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Centers and medians of distance-hereditary graphs

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Abstract

A graph is distance-hereditary if the distance between any two vertices in a connected induced subgraph is the same as in the original graph. In this paper, we study metric properties of distance-hereditary graphs. In particular, we determine the structures of centers and medians of distance-hereditary and related graphs. The relations between eccentricity, radius, and diameter of such graphs are also investigated.

Keywords: Distance; Eccentricity; Diameter; Radius; Center; Median; Distance-hereditary graph; Chordal graph; Ptolemaic graph

1. Introduction

This paper investigates metric properties of centers and medians of distance-hereditary graphs.

Suppose $G=(V,E)$ is a graph with vertex set V and edge set E. The *distance* $d_G(x,y)$ or $d(x,y)$ between two vertices x and y in the graph G is the minimum number of edges of an x–y path in G. The *eccentricity* $e_G(v)$ of a vertex v in G is $\max_{x\in V} d(v,x)$. The *diameter* $\mathrm{diam}(G)$ of G is the largest eccentricity of a vertex in G, and the *radius* $\mathrm{rad}(G)$ is the smallest. The *center* of G is the set

$$C(G)=\{v\in V:\ e_G(v)\leqslant e_G(x) \text{ for all } x\in V\}.$$

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The *distance sum* $D_G(v)$ of v is $\sum_{x\in V} d_G(v,x)$. The *median* of G is the set

$$M(G)=\{v\in V:\ D_G(v)\leqslant D_G(x) \text{ for all } x\in V\}.$$

Suppose w is a real-valued function on V. The *w-distance sum* $D_{G,w}(v)$ of v is $\sum_{x\in V} d_G(v,x)w(x)$. The *w-median* of G is the set

$$M_w(G)=\{v\in V:\ D_{G,w}(v)\leqslant D_{G,w}(x) \text{ for all } x\in V\}.$$

The *local w-median* of G is the set

$$LM_w(G)=\{v\in V:\ D_{G,w}(v)\leqslant D_{G,w}(x) \text{ for all } x \text{ adjacent to } v\}.$$

We very often also call the subgraph induced by the center (respectively, median, w-median, local w-median) simply the center (respectively, median, w-median, local w-median) of the graph.

The problem of determining shapes of centers and medians for different classes of graphs have been extensively studied in the literature, see Refs. [1,3–5,6,10,11,14,15,18–42]. The earliest such kind of result due to Jordan [18] is that the center or the median of a tree is either a single vertex or two adjacent vertices. Slater [30,31] examined the structure of variations of centers for trees. Proskurowski [26] proved that the center of a maximal outplanar graph is one of seven special graphs. As a generalization, he [27] found all possible centers of 2-trees and showed that the center of a 2-tree is biconnected. Laskar and Shier [19] proved that the center of a connected chordal graph is connected. Chang [5] showed that the center of a connected chordal graph is distance invariant and biconnected. He also gave a characterization of a biconnected chordal graph of diameter 2 and radius 1 to be the center of some chordal graph. Yushmanov [38,40] showed that the center of a connected chordal graph is m-convex and is either a complete graph or its connectivity is no smaller than the connectivity of the block in which it lies. Soltan and Chepoi [36] proved that the center of a connected chordal graph has diameter at most 3.

Slater [31] studied the structure of various types of medians for trees. He [32] also proved that for every graph H there exists a graph G whose median is H, and that the median of a 2-tree is isomorphic to K_1, K_2 or K_3. Yushmanov [41] and Nieminen [25] showed that the median of a Ptolemaic graph is a complete graph. Lee and Chang [20] proved that the w-median of a connected strongly chordal graph is a complete graph when the function w is positive. Wittenberg [37] proved that for any chordal graph the local w-median coincides with the w-median if a certain neighborhood condition holds.

Many results on centers and medians are on graphs with tree-like structures. In this paper, we focus on distance-hereditary graphs, which include trees and cographs. The paper is organized as follows. In Section 2, we survey some properties of distance-hereditary graphs. We also introduce the concept of concavity of a path, and derive two lemmas that are useful in this paper. Section 3 shows that the center of a distance-hereditary graph (respectively, bipartite distance-hereditary graph) is either a connected graph with diameter at most 3 or a cograph (respectively, an independent set of G). Section 4 shows that the w-median of a distance-hereditary graph with a positive function w is a cograph, and the w-median of a bipartite distance-hereditary graph G with a positive function w is either a connected cograph or an independent set of G. In Section 5, we show that the w-median of a distance-hereditary graph with a positive function w "nearly" coincides with its local w-median.

2. Preliminaries of distance-hereditary graphs

This section gives a brief introduction to distance hereditary and related graphs. We also introduce the concept of concavity of a path, and give two related lemmas that are useful in the paper.

Suppose S is a vertex subset of a graph $G=(V,E)$. Denote $G[S]$ the subgraph of G induced by S. The *deletion* of S from G, denoted by $G-S$, is the induced subgraph $G[V-S]$. The *neighborhood* $N_G(v)$ or $N(v)$ of a vertex v is the set of all vertices adjacent to v. An *induced* (or *chordless*) *path* is a path $v_1, v_2, \ldots, v_n$ in which v_i is not adjacent to v_j whenever $|i-j| \neq 1$. A vertex subset S of a graph G is called *m-convex* if all the vertices of all the induced path joining vertices of S lie in S. Two vertices x and y are *connected* if there is an x–y path in G; otherwise they are *disconnected*. The *distance* between two vertex subsets A and B is $d_G(A,B)=\min\{d_G(a,b)\colon a\in A$ and $b\in B\}$.

