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# Circular Chromatic Numbers and Fractional Chromatic Numbers of Distance Graphs

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This paper studies circular chromatic numbers and fractional chromatic numbers of distance graphs $G(Z, D)$ for various distance sets $D$. In particular, we determine these numbers for those $D$ sets of size two, for some special $D$ sets of size three, for $D = \{1, 2, \dots, m, n\}$ with $1 \le m < n$, for $D = \{q, q+1, \dots, p\}$ with $q \le p$, and for $D = \{1, 2, \dots, m\} - \{k\}$ with $1 \le k \le m$.



## 1. INTRODUCTION

Suppose $S$ is a subset of a metric space $\mathcal{M}$ with a metric $\delta$, and $D$ a subset of positive real numbers. The *distance graph* $G(S, D)$, with a *distance set* $D$, is the graph with vertex set $S$ in which two vertices $x$ and $y$ are adjacent iff $\delta(x, y) \in D$. Distance graphs, first studied by Eggleton *et al*. [7], were motivated by the well-known plane-coloring problem: What is the minimum number of colors needed to color all points of a euclidean plane so that points at unit distances are colored with different colors. This problem is equivalent to determining the chromatic number of the distance graph $G(R^2, \{1\})$. It is well-known that the chromatic number of this distance graph is between 4 and 7 (see [12, 15]). However, the exact number of colors needed remains unknown.

For distance graphs on the real line $R$ or the integer set $Z$, the problem of finding the chromatic numbers of $G(R, D)$ or $G(Z, D)$ for different $D$ sets has been studied extensively (see [3, 10, 13, 14, 17, 18, 20, 22]). Two recent papers [3, 14] related distance graphs to the $T$-coloring problem. Chromatic numbers and fractional chromatic numbers of distance graphs were used to derive bounds for $T$-spans of the corresponding $T$-colorings, and vice versa. In this paper, we study circular chromatic numbers and fractional chromatic numbers of distance graphs $G(Z, D)$ for various $D$ sets.

The circular chromatic number of a graph is a natural generalization of the chromatic number of a graph, introduced by Vince [16] under the name the 'star chromatic number' of a graph. Suppose $p$ and $q$ are positive integers such that $p \ge 2q$. A $(p, q)$-*coloring* of a graph $G = (V, E)$ is a mapping $c$ from $V$ to $\{0, 1, \dots, p-1\}$ such that $\|c(x) - c(y)\|_p \ge q$ for any edge $xy$ in $E$, where $\|a\|_p = \min\{a, p-a\}$. The *circular chromatic number* $\chi_c(G)$ of $G$ is the infimum of the ratios $p/q$ for which there exist $(p, q)$-colorings of $G$.

Note that a $(p, 1)$-coloring of a graph $G$ is simply an ordinary $p$-coloring of $G$. Therefore, $\chi_c(G) \le \chi(G)$ for any graph $G$. On the other hand, it has been shown [16] that for all graphs $G$, we have $\chi(G) - 1 < \chi_c(G)$. Therefore, $\chi(G) = \lceil \chi_c(G) \rceil$. In particular, two graphs with the same circular chromatic number also have the same chromatic number. However, two graphs with the same chromatic number may have different circular chromatic numbers. Thus $\chi_c(G)$ is a refinement of $\chi(G)$, and it contains more information about the structure of the graph. It is usually much more difficult to determine the circular chromatic number of a graph than to determine its chromatic number. The main results of this article determine the circular chromatic numbers of various distance graphs. These results may be viewed as improvements

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on previous results concerning the chromatic numbers of these distance graphs presented in [3, 4, 7, 13, 17, 21].

The fractional chromatic number of a graph is another well-known variation of the chromatic number. A *fractional coloring* of a graph $G$ is a mapping $c$ from $\mathcal{I}(G)$, the set of all independent sets of $G$, to the interval $[0, 1]$ such that $\sum_{x \in I \in \mathcal{I}(G)} c(I) \geq 1$ for all vertices $x$ in $G$. The *fractional chromatic number* $\chi_f(G)$ of $G$ is the infimum of the value $\sum_{I \in \mathcal{I}(G)} c(I)$ of a fractional coloring $c$ of $G$.

