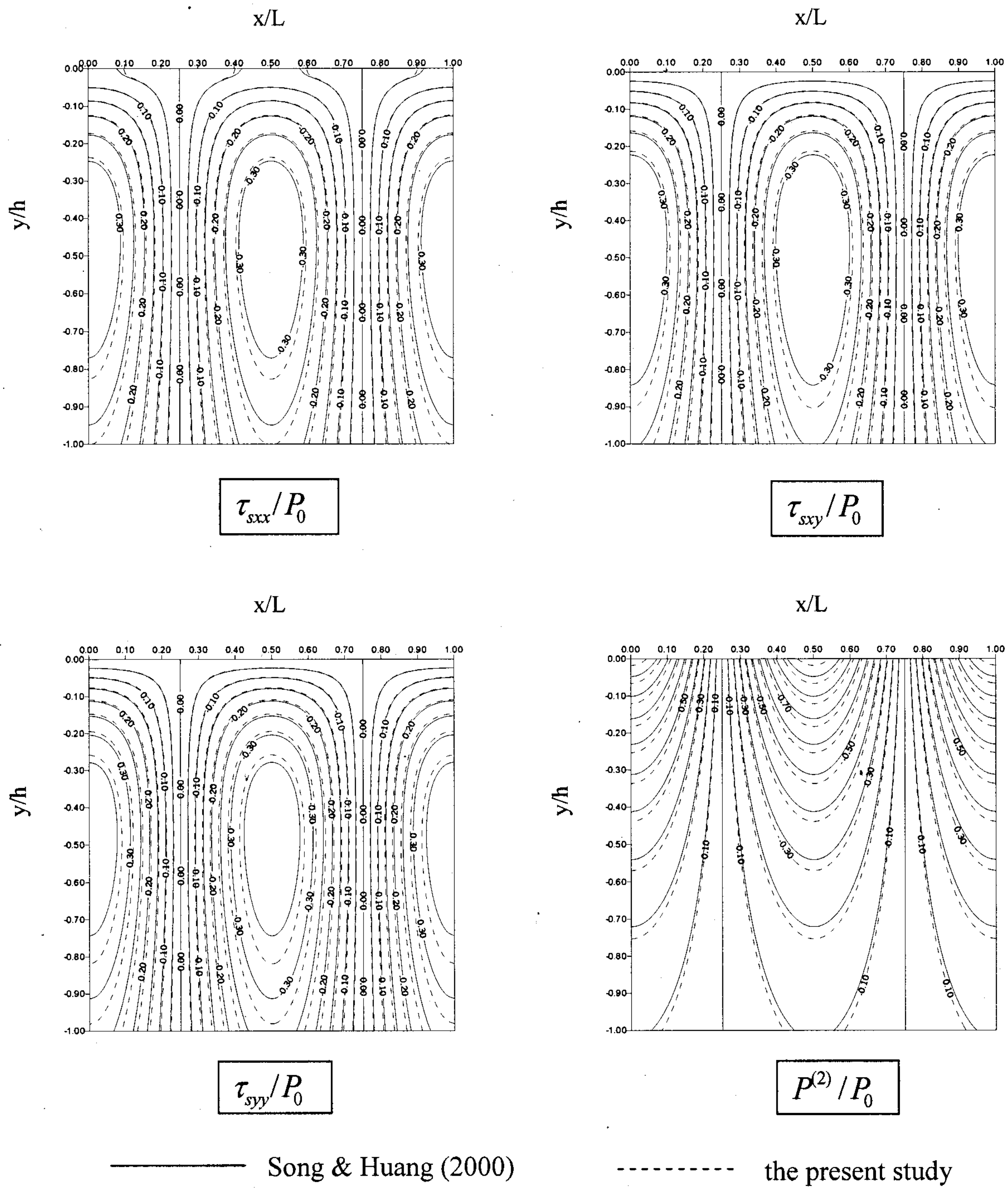

Fig. 2. Variation of wave-induced effective stresses and pore water pressures

need to be solved by the boundary layer correction approach inside the boundary layer. The results of the coefficients are as follows:

$$A_0 = -ig\eta_0/c_0\omega \tag{87}$$

$$B_0 = (ig\eta_0/c_0\omega)\left\{\frac{f}{h} - \left[(1-n_0+\alpha_1 n_0)H_2\left(\frac{k_1^2}{k_0^2}\right)(-h\sinh\xi + f\cosh\xi) - (1-n_0+\alpha_2 n_0)H_1\left(\frac{k_2^2}{k_0^2}\right)(-h\sinh\xi + f\cosh\xi)\right]\Big/(Dh)\right\} \tag{88}$$

$$a_1 = \frac{H_2}{D}(-h\sinh\xi + f\cosh\xi)\,\frac{n_0\rho g\eta_0}{c_0 K} \tag{89}$$

$$a_2 = -\frac{H_1}{D}(-h\sinh\xi + f\cosh\xi)\,\frac{n_0\rho g\eta_0}{c_0 K} \tag{90}$$

$$a_3 = -L_1 a_1 - L_2 a_2 \tag{91}$$

$$a_0 = -\frac{\omega}{\sqrt{1-k_0U/\omega}}[ia_1(1-n_0+\alpha_1 n_0)k_0 + ia_2(1-n_0+\alpha_2 n_0)k_0 + a_3(1-n_0+\alpha_3 n_0)K_3] - k_0^3[A_0\cosh(\xi) + B_0\sinh(\xi)] \tag{92}$$