The *hanging* h_u of a connected graph $G=(V,E)$ at a vertex $u\in V$ is the collection of sets $L_0(u), L_1(u), \ldots, L_t(u)$ (or $L_0, L_1, \ldots, L_t$ if there is no ambiguity), where $t=\max_{v\in V} d_G(u,v)$ and $L_i(u)=\{v\in V\colon d_G(u,v)=i\}$ for $0\leqslant i\leqslant t$. $L_i(u)$ is called the *level* i of the hanging h_u. For any $1\leqslant i\leqslant t$ and any vertex $v\in L_i$, let $N'(v)=N(v)\cap L_{i-1}$.

A graph G is *distance hereditary* if each connected induced subgraph F of G has the property that $d_F(u,v)=d_G(u,v)$ for every pair of vertices u and v in F. Distance-hereditary graphs were introduced by Howorka [16]. The characterizations and recognitions of distance-hereditary graphs have been studied in [2,8,13,16]. A graph is *chordal* if every cycle of length greater than three has a chord. A *cograph* is a graph containing no induced path of four vertices, see [7]. A graph is *Ptolemaic* if for any four vertices x, y, z, w of it, the Ptolemy inequality $\mathrm{d}(x,y)\mathrm{d}(z,w)\leqslant \mathrm{d}(x,z)\mathrm{d}(y,w)+\mathrm{d}(x,w)\mathrm{d}(y,z)$ holds. It was shown in [17] that G is Ptolemaic if and only if G is distance hereditary and chordal. Some containment relationships among these and other families of graphs are as follows:

trees $\subset$ block graphs $\subset$ Ptolemaic graphs $\subset$ distance-hereditary graphs,
trees $\subset$ bipartite distance-hereditary graphs $\subset$ distance-hereditary graphs,
cographs $\subset$ distance-hereditary graphs,
Ptolemaic graphs $\subset$ strongly chordal graphs $\subset$ chordal graphs.

Theorem 1 (Bandelt and Mulder [2], D'Atri and Moscarini [8], Hammer and Maffray [13], Howorka [16]). *For any connected graph $G=(V,E)$ the following statements are equivalent*:

(1) *G is a distance-hereditary graph.*
(2) *Every cycle of length at least five in G has two crossing chords.*

(3) *Every induced path in G is a shortest path.*
(4) *For every hanging $h_u=(L_0,L_1,\ldots,L_t)$ of G and every pair of vertices x, $y\in L_i$ $(1\leqslant i\leqslant t)$ that are in the same component of $G-L_{i-1}$, $N'(x)=N'(y)$.*

Theorem 2 (Bandelt and Mulder [2], D'Atri and Moscarini [8], Hammer and Maffray [13], Howorka [16]). *Suppose $h_u=(L_0,L_1,\ldots,L_t)$ is the hanging of a connected distance-hereditary graph G at u. Then, every L_i induces a cograph. Moreover, if G is bipartite, then every L_i is an independent set of G.*

Suppose $G=(V,E)$ is a graph. For two vertices x and y in V, let

$$V_{xy}=\{v\in V:\ d_G(x,v)<d_G(y,v)\}.$$

Define the function ℓ_u from V to non-negative integers by: $\ell_u(v)=k$ whenever $d_G(u,v)=k$, or equivalently, v is in L_k for the hanging $h_u=(L_0,L_1,\ldots,L_t)$ of G at u. A path $P:x_0,x_1,\ldots,x_k$ is *u-concave downward* if $\ell_u(x_0)>\ell_u(x_1)>\cdots>\ell_u(x_{r-1})>\ell_u(x_r)=\ell_u(x_{r+1})=\cdots=\ell_u(x_{r'})<\ell_u(x_{r'+1})<\cdots<\ell_u(x_{k-1})<\ell_u(x_k)$ for some $0\leqslant r\leqslant r'\leqslant k$ with $0\leqslant r'-r\leqslant 1$. P is *monotone* if $r=k$ or $r'=0$.

The following two lemmas are useful in this paper. Note that they are trivial for the case when the distance-hereditary graphs are trees.

Lemma 3. *Suppose $h_u=(L_0,L_1,\ldots,L_t)$ is the hanging of a connected distance-hereditary graph G at u. For any two vertices x and y in G, there exists a shortest x–y path which is u-concave downward.*

Proof. Suppose $P:x=x_0,x_1,\ldots,x_k=y$ is a shortest x–y path, where $d_G(x,y)=k$. We may assume that P is chosen such that $s(P)=\sum_{j=0}^{k}\ell_u(x_j)$ is as small as possible. Let i be the largest index such that $x_0,x_1,\ldots,x_i$ is u-concave downward. Note that $i\geqslant 1$. In fact $i=k$ and so the lemma holds. Suppose to the contrary that $i<k$. Then one of the following cases holds:

(1) $\ell_u(x_{i-1})=\ell_u(x_i)>\ell_u(x_{i+1})$.
(2) $\ell_u(x_{i-1})<\ell_u(x_i)=\ell_u(x_{i+1})$.
(3) $\ell_u(x_{i-1})=\ell_u(x_i)=\ell_u(x_{i+1})$.
(4) $\ell_u(x_{i-1})<\ell_u(x_i)>\ell_u(x_{i+1})$.

For case (1) or (2), by Theorem 1 (4), x_{i-1} is adjacent to x_{i+1}, a contradiction to that P is a shortest path. For case (3) or (4), by Theorem 1 (4), x_{i-1} and x_{i+1} have a common neighbor x_i' with $\ell_u(x_i')=\ell_u(x_{i-1})-1$. Then, the path P' resulting from P by replacing x_i with x_i' is also a shortest x–y path whose $s(P')<s(P)$, a contradiction to the choice of P. This completes the proof of the lemma. □

Lemma 4. *Suppose $P:x_0,x_1,\ldots,x_k$ is an induced path of a distance-hereditary graph $G=(V,E)$. If $k=2$ with G chordal or $k\geqslant 3$, then $V_{x_0x_1}$ is a proper subset of $V_{x_{k-1}x_k}$.*

Proof. Suppose $h_{x_0}=(L_0,L_1,\ldots,L_t)$ is the hanging of G at x_0. Assume that there exists some vertex $v\in V_{x_0x_1}-V_{x_{k-1}x_k}$. By Lemma 3, there exists an x_0-concave downward shortest v–x_k path $P': v = y_0, y_1, \ldots, y_r, \ldots, y_{r'}, \ldots, y_{k'} = x_k$, where y_r (respectively, $y_{r'}$) is the first (respectively, last) vertex of P' in a smallest level L_f with $0 \leqslant r' - r \leqslant 1$.