For any graph $G$, it is well known that

$$\max\{\omega(G), |G|/\alpha(G)\} \leq \chi_f(G) \leq \chi_c(G) \leq \lceil \chi_c(G) \rceil = \chi(G). \qquad (*)$$

For simplicity, let $\omega(S, D)$, $\alpha(S, D)$, $\chi_f(S, D)$, $\chi_c(S, D)$ and $\chi(S, D)$ denote, respectively, the clique number, the independence number, the fractional chromatic number, the circular chromatic number, and the chromatic number of a distance graph $G(S, D)$.

Chromatic numbers of distance graphs with distance sets $|D| \leq 2$ were determined by Chen *et al.* [4] and Voigt [17]. Chromatic numbers of distance graphs with $|D| = 3$ were determined by Zhu [21]. In Section 2, we use a 'multiplier method' to establish an upper bound for the circular chromatic number of a distance graph $G(Z, D)$ with an arbitrary distance set $D$. This upper bound is then used to determine the circular chromatic numbers and the fractional chromatic numbers of those distance graphs with distance sets $D$ for $|D| = 2$, for some special $D$ with $|D| = 3$, for $D = \{1, 2, \ldots, m, n\}$ with $1 \leq m < n$, and for $D = \{q, q + 1, \ldots, p\}$ with $q \leq p$. The chromatic number for $G(Z, D)$ with $D = \{q, q+1, \ldots, p\}$ was determined in [7, 13].

Chromatic numbers of distance graphs with distance sets of the form $D_{m,k} = \{1, 2, \ldots, m\} - \{k\}$, with $1 \leq k \leq m$, were studied in [3, 7, 13, 14]. Partial results concerning chromatic numbers of such distance graphs were obtained in [7, 13, 14], and a complete solution was recently obtained by Chang *et al.* [3]. The authors of [3] also obtained circular chromatic numbers of such distance graphs for some special values of $m$ and $k$. In Section 3, we determine the circular chromatic numbers $\chi_c(Z, D_{m,k})$ for all integer pairs $m, k$.

## 2. MULTIPLIER METHOD FOR $\chi_f(Z, D)$ AND $\chi_c(Z, D)$

In this section we use a 'multiplier method' to establish an upper bound on $\chi_c(Z, D)$ for an arbitrary $D$ set. We then use this upper bound to determine circular chromatic numbers for some $D$ sets.

The multiplier method was used in [2] to study the density of $D$-sets, and was also used in [11] to study fractional chromatic numbers and circular chromatic numbers of circulant graphs. In taking distance graphs to be 'infinite' circulant graphs, Theorem 2.2 is parallel to a result in [11]. Half of the proof of Theorem 2.3 is parallel to an argument in [2].

LEMMA 2.1. *Suppose $D$ is a set of positive integers, and that $p$ and $r$ are positive integers. Let*

$$d_D(p, r) = \min\{\|ri \bmod p\|_p : i \in D\}.$$

*If $d_D(p, r) \geq 1$, then $\chi_c(Z, D) \leq p/d_D(p, r)$.*

PROOF. It is straightforward to verify that the coloring defined as $c(i) = (ri \bmod p)$ for $i \in Z$ is a $(p, d_D(p, r))$-coloring of the distance graph $G(Z, D)$. □

Let $f_D = \inf\{p/d_D(p,r) : d_D(p,r) \geq 1\}$. The function is well defined since $d_D(p,r)$ is always an integer between 0 and $\lfloor p/2 \rfloor$. Theorem 2.2 follows from Lemma 2.1.

THEOREM 2.2. *For any set $D$ of positive integers, $\chi_c(Z, D) \leq f_D$.*

It is known [4, 17] that if $D$ contains exactly two relatively prime integers, then $\chi(Z, D) = 2$ when the two integers are odd and $\chi(Z, D) = 3$ when the two integers have different parities. We first use $f_D$ to determine $\chi_c(Z, D)$ and $\chi_f(Z, D)$ for $D$ with $|D| = 2$.

THEOREM 2.3. *If $D = \{a, b\}$ and $\gcd(a, b) = 1$, then*

$$\chi_f(Z, D) = \chi_c(Z, D) = f_D = (a+b)/\lfloor (a+b)/2 \rfloor.$$

PROOF. Suppose both $a$ and $b$ are odd. Since $2 \leq \omega(Z, D)$ and $d_D(2, 1) = 1$, the theorem follows from $(*)$ and Theorem 2.2.