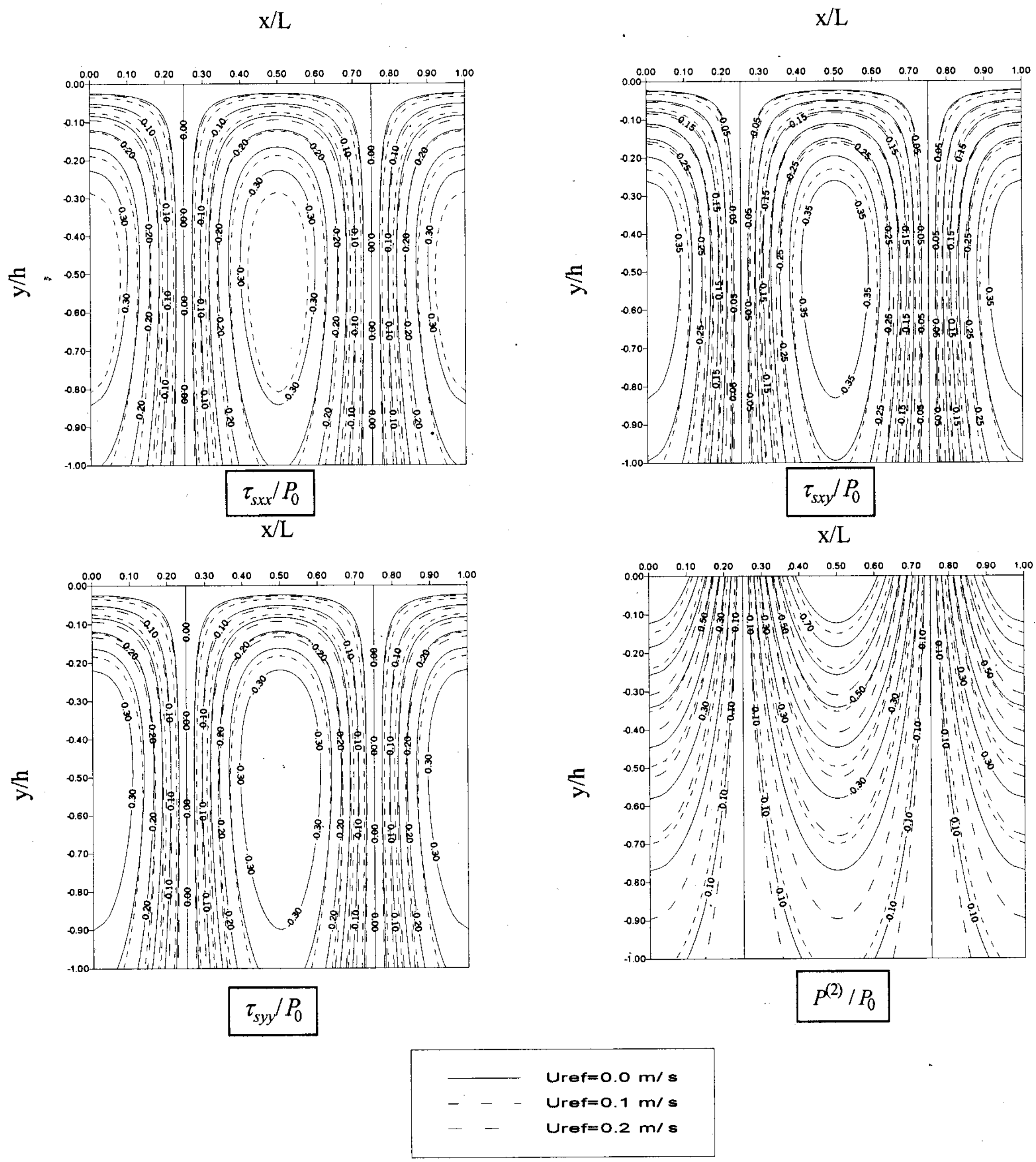

Fig. 3. Variation of effective stresses and pore water pressures under different currents

$$a_4 = i\,\frac{(1-k_0U/\omega)a_0}{\omega\alpha_4} \tag{93}$$

$$c_0 = 1-\frac{k_0U_{\mathrm{ref}}}{\omega} \tag{94}$$

$$f=f(U)=\cosh\xi-\frac{Uk_0}{\omega}\cosh\xi-\frac{U_{,y}}{\omega}\sinh\xi+\frac{U_{,yy}}{\omega k_0}(1-\cosh\xi) \tag{95}$$

$$q=q(U)=\sinh\xi-\frac{Uk_0}{\omega}\sinh\xi-\frac{U_{,y}}{\omega}\cosh\xi+\frac{U_{,yy}}{\omega k_0}\sinh\xi \tag{96}$$

$$L_1=2i(K_1/k_0)/(1+K_3^2/k_0^2) \tag{97}$$

$$L_2=2i(K_2/k_0)/(1+K_3^2/k_0^2) \tag{98}$$

$$H_1=(K_1^2/k_0^2)-2(K_1K_3/k_0^2)/(1+K_3^2/k_0^2)-\frac{\lambda}{2G}(k_1^2/k_0^2) \tag{99}$$

$$H_2=(K_2^2/k_0^2)-2(K_2K_3/k_0^2)/(1+K_3^2/k_0^2)-\frac{\lambda}{2G}(k_2^2/k_0^2) \tag{100}$$

$$\begin{aligned}D=&(1-n_0+\alpha_1n_0)H_2[(k_1^2/k_0^2)\cosh\xi+(n_0\rho\omega^2/Kk_0^2)(K_1/k_0)h]\\&-(1-n_0+\alpha_2n_0)H_1[(k_2^2/k_0^2)\cosh\xi+(n_0\rho g\omega^2/Kk_0^2)\\&\times(K_2/k_0)h]+i(1-n_0+\alpha_3n_0)(L_1H_2-L_2H_1)\\&\times(n_0\rho\omega^2/Kk_0^2)h\end{aligned} \tag{101}$$