For the case when $f \geqslant 2$, y_r and x_f are in the same component of $G - L_{f-1}$. According to Theorem 1 (4), y_r is adjacent to x_{f-1} and so $v = y_0, y_1, \ldots, y_r, x_{f-1}, x_{f-2}, \ldots, x_1, x_0$ is a shortest v–x_0 path, contradicting that $v \in V_{x_0x_1}$.

For the case when $f = 0$, $v = y_0, y_1, \ldots, y_r, x_1, x_2, \ldots, x_{k-1}, x_k$ is a shortest v–x_k path, contradicting that $v \notin V_{x_{k-1}x_k}$.

Now, suppose $f = 1$. For the case when $k \geqslant 3$, $y_{r'+1}$ and x_2 are in the same component of $G - L_1$. According to Theorem 1 (4), $y_{r'}$ is adjacent to x_2 and so $v = y_0, y_1, \ldots, y_{r'}, x_2, x_3, \ldots, x_{k-1}, x_k$ is a shortest v–x_k, contradicting that $v \notin V_{x_{k-1}x_k}$. For the case when $k = 2$ and G is chordal, $y_{r'}$ and x_1 adjacent to x_2 imply that they are also adjacent to x_0, i.e., $y_{r'}x_0x_1x_2y_{r'}$ is a cycle of G. By the definition of a chordal graph, $y_{r'}$ is adjacent to x_1. If $r = r'$, then $d_G(v, x_1) \leqslant d_G(v, x_0)$, contradicting that $v \in V_{x_0x_1}$. If $r = r' - 1$, then $x_0x_1x_2y_{r'}y_rx_0$ is a cycle of G. By Theorem 1 (2), y_r is adjacent to x_1, and hence $d_G(v, x_1) < d_G(v, x_2)$ contradicting that $v \notin V_{x_{k-1}x_k}$.

Therefore, $V_{x_0x_1}$ is a subset of $V_{x_{k-1}x_k}$; and in fact a proper subset as $x_{k-1} \in V_{x_{k-1}x_k} - V_{x_0x_1}$. This completes the proof of the lemma. □

3. Centers

The purpose of this section is to investigate the shapes of centers of distance-hereditary graphs. We in fact study centers in a more general setting as follows. Suppose S is a non-empty subset of V in a graph $G = (V, E)$. The *S-eccentricity* $e_{G,S}(v)$ of a vertex v in G is $\max_{x \in S} d(v, x)$.

The *S-center* of G is $C_S(G) = \{v \in V:\ e_{G,S}(v) \leqslant e_{G,S}(x) \text{ for all } x \in V\}$.

The *anticenter* of G is $AC(G) = \{v \in V:\ e_G(v) \geqslant e_G(x) \text{ for all } x \in V\}$.

The *S-anticenter* of G is $AC_S(G) = \{v \in V:\ e_{G,S}(v) \geqslant e_{G,S}(x) \text{ for all } x \in V\}$.

Theorem 5. *Suppose S is a non-empty vertex set of a distance-hereditary graph $G = (V, E)$. If H is a connected component of $G[T]$, where $T \subseteq V$ with $e_{G,S}(x) = e_{G,S}(y)$ for every two vertices x and y in T, then* $\operatorname{diam}(H) \leqslant 3$. *If moreover G is Ptolemaic, then* $\operatorname{diam}(H) \leqslant 2$.

Proof. Suppose x and y are two vertices in $V(H)$ such that $d_H(x, y) = \operatorname{diam}(H) = k$. Choose an induced x–y path $P: x = x_0, x_1, \ldots, x_k = y$ in H. Note that P is also an induced path of G and $e_{G,S}(x_0) = e_{G,S}(x_1) = \cdots = e_{G,S}(x_k)$. Let $e_{G,S}(x_1) = d_G(x_1, z)$ for some vertex $z \in S$. Then, for $0 \leqslant i \leqslant k$, we have

$$d_G(x_i, z) \leqslant e_{G,S}(x_i) = e_{G,S}(x_1) = d_G(x_1, z), \text{ i.e., } z \in V_{x_ix_1} \text{ or } d_G(x_i, z) = d_G(x_1, z).$$

We first prove that $k \leqslant 3$. Suppose to the contrary that $k \geqslant 4$. If $z \in V_{x_0x_1}$, then $z \in V_{x_2x_3}$ and $z \in V_{x_3x_4}$ by Lemma 4. These imply that $e_{G,S}(x_4) \geqslant d_G(x_4, z) = d_G(x_3, z) + 1 = d_G(x_2, z) + 2 > d_G(x_1, z) = e_{G,S}(x_1)$, a contradiction. Hence $d_G(x_0, z) = d_G(x_1, z)$. We then hang G at z. Note that x_0 and x_1 are in the same level. If x_2 is in level $d_G(x_2, z) = d_G(x_1, z) - 1$, then x_0 is adjacent to x_2 by Theorem 1 (4), a contraction. Hence

$d_G(x_1,z)\leqslant d_G(x_2,z)$. Since $d_G(x_0,z)=d_G(x_1,z)$, we have $z\notin V_{x_1x_0}$. By Lemma 4, $z\notin V_{x_3x_2}$ and so $d_G(x_2,z)\leqslant d_G(x_3,z)$. Therefore, $d_G(x_0,z)=d_G(x_1,z)\leqslant d_G(x_2,z)\leqslant d_G(x_3,z)\leqslant d_G(x_1,z)$ and so $d_G(x_0,z)=d_G(x_1,z)=d_G(x_2,z)=d_G(x_3,z)$. Now, consider the hanging of graph G at z. Then x_0,x_1,x_2,x_3 are in the same level. By Theorem 1 (4), there exists a vertex z^* adjacent to x_0, x_1, x_2, and x_3. Note that $z^*x_0x_1x_2x_3z^*$ is a cycle of length 5 without crossing chords, a contradiction. Hence $k\leqslant 3$, i.e., $\operatorname{diam}(H)\leqslant 3$.