Suppose that $a$ and $b$ have different parities, i.e., $a + b$ is odd. Assume that $a + b = p$. Since $\gcd(p, b - a) = 1$, there exists a positive integer $r$ such that $r(b - a) \equiv 1 \pmod{p}$. Since $r(b + a) \equiv 0 \pmod{p}$, it follows that $2rb \equiv -2ra \equiv 1 \pmod{p}$. Hence, $ra \equiv -rb \equiv (p-1)/2 \pmod{p}$, which implies that $d_D(p, r) = (p - 1)/2$. Hence, according to Theorem 2.2, $\chi_c(Z, D) \leq f_D \leq 2p/(p-1) = (a+b)/\lfloor (a+b)/2 \rfloor$. On the other hand, it is easy to see that $G(Z, D)$ contains the odd cycle $C_p$. Thus, $2p/(p-1) \leq p/\alpha(C_p) \leq \chi_f(C_p) \leq \chi_f(Z, D) \leq \chi_c(Z, D)$. This completes the proof of the theorem. □

Note that precisely the same arguments in the first two lines of the proof above also give that $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 2$ if $D$ contains only odd integers.

We now consider circular chromatic numbers and fractional chromatic numbers of distance graphs $G(Z, D)$ with $|D| = 3$. Zhu [21] proved the following result for chromatic numbers, which provides a range for circular chromatic numbers.

THEOREM 2.4 ([21]). *If $D = \{a, b, c\}$, where $a < b < c$ are positive integers with $\gcd(a, b, c) = 1$, then*

$$\chi(Z, D) = \begin{cases} 2, & \text{if } a, b, c \text{ are odd,} \\ 4, & \text{if } a = 1 \text{ and } b = 2 \text{ and } c \equiv 0 \pmod{3}, \\ 4, & \text{if } a + b = c \text{ and } a \not\equiv b \pmod{3}, \\ 3, & \text{otherwise.} \end{cases}$$

THEOREM 2.5. *If $D = \{a, a + 1, c\}$, with $a + 1 < c$, where $c + a = (2a + 1)k + r$, with $k \geq 1$ and $0 \leq r \leq 2a$, then*

$$\chi_f(Z, D) \leq \chi_c(Z, D) \leq f_D \leq \begin{cases} (c+a)/(ak), & \text{if } 0 \leq r \leq a, \\ (c+a+1)/(ak+r-a), & \text{if } a+1 \leq r \leq 2a. \end{cases}$$

PROOF. Note that $ck \equiv -ak \pmod{c+a}$ and $c(k+1) \equiv -(a+1)(k+1) \pmod{c+a+1}$. Therefore, $d_D(c + a, k) = ak$ for all $r$, and $d_D(c + a + 1, k + 1) = ak + r - a$ when $a + 1 \leq r \leq 2a$. The theorem then follows. □

THEOREM 2.6. *If $D = \{a, a + 1, c\}$ with $a + 1 < c$ and $c + a \equiv 2a$ or $0 \pmod{2a+1}$, then $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 2 + 1/a$.*

PROOF. Since $G(Z, D)$ contains the odd cycle $C_{2a+1}$, according to $(*)$, $2 + 1/a = (2a + 1)/\alpha(C_{2a+1}) \le \chi_f(C_{2a+1}) \le \chi_f(Z, D)$. On the other hand, since $c + a \equiv 2a$ or $0 \pmod{2a + 1}$, it follows from Theorem 2.5 that $f_D \le 2 + 1/a$. □

Denote the subgraph of $G(Z, D)$ induced by $V_i = \{0, 1, \cdots, i\}$ as $G_i$.

THEOREM 2.7. *If $D = \{2, 3, c\}$, with $3 < c$, where $c + 2 = 5k + r$, with $k \ge 1$ and $0 \le r \le 4$, then*

$$\chi_f(Z, D) = \chi_c(Z, D) = f_D = \begin{cases} (c+2)/2k, & \text{if } r = 1, 2, \\ (c+3)/(2k+1), & \text{if } r = 3, \\ 5/2, & \text{if } r = 4, 0. \end{cases}$$

PROOF. The case in which $r = 4$ or $0$ follows from Theorem 2.6. For the other cases, Theorem 2.5 implies that $f_D \le (c + 2)/2k$ when $r = 1, 2$, and $f_D \le (c + 3)/(2k + 1)$ when $r = 3$. Therefore it suffices to show that $\alpha(G_{c+1}) \le 2k$ when $r = 1, 2$ and $\alpha(G_{c+2}) \le 2k + 1$ when $r = 3$.