$$\xi=k_0h \tag{102}$$

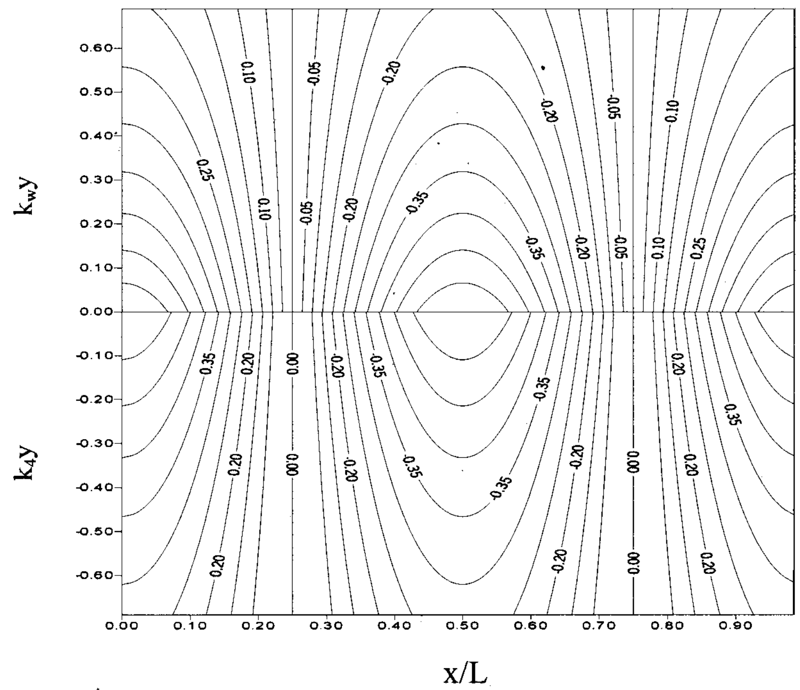

Fig. 4. Distribution of vorticity in homogeneous water (upper part) and pore water (lower part)

The dispersion relation, which is used to find the complex wave number k_0, can be obtained

$$\sinh^2 \xi - q^2 r T = q(r \cosh \xi - fT) + (\cosh^2 \xi - f \cosh \xi) \tag{103}$$

where

$$r = r(\omega, U_{\mathrm{ref}}) = \frac{\omega^2}{g k_0} - \frac{2 U_{\mathrm{ref}} \omega}{g} + \frac{U_{\mathrm{ref}}^2 k_0}{g} \tag{104}$$

$$T = (n_0 \rho \omega^2 / K k_0^2) / (T_1 / T_2) \tag{105}$$

$$T_1 = (1 - n_0 + \alpha_1 n_0) H_2 K_1 / k_0 - (1 - n_0 + \alpha_2 n_0) H_1 K_2 / k_0 + i(1 - n_0 + \alpha_3 n_0)(L_1 H_2 - L_2 H_1) \tag{106}$$

$$T_2 = (1 - n_0 + \alpha_1 n_0) H_2 k_1^2 / k_0^2 - (1 - n_0 + \alpha_2 n_0) H_1 k_2^2 / k_0^2 \tag{107}$$

After solving the displacement potentials, all the other variables can be obtained.

Results and Comments

Since the present work takes the viscosity of water into account, there exists a boundary layer of the homogeneous water near the bed surface. And since the second transverse wave inside the poroelastic bed decays very quickly in the vertical direction near the interface, a multiparameter expansion based on $\varepsilon_2 = k_0 / k_w$ and $\varepsilon_4 = k_0 / k_4$ needs to be proposed for the boundary layer correction.

The present laminar model adopts the values of each parameter as $K = 2.3 \times 10^9$ N/m^2, $\mu = 0.001$ N s/m^2, $h = 2.0$ m, $a = 0.2$ m, $T = 2.0$ s, $n_0 = 0.4$, $\rho = 2{,}650$ kg/m^3, $k_p = 1.0$