Next, we prove that $k\leqslant 2$ when G is Ptolemaic. Suppose to the contrary that $k\geqslant 3$. If $z\in V_{x_0x_1}$, then $z\in V_{x_1x_2}$ by Lemma 4. This implies that $e_{G,S}(x_2)\geqslant d_G(x_2,z)>d_G(x_1,z)=e_{G,S}(x_1)$, a contradiction. Hence $z\notin V_{x_0x_1}$ and so $d_G(x_0,z)=d_G(x_1,z)$ and $z\notin V_{x_1x_0}$. Then, by Lemma 4, $z\notin V_{x_2x_1}$ and $z\notin V_{x_3x_2}$. Therefore, $d_G(x_0,z)=d_G(x_1,z)\leqslant d_G(x_2,z)\leqslant d_G(x_3,z)\leqslant d_G(x_1,z)$ and so $d_G(x_0,z)=d_G(x_1,z)=d_G(x_2,z)=d_G(x_3,z)$. Again, we can get a cycle $z^*x_0x_1x_2x_3z^*$ of length 5 without crossing chords, a contradiction. Hence $k\leqslant 2$, i.e., $\operatorname{diam}(H)\leqslant 2$. □

Corollary 6. *Suppose S is a non-empty vertex set in a distance-hereditary graph G. If H is a connected component of the subgraph induced by $C_S(G)$ or $AC_S(G)$, then* $\operatorname{diam}(H)\leqslant 3$. *If moreover G is Ptolemaic, then* $\operatorname{diam}(H)\leqslant 2$.

The distance-hereditary graphs G_1 and G_2 and the Ptolemaic graphs G_3 and G_4 in Fig. 1 show that the bounds in Corollary 6 are sharp. Note that $C(G_1)=\{a_1,b_1,c_1,d_1,e_1,f_1\}$, $AC(G_2)$ has a connected component $G_2[\{a_2,b_2,c_2,d_2\}]$, $C(G_3)=\{a_3,b_3,c_3,d_3,e_3\}$, and $AC(G_4)$ has a connected component $G_4[\{a_4,b_4,c_4,d_4,e_4\}]$.

Because distance-hereditary graphs have a "tree like" structure of adjacency, one may expect that their centers are "small" and "compact". The following lemma supports such expectations.

Lemma 7. *If V_1 and V_2 are the vertex sets of two distinct components of the S-center of a distance-hereditary graph $G=(V,E)$, then $d_G(V_1,V_2)=2$.*

Proof. Assume that $d_G(V_1,V_2)\geqslant 3$. Then there exists an induced path $P: x_0,x_1,\ldots,x_k$, where $k\geqslant 3$, $x_0\in V_1$, $x_k\in V_2$, but $x_1,x_{k-1}\notin C_S(G)$. Assume z is a vertex in S with $d_G(x_{k-1},z)=e_{G,S}(x_{k-1})$. Since $x_k\in C_S(G)$ and $x_{k-1}\notin C_S(G)$, $d_G(x_k,z)\leqslant e_{G,S}(x_k)<e_{G,S}(x_{k-1})=d_G(x_{k-1},z)$ and so $z\in V_{x_kx_{k-1}}$. By Lemma 4, we then have $z\in V_{x_1x_0}$. Let $h_z=(L_0,L_1,\ldots,L_t)$ be the hanging of G at z. Since $x_0\in C_S(G)$ and $x_{k-1}\notin C_S(G)$, we have $d_G(x_0,z)\leqslant e_{G,S}(x_0)<e_{G,S}(x_{k-1})=d_G(x_{k-1},z)$. Therefore, the relative positions of x_0,x_1, x_{k-1},x_k are as shown in Fig. 2(a). Thus, there exists a vertex x_j in the path P such that x_j and x_k are in the same level, say L_i, of h_z, and $x_jx_{j+1}x_{j+2}\ldots x_k$ is a path in $G-L_{i-1}$ (see Fig. 2(b)). Then, by Theorem 1 (4), x_k is adjacent to x_{j-1}, contrary to that P is an induced path. Therefore, $d_G(V_1,V_2)\leqslant 2$. □

Theorem 8. *If G is a distance-hereditary graph, then the S-center $C_S(G)$ is either a connected graph of diameter 3 or a cograph. If moreover G is a bipartite distance-hereditary graph, then the S-center $C_S(G)$ is either a connected graph of diameter $\leqslant 3$ or an independent set of G.*