Consider the graph $G_{c+2}$ for $r = 1, 2, 3$. Decompose the vertex set $\{0, 1, \cdots, c+2\}$ into $k+1$ subsets $I_i = \{5i, 5i+1, \ldots, 5i+4\}$ for $0 \le i \le k-1$, and $J = \{5k, \ldots, 5k+r = c+2\}$. Then, $J = \{c+1, c+2\}$ when $r = 1$, $J = \{c, c+1, c+2\}$ when $r = 2$, and $J = \{c-1, c, c+1, c+2\}$ when $r = 3$. Suppose that $G_{c+2}$ has an independent set $S$ of size $2k + 2$. We may assume that $0 \in S$ and then $c \notin S$. Since every five consecutive vertices in $G_{c+2}$ form a 5-cycle, we conclude that $|I_i \cap S| = |J \cap S| = 2$ for $0 \le i \le k - 1$. Then $c + 1 \in S$, and hence, $1 \notin S$.

Since $|I_0 \cap S| = 2$, 2 and 3 are not in $S$. We therefore conclude that $4 \in S$. In a general step, using the fact that $|I_i \cap S| = 2$ and $5(i - 1) - 1 \in S$, it is straightforward to derive that $5i - 1 \in S$. Therefore, $5k - 1 \in S$. Since $5k - 1 = c$ when $r = 1$, and $5k - 1$ is adjacent to $c + 1$ when $r = 2$ or 3, we have contradictions. Hence, $\alpha(G_{c+2}) \le 2k + 1$ for $r = 1, 2, 3$. Moreover, for the case in which $r = 1$ or 2, any independent set $S'$ of $G_{c+2}$ of size $2k + 1$ that contains the vertex 0 does not contain the vertex $c + 1$. Hence, $c + 2 \in S'$ and $\alpha(G_{c+1}) \le 2k$. This completes the proof of the theorem. □

THEOREM 2.8. *Suppose $D = \{a, b, a + b\}$, with $0 < a < b$ and $\gcd(a, b) = 1$. If $a \equiv b \pmod 3$, then $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 3$.*

PROOF. Since $\gcd(a, b) = 1$ and $a \equiv b \pmod 3$, we have $a, b, c \not\equiv 0 \pmod 3$ and so $d_D(3, 1) = 1$. The theorem then follows from $(*)$ and the fact that $\{0, a, a + b\}$ is a clique. □

THEOREM 2.9. *If $D = \{1, 2, \cdots, m, n\}$, with $1 \le m < n$, then*

$$\chi_f(Z, D) = \chi_c(Z, D) = f_D = \begin{cases} m + 1, & \text{if } n \not\equiv 0 \pmod{m + 1}, \\ m + 1 + 1/k, & \text{if } n = k(m + 1). \end{cases}$$

PROOF. Suppose $n \not\equiv 0 \pmod{m + 1}$. Since $m + 1 \le \omega(G)$ and $d_D(m + 1, 1) = 1$, the theorem follows from $(*)$ and Theorem 2.2.

Suppose $n = k(m + 1)$. Since every independent set of $G_n$ contains at most one vertex from any $m + 1$ consecutive vertices, and at most one vertex from $\{0, n\}$, $\alpha(G_n) = k$. Consequently, $m + 1 + 1/k = (n + 1)/\alpha(G_n) \le \chi_f(G_n) \le \chi_f(Z, D)$. Also, $f_D \le (n + 1)/d_D(n + 1, k) = m + 1 + 1/k$. The theorem then follows. □

COROLLARY 2.10. *If $D = \{1, 2, 3k\}$, where $k \ge 1$, then $\chi_f(Z, D) = \chi_c(Z, D) = 3 + 1/k$.*

TABLE 1.

<table>
<tr><th colspan="2">Conditions of $a, b, c$</th><th>$\chi_f(Z, D)$, $\chi_c(Z, D)$, $f_D$</th><th>$\chi(Z, D)$</th></tr>
<tr><td colspan="2">$a, b, c$ are odd</td><td>2</td><td>2</td></tr>
<tr><td colspan="2">$a = 1, b = 2, c = 3k$</td><td>$3 + \frac{1}{k}$ (Corollary 2.10)</td><td>4</td></tr>
<tr><td colspan="2">$c = a + b, a \not\equiv b \pmod 3$</td><td>?</td><td>4</td></tr>
<tr><td colspan="2">$c = a + b, a \equiv b \pmod 3$</td><td>3 (Theorem 2.8)</td><td rowspan="5">3</td></tr>
<tr><td colspan="2">$b = a + 1, c \equiv a$ or $a + 1 \pmod{2a + 1}$</td><td>$2 + \frac{1}{a}$ (Theorem 2.6)</td></tr>
<tr><td rowspan="2">$a = 2, b = 3, c + 2 = 5k + r$</td><td>$r = 1, 2$</td><td>$\frac{c+2}{2k}$ (Theorem 2.7)</td></tr>
<tr><td>$r = 3$</td><td>$\frac{c+3}{2k+1}$ (Theorem 2.7)</td></tr>
<tr><td colspan="2">Otherwise</td><td>?</td></tr>
</table>