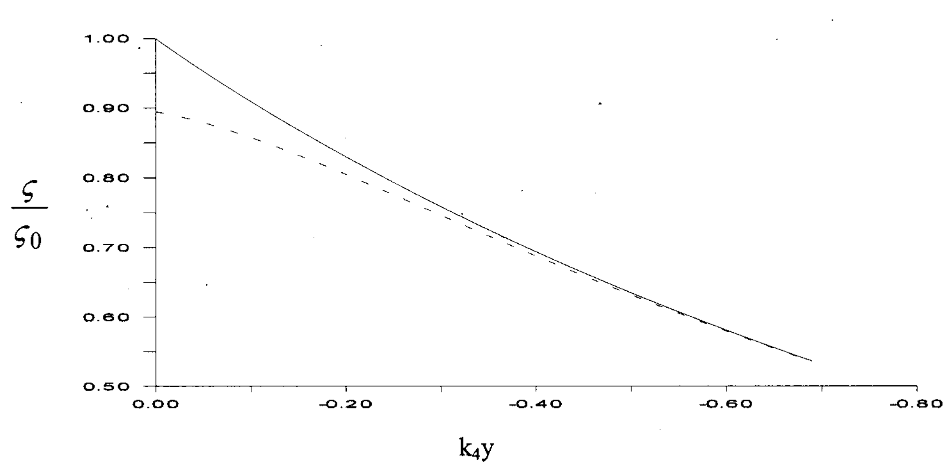

Fig. 5. Weight of vorticity induced by first transverse wave and all waves inside porous bed

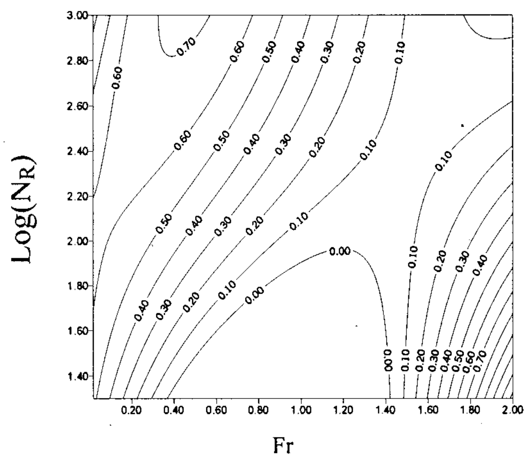

Fig. 6. Contour lines of parameter γ's

$\times 10^{-12}$ m^2, $G=5.0\times 10^{10}$ N/m^2, and $\lambda=1.0\times 10^{11}$ Ns/m^2, which is constrained to $O(\varepsilon_4)<O(\varepsilon_2)<O(\varepsilon_1)$ and $2G+\lambda \geqslant O(\mu\omega/k_p)$. When the surface velocity $U_{\text{ref}}=0$, the present results obtained by the boundary layer correction approach are compared with those of Song and Huang (2000) as shown in Fig. 2. In Fig. 2, L is the wavelength of water and P_0 is the perturbed water pressure on the mean bed surface. It is clear that the dynamic response is close to that of Song and Huang (2000) near the bed surface; however, they are slightly different at a deeper location. This is because the present study is merely the result of leading order. If higher order perturbation expansion is performed, the results are supposed to fit better.

To show the current effect, $U_{\text{ref}}=0$, $U_{\text{ref}}=0.1$ m/s, and $U_{\text{ref}}=0.2$ m/s are chosen to add into the simulation. The results due to the interaction of water waves and current are shown in Fig. 3. In the figure, it shows the solid stresses and pore water pressure under the three different currents. They are getting larger when the current effect is getting stronger. The displacements of porous bed also fluctuate more drastically with increasing current effect.

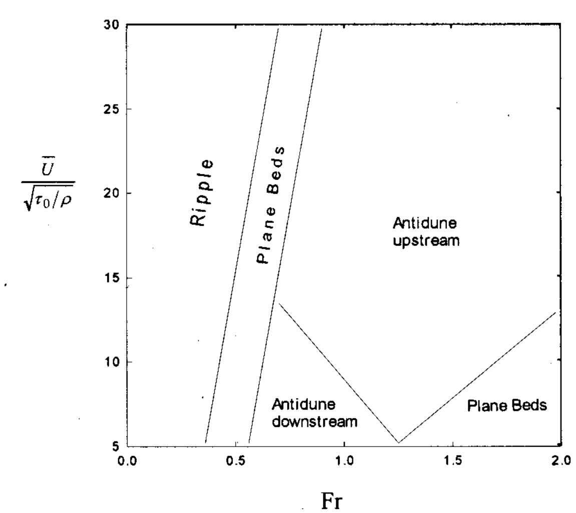

Fig. 7. Distribution of various bed forms [after Englund and Hansen (1966) in Allen John (1968)]