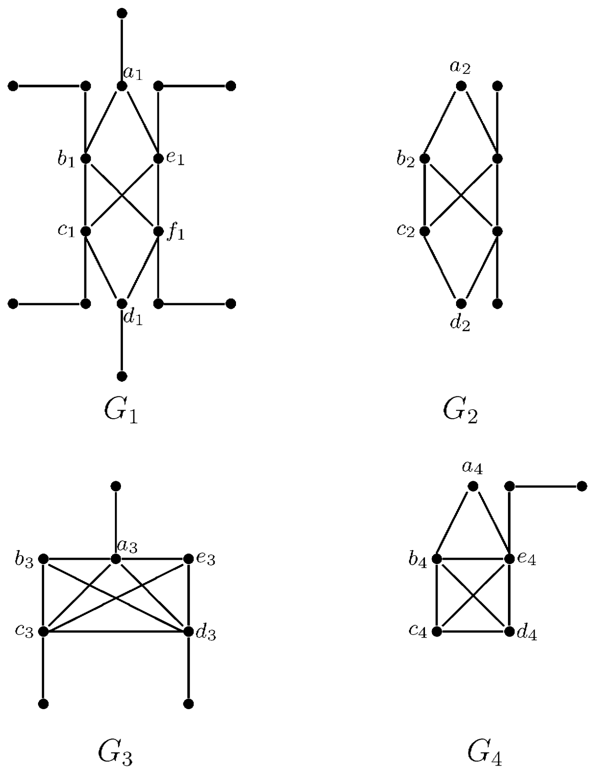

Fig. 1. Examples for which the bounds in Corollary 6 are sharp.

Proof. First, if $H=G[C_S(G)]$ is connected, then the theorem follows immediately from Theorem 5. Hence, we may assume that H is disconnected. Choose any two distinct components H_1 and H_2. Then, by Lemma 7, there exists an induced path xzy in G such that $x\in V(H_1)$, $y\in V(H_2)$ and $z\notin C_S(G)$. Suppose w is a vertex in S with $d_G(w,z)=e_{G,S}(z)$. Then $d_G(w,x)\leqslant e_{G,S}(x)<e_{G,S}(z)=d_G(w,z)$ and so $d_G(w,z)=d_G(w,x)+1$. Similarly, $d_G(w,z)=d_G(w,y)+1$.

Let $h_w=(L_0,L_1,\ldots,L_t)$ be the hanging of G at w. Note that x and y lie on the same level L_i of h_w and $z\in L_{i+1}$. Now for every vertex $x'\in H_1$, $d_G(w,x')\leqslant e_{G,S}(x')<e_{G,S}(z)=d_G(w,z)$ and so $x'\in L_r$ for some $r\leqslant i$. In fact $r=i$. Suppose to the contrary that $x'\in L_r$ for some $r\leqslant i-1$. Then H_1 has an x–x' path P laying above level $i+1$. Hence P contains an edge uv such that y and u are in the same component of $G-L_{i-1}$ and $v\in L_{i-1}$. Thus, by Theorem 1 (4), y is adjacent to v and hence $y\in H_1$, a contradiction. Therefore, every vertex of H_1 lies on level L_i. Hence H_1 is a cograph by Theorem 2. Moreover, if G is also bipartite, then L_i is an independent set of G by Theorem 2, and so is H_1. This completes the proof of the theorem. □

As described in [5], Hedetniemi proved that any graph H is isomorphic to the center of some graph G of diameter 4 and radius 2. When H is a cograph, an analogous

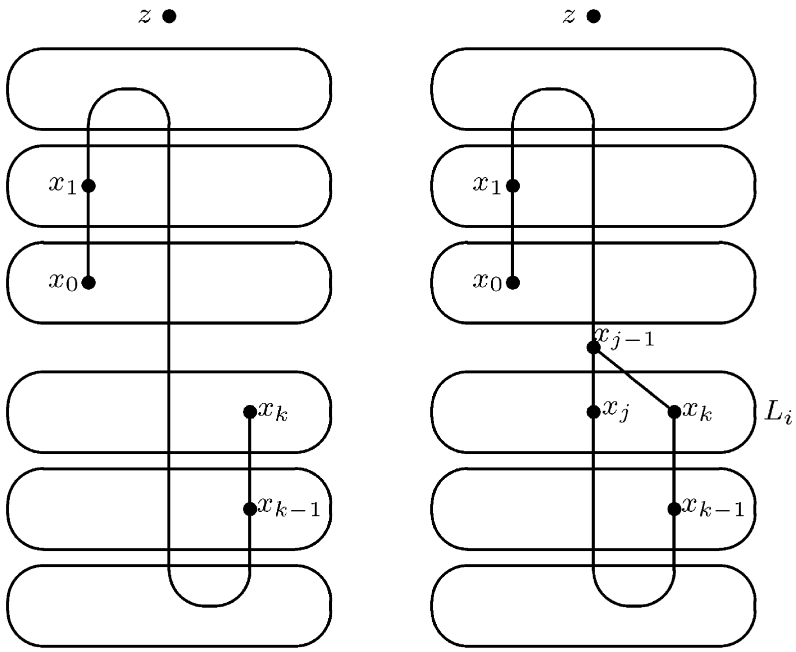

Fig. 2. For the proof of Lemma 7.

result for distance-hereditary graph G is the following theorem. Using the proof of this theorem, we can construct a distance-hereditary graph whose center induces a graph with arbitrary number of components. Also the center of a bipartite distance-hereditary graph can be an independent set of arbitrarily large size.

Theorem 9. *For any given cograph H there exists a connected distance-hereditary graph G whose center is isomorphic to H.*

Proof. We construct G by adding four new vertices u, v, w, x into H such that v and w are adjacent to all vertices of H, u is adjacent only to v and x only to w. It is clear that G is a distance-hereditary graph whose center is isomorphic to H. □

4. Medians

This section discusses the structures of medians of distance-hereditary graphs. We again study medians in a more general setting. Suppose S is a non-empty subset of V. The *S-distance sum* $D_{G,S}(v)$ is equal to $\sum_{x\in S} d_G(v,x)$. The *S-w-distance sum* $D_{G,S,w}(v)$ of v is $\sum_{x\in S} d_G(v,x)w(x)$. The *S-median* (also called the *S-centroid* [31]) of G is the set

$$M_S(G)=\{v\in V:\ D_{G,S}(v)\leqslant D_{G,S}(x) \text{ for all } x\in V\}.$$

The *S-w-median* of G is the set

$$M_{S,w}(G)=\{v\in V:\ D_{G,S,w}(v)\leqslant D_{G,S,w}(x) \text{ for all } x\in V\}.$$