This is one of the two cases covered by Theorem 2.4 in which we have $\chi(Z, D) = 4$. The other is that in which $D = \{a, b, c\}$, $a + b = c$ and $a \not\equiv b \pmod 3$. We note that, in this case, the chromatic number of $G(Z, D)$ is easily determined. However, the circular chromatic numbers of $G(Z, D)$ are still unknown, except for some special values of $a$ and $b$.

We summarize the results for $D = \{a, b, c\}$ with $a < b < c$ and $\gcd(a, b, c) = 1$ in Table 1.

THEOREM 2.11. *If $D = \{q, q + 1, \dots, p\}$, with $q \le p$, then $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 1 + p/q$.*

PROOF. Since $d_D(p + q, 1) = q$, we conclude that $f_D \le (p + q)/q$. On the other hand, it is quite obvious that $\alpha(G_{p+q-1}) = q$. Hence, $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 1 + p/q$. □

THEOREM 2.12. *If $D = [1, r]$, where $r$ is any real number greater than or equal to 1, then $\chi_f(R, D) = \chi_c(R, D) = 1 + r$.*

PROOF. We first consider the case in which $r = p/q$ is rational. Let $D' = \{q, q+1, \dots, p\}$. It is then straightforward to verify that each connected component of $G(R, D)$ is isomorphic to $G(Z, D')$. According to Theorem 2.11, $\chi_f(R, D) = \chi_c(R, D) = 1 + r$.

When $r$ is irrational, then let $(r_i : i = 1, 2, \dots)$ and $(r'_i : i = 1, 2, \dots)$ be sequences of rational numbers such that $r'_i \le r \le r_i$ for each $i$ and $\lim_{i\to\infty} r'_i = \lim_{i\to\infty} r_i = r$. The above argument then shows that $1 + r'_i \le \chi_f(R, D) \le \chi_c(R, D) \le 1 + r_i$ for each $i$. Therefore, $\chi_f(R, D) = \chi_c(R, D) = 1 + r$. □

It was shown by Eggleton *et al.* [10] (Theorem 2) that if a prime distance graph has a proper $k$-coloring, then it has a periodic $k$-coloring. The proof in fact shows that any $k$-colorable distance graph has a periodic $k$-coloring. We remark that an argument parallel to the proof of Theorem 2 of [10] shows that if a distance graph $G(Z, D)$ has a $(p, q)$-coloring, then it has a periodic $(p, q)$-coloring. Also we note that a $(p, q)$-coloring derived by the multiplier method is always a periodic $(p, q)$-coloring.

## 3. CIRCULAR CHROMATIC NUMBER $\chi_c(Z, D_{m,k})$

As mentioned in the introduction, Chang *et al.* [3] determined the chromatic number and the fractional chromatic number of the distance graph $G(Z, D_{m,k})$, where $D_{m,k} = \{1, 2, \dots, m\}-$

TABLE 2.

<table>
<tr><td colspan="3">Conditions of $m, k, r, s$</td><td>$\chi_f(Z, D_{m,k})$</td><td>$\chi_c(Z, D_{m,k})$</td><td>$\chi(Z, D_{m,k})$</td></tr>
<tr><td colspan="3">$2k > m$</td><td>$k$</td><td>$k$</td><td>$k$</td></tr>
<tr><td rowspan="4">$2k \leq m$</td><td colspan="2">$r > s$</td><td rowspan="4">$\frac{m+k+1}{2}$</td><td rowspan="2">$\frac{m+k+1}{2}$</td><td>$\frac{m+k+1}{2}$</td></tr>
<tr><td rowspan="2">$r = 0$</td><td>$\gcd(m + k + 1, k) = 1$</td><td rowspan="2">$\frac{m+k+2}{2}$</td></tr>
<tr><td>$\gcd(m + k + 1, k) \neq 1$</td><td rowspan="2">?</td></tr>
<tr><td colspan="2">$1 \leq r \leq s$</td><td>$\frac{m+k+3}{2}$</td></tr>
</table>

$\{k\}$ and $1 \leq k \leq m$. They also determined the circular chromatic number of $G(Z, D_{m,k})$ for some pairs of integers $m$ and $k$.