Since the laminar flow will induce the vorticity inside the porous media, denoted as ς, due to the unsteady perturbed velocity, we can use the following equation:

$$\varsigma=\alpha_3\nabla\times(\nabla\times \underline{H}_3^{(2)})+\alpha_4\nabla\times(\nabla\times \underline{H}_4^{(2)}) \qquad (108)$$

to calculate the vorticity, and its distribution is shown in Fig. 4 [upper vorticity, $\varsigma=\nabla\times(\nabla\times \underline{V}^{(1)})$, is due to unsteady flow within homogeneous water] with $U_{\text{ref}}=0.2$ m/s. Although the vorticity distributions of homogeneous water and porous material bed are not continuous at the interface, they will fit better if higher order perturbation expansion is performed. Note that the vorticity inside the porous bed is mainly contributed by the first transverse wave, while relatively little by the second transverse wave. This can be found from Fig. 5, which indicates that only 10% of vorticity contributed from the second transverse wave in this example. This phenomenon is due to the fact that the first transverse wave is arising from the displacement of fluid induced by the tangential stresses of porous bed, while the second one is due to the viscosity of pore water and the coupled effect of shear stresses and permeability. The latter effect is significant inside the boundary layer, but it dissipates very quickly outside the boundary layer.

In addition, defining a parameter γ as the ratio of fluid shear stress to pore pressure on the bed surface under the water waves and current effects, various flows are simulated to show their mechanism of bed form formation. Every simulation can be denoted as the Reynolds number and Froude number, i.e., $(\mathsf{R}_N,\mathsf{F}_r)$. In the study, Reynolds number is defined as $\mathsf{R}_N=U_{\text{ref}}h/\nu$, and Froude number is defined as $\mathsf{F}_r=U_{\text{ref}}/\sqrt{gh}$. Fig. 6 shows the contour lines of γ's under various flows, which indicates that the values of γ's are large when F_r's are less than 0.5 or greater than 1.5; otherwise, γ's are small. The results have the same trend as that of the work of Engelund and Hansen (1966) [referring to Allen John (1968)] as shown in Fig. 7. From Figs. 6 and 7, we can conclude that the formation mechanism of antidune and plane bed is dominated by Froude number, i.e., pressure gradient is important; while that of ripple is dominated by Reynolds number, i.e., viscosity and shear effect are important. Since the formation mechanism of ripple is mainly due to the effects of the fluid viscosity and the shear stress near the fluid/porous bed interface, this bed form, ripple, cannot be produced in the work of Hsieh et al. (2001) which simulated the bed forms caused by water waves accompanied with a constant current by potential theory.

Conclusions

The present study investigates the interaction of oscillatory laminar water waves and a steady nonuniform current passing over a hard poroelastic bed. Since the viscosity of water is involved, the second transverse wave of the porous media is induced and makes the problem more complicated. Based on the fact that there exist two Stokes boundary layers near the porous bed surface, multiple length scales are applied, and thus a multiparameter perturbation expansion is proposed. The boundary layer correction approach makes the analytical solution of water waves and current passing over a hard poroelastic bed possible. The complete solution is composed of solving the problem without boundary layer first, and then adding the boundary-layer correction. The results of different shear stress/pressure ratios, which reveal a trend similar to Engelund and Hansen (1966) (Allen John 1968), indicate the hydraulic mechanism of bed form formation under different flows, and will be helpful to realize the cause of ripple formation. Finally, since the boundary-layer correction can be solved uncoupled by this approach, it is expected that it will be very helpful in computation for further studies.

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