The *antimedian* of G is the set

$$\mathrm{AM}(G)=\{v\in V:\ D_G(v)\geqslant D_G(x) \text{ for all } x\in V\}.$$

Lemma 10 (Entringer et al. [10], Slater [31]). *If a and b are two adjacent vertices of a graph $G=(V,E)$ with a function w on V, then $D_{G,w}(a)-D_{G,w}(b)=w(V_{ba})-w(V_{ab})$.*

Proof. The lemma follows immediately from the fact that

$$D_{G,w}(a)-D_{G,w}(b)=\sum_{x\in V_{ab}}\{d_G(x,a)-d_G(x,b)\}w(x) - \sum_{x\in V_{ba}}\{d_G(x,a)-d_G(x,b)\}w(x). \qquad \square$$

Lemma 11. *Suppose $P:x_0,x_1,\ldots,x_k$ is an induced path of a distance-hereditary graph $G=(V,E)$ with a function $w>0$ (respectively, $w\geqslant 0$) on V. If either $k\geqslant 2$ with G chordal or $k\geqslant 3$, then $D_{G,w}(x_0)-D_{G,w}(x_1)>$ (respectively, $\geqslant$) $D_{G,w}(x_{k-1})-D_{G,w}(x_k)$.*

Proof. The lemma follows immediately from Lemmas 4 and 10. □

Theorem 12. *For any $S\subseteq V$ of a Ptolemaic graph $G=(V,E)$ with a function $w\geqslant 0$, the S-w-median $M_{S,w}(G)$ is m-convex.*

Proof. Assume $M_{S,w}(G)$ is not m-convex. Then there exists an induced path $P:x_0,x_1,\ldots,x_k$ in G with $k\geqslant 2$ such that $x_0,x_k\in M_{S,w}(G)$ but $x_1,x_{k-1}\notin M_{S,w}(G)$. Then, by Lemmas 10 and 4, we have

$$\begin{aligned}0 &> D_{G,S,w}(x_0)-D_{G,S,w}(x_1)\\ &= w(V_{x_1x_0}\cap S)-w(V_{x_0x_1}\cap S)\\ &\geqslant w(V_{x_kx_{k-1}}\cap S)-w(V_{x_{k-1}x_k}\cap S)\\ &= D_{G,S,w}(x_{k-1})-D_{G,S,w}(x_k)>0,\end{aligned}$$

a contradiction. So, $M_{S,w}(G)$ is m-convex. □

Corollary 13 (Soltan [35]). *The median of a Ptolemaic graph is connected.*

Corollary 14 (Slater [31]). *For any subset S of the vertices of a tree T, the S-median of T is connected.*

Theorem 15. *Suppose $G=(V,E)$ is a distance-hereditary graph with a function $w>0$. If H is a connected component of $G[T]$, where $T\subseteq V$ having $D_{G,w}(x)=D_{G,w}(y)$ for* *every two vertices x and y in T, then H is a cograph. If moreover G is Ptolemaic, then H is a clique.*

Proof. Suppose $P: x_0, x_1, \dots, x_k$ is an induced path in H. Note that P is also an induced path of G. If G is distance hereditary (respectively, Ptolemaic) and $k \geqslant 3$ (respectively, $k \geqslant 2$), then by Lemma 11 and the fact that $w > 0$, we have $D_{G,w}(x_0) - D_{G,w}(x_1) > D_{G,w}(x_{k-1}) - D_{G,w}(x_k)$, contrary to $D_{G,w}(x_0) = D_{G,w}(x_1) = D_{G,w}(x_{k-1}) = D_{G,w}(x_k)$. So, $k \leqslant 2$ (respectively, $\leqslant 1$) and hence H is a cograph (respectively, a clique). □

Corollary 16 (Nieminen [25], Yushmanov [38]). *The median of a Ptolemaic graph is a clique.*

Corollary 17 (Yushmanov [38]). *Every connected component of the subgraph induced by the antimedian of a Ptolemaic graph is a clique.*

It is worth pointing out that the S-median of a Ptolemaic graph does not have a theorem like Theorem 15. As shown in [31], there exist trees whose S-median contains a path of n vertices for any n. The following theorem shows that the median of a distance-hereditary graph nearly coincides with its local median.

Theorem 18. *Suppose G is a distance-hereditary graph with a function $w > 0$. If $x \in M_w(G)$ and $y \in LM_w(G)$, then $d(x, y) \leqslant 2$.*

Proof. Suppose $P: x = x_0, x_1, \dots, x_k = y$ is an induced x–y path of G. Assume $k \geqslant 3$. By Lemma 11, we have $0 \geqslant D_{G,w}(x_0) - D_{G,w}(x_1) > D_{G,w}(x_{k-1}) - D_{G,w}(x_k) \geqslant 0$, a contradiction. So $d_G(x, y) \leqslant 2$. □

Theorem 19. *If G is a Ptolemaic graph with a function $w \geqslant 0$, then $M_w(G) = LM_w(G)$.*

Proof. Assume that $LM_w(G) - M_w(G) \neq \emptyset$. Pick $y \in LM_w(G) - M_w(G)$ and $x \in M_w(G)$ such that $d_G(x, y) = d_G(LM_w(G) - M_w(G), M_w(G))$. Suppose $P: x = x_0, x_1, \dots, x_k = y$ is an induced x–y path of G. Note that $k \geqslant 2$ and $x_1 \notin M_w(G)$. Hence, by Lemma 11, $0 > D_{G,w}(x_0) - D_{G,w}(x_1) \geqslant D_{G,w}(x_{k-1}) - D_{G,w}(x_k) \geqslant 0$, a contradiction. Therefore, $LM_w(G) - M_w(G) = \emptyset$ and so $M_w(G) - LM_w(G)$. □