Let $m + k + 1 = 2^r m'$ and $k = 2^s k'$, where $m'$ and $k'$ are both odd. Table 2 shows their results.

The circular chromatic numbers $\chi_c(Z, D_{m,k})$ remain unknown for those pairs of integers $m, k$ corresponding to the question mark in Table 2. In this section, we shall fill in the unknown part of Table 2 by showing that $\chi_c(Z, D_{m,k}) = \frac{m+k+2}{2}$ when $2k \leq m$ and $r \leq s$ and $\gcd(m + k + 1, k) \neq 1$.

The following lemma was proven in [16] and is used frequently in our proofs.

LEMMA 3.1 ([16]). *If $G$ has a circular chromatic number $\frac{p}{q}$ (where $p$ and $q$ are relatively prime), then $p \leq |V(G)|$, and any $(p, q)$-coloring $c$ of $G$ is an onto mapping from $V(G)$ to $\{0, 1, \ldots, p - 1\}$.*

As in the preceding section, we denote the subgraph of $G(Z, D_{m,k})$ induced by $V_i = \{0, 1, \ldots, i\}$ as $G_i$. We shall first derive a lower bound for $\chi_c(Z, D_{m,k})$.

LEMMA 3.2. *Suppose $2k \leq m$. Let $m+k+1 = 2^r m'$ and $k = 2^s k'$, where $r$ and $s$ are non-negative integers and $m'$ and $k'$ are odd integers. If $1 \leq r \leq s$, then $\chi_c(G_{m+2k-1}) > \frac{m+k+1}{2}$.*

PROOF. Since $m + k + 1$ is even and $\chi_c(G_{m+2k-1}) > \chi(G_{m+2k-1}) - 1$, it suffices to show that $\chi(G_{m+2k-1}) > \frac{m+k+1}{2}$. Assume to the contrary that $\chi(G_{m+2k-1}) \leq \frac{m+k+1}{2}$, and that $c$ is a $\frac{m+k+1}{2}$-coloring of $G_{m+2k-1}$.

For each integer $i$ with $0 \leq i \leq k - 2$, consider the subgraph of $G_{m+2k-1}$ induced by the $m + k + 1$ vertices $\{i, i + 1, \ldots, i + m + k\}$. This graph has an independence number 2. Therefore, each of the $\frac{m+k+1}{2}$ colors is used at most, and thus exactly, twice in this subgraph. Consequently, vertices $i$ and $i + m + k + 1$ have the same colors for all $0 \leq i \leq k - 2$. Therefore, for each $j \in S := \{0, 1, \ldots, m + k\}$, the only possible vertices in $S$ having the same color as $j$ are $j + k$ and $j - k$.

Consider the *circulant graph* $C(m + k + 1, k)$, with vertex set $S$ and in which vertex $i$ is adjacent to vertex $j$ iff $j \equiv i + k$ or $i - k \pmod{m + k + 1}$. It follows from the discussion in the preceding paragraph that two vertices $x$ and $y$ in $S$ have the same color only if $xy$ is an edge of the circulant graph $C(m + k + 1, k)$. Since the intersection of each color class with $S$ contains exactly two vertices, the coloring induces a perfect matching of $C(m + k + 1, k)$. However, $C(m + k + 1, k)$ is the disjoint union of $d$ cycles of length $\frac{m+k+1}{d}$, where $d = \gcd(m + k + 1, k)$.

Since $C(m+k+1, k)$ has a perfect matching, each cycle has an even length. This implies that $r > s$, contrary to the assumption $r \leq s$. Hence, $\chi(G_{m+2k-1}) > \frac{m+k+1}{2}$. $\square$

LEMMA 3.3. *Suppose* $2k \leq m$. *If* $m+k+1$ *is odd and* $\gcd(m+k+1, k) \neq 1$, *then* $\chi_c(G_{m+k}) > \frac{m+k+1}{2}$, *and hence,* $\chi_c(G_{m+2k-1}) > \frac{m+k+1}{2}$.

PROOF. First, it is clear that $\chi_c(G_{m+k}) \geq \frac{m+k+1}{\alpha(G_{m+k})} = \frac{m+k+1}{2}$. Suppose $\chi_c(G_{m+k}) = \frac{m+k+1}{2}$. Since $m+k+1$ and 2 are relatively prime, every $(m+k+1, 2)$-coloring $c$ of $G_{m+k}$ is onto and hence is one-to-one; i.e., there exists an ordering $x_0, x_1, x_2, \ldots, x_{m+k}$ of $V_{m+k}$ such that $c(x_i) = i$ for $0 \leq i \leq m+k$. Therefore, $X = (x_0, x_1, \ldots, x_{m+k}, x_0)$ is a cycle in the complement $G'$ of $G_{m+k}$.

Let $m = ak + b$, where $0 \leq b < k$. Since all vertices of $\{k-1, k, \ldots, m+1\}$ are of degree two in $G'$, the following paths must be on the cycle $X$:

$$P_i : i, k+i, 2k+i, \ldots, ak+i, (a+1)k+i \qquad \text{for } 0 \leq i \leq b;$$

$$P_j : j, k+j, 2k+j, \ldots, ak+j \qquad \text{for } b+1 \leq j \leq k-1.$$

For each vertex $u$, let $N(u) = \{v \in V_{m+k} : uv \in E(G')\}$. Since $N(k-1) = \{2k-1, m+k\}$ and $m+k = (a+1)k+b$, we have that $P_b P_{k-1}$ is a path of the cycle $X$. Since $N(k-2) = \{2k-2, m+k-1, m+k\}$ and vertex $m+k$ is on the path $P_b P_{k-1}$, we have that $P_{b-1} P_{k-2}$ is a path of the cycle $X$. Continuing this process, we have that $P'_t = P_{b+1+t} P_t$, where the index $b+1+t$ is taken modulo $k$, is a path of the cycle $X$ for $0 \leq t \leq k-1$. Since $\gcd(m+k+1, k) \neq 1$, we have $\gcd(b+1, k) \neq 1$. Therefore, these paths $P'_t$ form at least 2 disjoint cycles, contrary to our assumption that $X$ is a cycle. Thus, the coloring $c$ does not exist and $\chi_c(G_{m+k}) > \frac{m+k+1}{2}$.

Since $G_{m+k}$ is a subgraph of $G_{m+2k-1}$, we conclude that $\chi_c(G_{m+2k-1}) > \frac{m+k+1}{2}$. $\square$

THEOREM 3.4. *Suppose* $2k \leq m$. *Let* $m+k+1 = 2^r m'$ *and* $k = 2^s k'$, *where* $r$ *and* $s$ *are non-negative integers and* $m'$ *and* $k'$ *are odd integers. If* $r \leq s$ *and* $\gcd(m+k+1, k) \neq 1$, *then* $\chi_c(Z, D_{m,k}) \geq \frac{m+k+2}{2}$.

PROOF. Suppose $\chi_c(G_{m+2k-1}) = \frac{p}{q}$, where $p$ and $q$ are relatively prime. Then, $p \leq |V_{m+2k-1}| = m+2k$ and $\frac{p}{q} > \frac{m+k+1}{2}$ according to Lemmas 3.2 and 3.3. If $q \geq 3$, then $p > \frac{q}{2}(m+k+1) \geq \frac{3}{2}(m+k+1) > m+2k$, a contradiction. Hence, $q \leq 2$ and so $\chi_c(Z, D_{m,k}) \geq \frac{p}{q} \geq \frac{m+k+2}{2}$. $\square$

Now we give an $(m+k+2, 2)$-coloring of $G(Z, D_{m,k})$ to show that $\chi_c(Z, D_{m,k}) \leq \frac{m+k+2}{2}$. We first give an $(m+k+2, 2)$-coloring of $G_{m+k}$ that is a variation of the coloring given in Theorem 2.1 after a shift operation. It is then extended to an $(m+k+2, 2)$-coloring of $G(Z, D_{m,k})$.

LEMMA 3.5. *If* $2k \leq m$, *then* $G_{m+k}$ *has an* $(m+k+2, 2)$*-coloring* $c$ *such that* $c(x) = c(x-k)+1$ *for* $k \leq x \leq m+k$.