Theorem 20. *If $G = (V, E)$ is a distance-hereditary graph with a function $w > 0$, then its w-median is a cograph. If moreover G is bipartite distance hereditary, then its w-median is either a connected cograph or an independent set of G.*

Proof. The first part of the theorem follows from Theorem 15 immediately. To prove the second part, suppose V_1 and V_2 are the vertex sets of two distinct components of the w-median of G. For any two vertices $x \in V_1$ and $y \in V_2$, by Theorem 18, we have $d_G(x, y) = 2$. Now consider the hanging $h_x = (L_0, L_1, \dots, L_t)$ of G at x. Clearly $V_2 \subseteq L_2$ and hence V_2 is an independent set of G by Theorem 2. □

5. Convexity and diameters

This section investigates metric properties for chordal graphs and distance-hereditary graphs.

A vertex subset S is called an *x–y separator* of G if x and y are in different components of $G-S$. An x–y separator S is said to be *minimal* if no proper subset of S is an x–y separator of G.

Theorem 21 (Dirac [9]). *Every minimal x–y separator of a chordal graph is a clique.*

Theorem 22. *If $G=(V,E)$ is a chordal graph and $S\subseteq V$, then the S-center $C_S(G)$ of G is m-convex.*

Proof. Assume to the contrary that $C_S(G)$ is not m-convex. Then there exist x and y in $C_S(G)$ with an induced x–y path P of G such that $|V(P)|\geqslant 3$ and $P\cap C_S(G)=\{x,y\}$. Suppose C is a minimal x–y separator of G. So, there exists a vertex $r\in P\cap C$ and hence $r\in C-C_S(G)$. Suppose $s'\in S$ with $d_G(s',r)=e_{G,S}(r)$. If $s'\in C$ then $e_{G,S}(r)=1$ contradicting that $r\notin C_S(G)$. Hence, without loss of generality, we may assume that s' and x are in different components of $G-C$. By the fact that C is a clique, there exists a vertex $t\in C$ such that $e_{G,S}(x)\geqslant d_G(s',x)=d_G(s',t)+d_G(t,x)\geqslant d_G(s',t)+1\geqslant d_G(s',r)=e_{G,S}(r)$ contrary to that $e_{G,S}(x)<e_{G,S}(r)$. Therefore, $C_S(G)$ is m-convex. □

Corollary 23 (Yushmanov [38,40]). *The center $C(G)$ of a chordal graph is m-convex.*

Suppose x and y are two vertices in a graph G with $d_G(x,y)=e_G(y)$. It is easily seen that if G is a tree then $e_G(x)=\mathrm{diam}(G)$. In general, $e_G(x)\neq\mathrm{diam}(G)$ for a distance-hereditary graph G. Moreover, the difference between $e_G(x)$ and $\mathrm{diam}(G)$ may be arbitrarily large for a general graph. However, in the following theorem we show that $e_G(x)$ is nearly equal to $\mathrm{diam}(G)$ for a distance-hereditary graph G. The graphs given in Fig. 3 show that the bounds in the following theorem are sharp.

Theorem 24. *For any vertex y in a distance-hereditary graph $G=(V,E)$. If x is a vertex with $d_G(y,x)=e_G(y)$, then $e_G(x)\geqslant\mathrm{diam}(G)-2$. If moreover G is Ptolemaic, then $e_G(x)\geqslant\mathrm{diam}(G)-1$.*

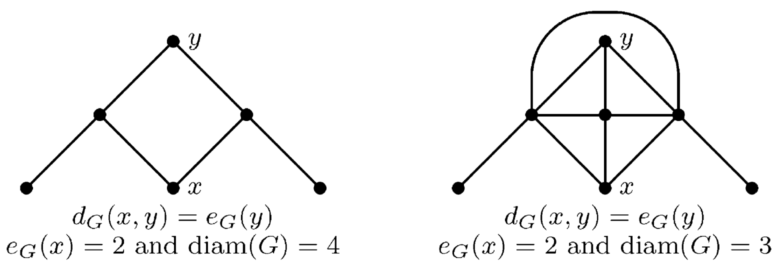

Fig. 3. Examples for which the bounds in Theorem 24 are sharp.

Proof. Let $h_u = (L_0, L_1, \ldots, L_t)$ be the hanging of G at a vertex u having $e_G(u) = \text{diam}(G) = t$. Choose a vertex z in L_t. By Lemma 3, there exists a u-concave downward shortest x–y path

$$P_x : x = x_0, x_1, \ldots, x_r, \ldots, x_{r'}, \ldots, x_k = y,$$

where x_r (respectively, $x_{r'}$) is the first (respectively, last) vertex of P_x in a smallest level L_{f_x} with $0 \leqslant r' - r \leqslant 1$; and a u-concave downward shortest z–y path

$$P_z : z = z_0, z_1, \ldots, z_s, \ldots, z_{s'}, \ldots, z_m = y,$$

where z_s (respectively, $z_{s'}$) is the first (respectively, last) vertex of P_z in a smallest level L_{f_z} with $0 \leqslant s' - s \leqslant 1$.

We may assume that $\ell_u(z) - 2 \geqslant \ell_u(x)$, for otherwise $\text{diam}(G) - 1 = \ell_u(z) - 1 \leqslant \ell_u(x) \leqslant e_G(x)$ and so the theorem holds.