PROOF. Suppose $m+k+1 = dm'$ and $k = dk'$, where $\gcd(m+k+1, k) = d$. Since $\gcd(m', k') = 1$, there exists an integer $n$ such that $nk' \equiv 1 \pmod{m'}$. Let $a_i = in \pmod{m'}$ for $0 \leq i \leq m'-1$. Consider the mapping $c$ from $V_{m+k}$ to $\{0, 1, \ldots, dm'-1 = m+k\}$ defined by $c(x) = a_i + jm'$, where $x = id + (d-1-j)$, with $0 \leq i \leq m'-1$ and $0 \leq j \leq d-1$.

For any edge $xy$ in $G_{m+k}$, we shall prove that $\|c(x) - c(y)\|_{m+k+2} \geq 2$. Suppose to the contrary that $c(x) = c(y)$, or $c(x) = c(y)+1$, or $c(x)+1 = c(y)$. Let $x = i_1 d + (d-1-j_1)$

and $y = i_2d + (d-1-j_2)$. For the case in which $c(x) = c(y)$, we have $a_{i_1} = a_{i_2}$ and $j_1 = j_2$, which imply $i_1 = i_2$ and $x = y$, a contradiction to $xy$ being an edge. For the case in which $c(x) = c(y) + 1$, either (1) $a_{i_1} = a_{i_2} + 1$ and $j_1 = j_2$, or (2) $a_{i_1} = 0$ and $a_{i_2} = m' - 1$ and $j_1 = j_2 + 1$. In subcase (1), we have $i_1 \equiv i_2 + k' \pmod{m'}$. Thus, $x - y = k$ or $y - x = m + 1$, a contradiction. In subcase (2), we have $i_1 = 0$ and $i_2 = m' - k'$. Thus, $y - x = m + 2$, a contradiction. Similarly, it is impossible that $c(x) + 1 = c(y)$. This completes the proof of the lemma. □

THEOREM 3.6. *If* $2k \le m$, *then* $\chi_c(Z, D_{m,k}) \le \frac{m+k+2}{2}$.

PROOF. Let $c$ be the coloring of $G_{m+k}$ given in Lemma 3.5. Consider the mapping $c'$ of $G(Z, D_{m,k})$ defined by

$$c'(x) = \begin{cases} c(x), & \text{for } 0 \le x \le m+k, \\ (c'(x-k)+1) \bmod (m+k+2), & \text{for } m+k+1 \le x, \\ (c'(x+k)-1) \bmod (m+k+2), & \text{for } 0 > x. \end{cases}$$

We now show that $c'$ is a proper $(m+k+2, 2)$-coloring of $G(Z, D_{m,k})$ by induction. According to Lemma 3.5, $c'$ is proper in $G_{m+k}$. Suppose $c'$ is proper in $G_{x-1}$ for $x \ge m+k+1$. Let $xy$ be an edge in $G_x$; i.e., $y = x - i$ for some $i \in D_{m,k}$. First, $c'(y)$ is not equal to $c'(x)$ mod $(m+k+2)$ or $(c'(x)-1)$ mod $(m+k+2)$, since $c'(y) \equiv c'(x) - 2 \pmod{m+k+2}$ when $i = 2k$, and $y = x - i$ is adjacent to $x - k$ in $G_{x-1}$ when $i \ne 2k$, where $c'(x-k) = (c'(x)-1)$ mod $(m+k+2)$. Also, $c'(y-k)$ is not equal to $c'(x)$ mod $(m+k+2)$, since $x-k$ is adjacent to $y-k$ in $G_{x-1}$ and $c'(x-k) = (c'(x)-1)$ mod $(m+k+2)$. Hence, $c'(y)$ is not equal to $(c'(x)+1)$ mod $(m+k+2)$. By induction, $c'$ is proper for non-negative vertices in $G(Z^+, D_{m,k})$. Similar arguments work for negative vertices. This completes the proof of the theorem. □

Combining Theorems 3.4 and 3.6 and results in [3], we have

THEOREM 3.7. *Suppose* $2k \le m$. *Let* $m+k+1 = 2^r m'$ *and* $k = 2^s k'$, *where* $r$ *and* $s$ *are non-negative integers and* $m'$ *and* $k'$ *are odd integers. If* $r \le s$ *and* $\gcd(m+k+1, k) \ne 1$, *then* $\chi_c(Z, D_{m,k}) = \frac{m+k+2}{2}$; *otherwise,* $\chi_c(Z, D_{m,k}) = \frac{m+k+1}{2}$.

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