Suppose $x_i = z_j$ for some $i \leqslant r$ and $j \leqslant s$. Since $d_G(z_j, z) = \ell_u(z) - \ell_u(z_j) > \ell_u(x) - \ell_u(x_i) = d_G(x_i, x)$, we have $d_G(y, z) = d_G(y, z_j) + d_G(z_j, z) > d_G(y, x_i) + d_G(x_i, x) = d_G(y, x) = e_G(y)$, a contradiction. Therefore, the two paths $x_0, x_1, \ldots, x_r$ and $z_0, z_1, \ldots, z_s$ have no vertex in common.

Next, $(\ell_u(x) - f_x) + (r' - r) + (\ell_u(y) - f_x) = d_G(x, y) = e_G(y) \geqslant d_G(z, y) = (\ell_u(z) - f_z) + (s' - s) + (\ell_u(y) - f_z)$. Therefore, $\ell_u(z) - 2 \geqslant \ell_u(x)$ and $r' - r \leqslant 1$ and $0 \leqslant s' - s$ imply $f_x < f_z$. Let x_q (respectively, $x_{q'}$) be the first (respectively, last) vertex of P_x in level $L_{f_z - 1}$. Since $x_{q'+1}$ and z_s are connected in $G - L_{f_z - 1}$, by Theorem 1 (4), $x_{q'}$ is adjacent to z_s. Consider the x–z path

$$P_1 : x = x_0, x_1, x_2, \ldots, x_{q'}, z_s, z_{s-1}, \ldots, z_0 = z.$$

Suppose P_1 is an induced path. Note that $d_G(x, x_{q'}) + d_G(x_{q'}, y) = d_G(x, y) = e_G(y) \geqslant d_G(u, y) = d_G(u, x_{q'}) + d_G(x_{q'}, y)$. Then, $d_G(x, x_{q'}) \geqslant d_G(u, x_{q'})$ and so $e_G(x) \geqslant d_G(x, z) = d_G(x, x_{q'}) + d_G(x_{q'}, z) \geqslant d_G(u, x_{q'}) + d_G(x_{q'}, z) \geqslant d_G(u, z) = \text{diam}(G)$. In this case, the theorem holds.

We then may assume that P_1 is not an induced path, say P_1 has a chord joining some vertex x_i to some vertex z_j. Note that in this case $0 < q \leqslant r \leqslant r' \leqslant q' < k$. Then, either $i = q$ with $j = s$, or $i \leqslant q - 1$ with $j \leqslant s$. For the first case, $x_q z_s \in E$. For the second case, x_{q-1} and z_s are connected in $G - L_{f_z - 1}$, and so again $x_q z_s \in E$ by Theorem 1 (4). In any case, $d_G(z_s, x_q) = 1$.

Since $d_G(y, z_s) + d_G(z_s, x_q) + d_G(x_q, x) \geqslant d_G(y, x) = e_G(y) \geqslant d_G(y, z) = d_G(y, z_s) + d_G(z_s, z)$, we have $1 + d_G(x_q, x) \geqslant d_G(z_s, z)$. By the fact that $d_G(x_q, x) = \ell_u(x) - \ell_u(x_q) \leqslant e_G(x) - (f_z - 1)$ and $d_G(z_s, z) = \text{diam}(G) - f_z$, we then have $e_G(x) \geqslant \text{diam}(G) - 2$. This proves the first part of the theorem.

To prove the second part of the theorem, suppose G is Ptolemaic, i.e., G is chordal and distance hereditary. For the case when $x_q = x_{q'}$, we have $q = r = r' = q'$ and $f_x = f_z - 1$. For the case when $x_q \neq x_{q'}$, since the two vertices x_q and $x_{q'}$ in $L_{f_z - 1}$ are adjacent to $z_s \in L_{f_z}$, they are also adjacent to some $w \in L_{f_z - 2}$ according to Theorem 1 (4). By the chordality of G, the cycle $w, x_q, z_s, x_{q'}, w$ has a chord, which must be $x_q x_{q'}$. So, $q = r < r' = q'$ and $f_x = f_z - 1$. In any case, $d_G(x_q, x_{q'}) \leqslant 1$. Consider the x–z path

$$P_2 : x = x_0, x_1, \ldots, x_q, z_s, z_{s-1}, \ldots, z_0 = z.$$

Suppose P_2 is an induced path. Note that $d_G(x,x_q)+d_G(x_q,x_{q'})+d_G(x_{q'},y)=d_G(x,y)=e_G(y)\geqslant d_G(u,y)=d_G(u,x_{q'})+d_G(x_{q'},y)$ and so $d_G(x,x_q)\geqslant d_G(u,x_{q'})-1$ since $d_G(x_q,x_{q'})\leqslant 1$. Therefore, $e_G(x)\geqslant d_G(x,z)=d_G(x,x_q)+d_G(x_q,z_s)+d_G(z_s,z)\geqslant d_G(u,x_{q'})-1+d_G(x_{q'},z_s)+d_G(z_s,z)\geqslant d_G(u,z)-1=\mathrm{diam}(G)-1$, since $d_G(x_{q'},z_s)=d_G(x_q,z_s)=1$. In this case, the second part of the theorem holds.

We then may assume that P_2 has a chord joining some vertex x_i to some vertex z_j with $i\leqslant q-1$ and $j\leqslant s$. Then x_{q-1} and $z_{s'}$ are connected in $G-L_{f_z-1}$. Again, by Theorem 1 (4) and (2) and the chordality of G, $d_G(x_{q-1},z_{s'})\leqslant 1$. Thus, $d_G(y,x)\leqslant d_G(y,z_{s'})+d_G(z_{s'},x_{q-1})+d_G(x_{q-1},x)\leqslant d_G(y,z_{s'})+1+d_G(x_{q-1},x)=d_G(y,x_{q'+1})+1+d_G(x_{q-1},x)<d_G(y,x_{q'+1})+d_G(x_{q'+1},x_{q-1})+d_G(x_{q-1},x)=d_G(y,x)$, a contradiction. This completes the proof of the theorem. □

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