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Circular Chromatic Numbers of Distance Graphs with Distance Sets Missing Multiples†

LINGLING HUANG AND GERARD J. CHANG

Given positive integers m, k, s with $m > sk$, let $D_{m,k,s}$ represent the set $\{1, 2, \dots, m\} \setminus \{k, 2k, \dots, sk\}$. The distance graph $G(Z, D_{m,k,s})$ has as vertex set all integers Z and edges connecting i and j whenever $|i - j| \in D_{m,k,s}$. This paper investigates chromatic numbers and circular chromatic numbers of the distance graphs $G(Z, D_{m,k,s})$. Deuber and Zhu [8] and Liu [13] have shown that $\lceil \frac{m+sk+1}{s+1} \rceil \le \chi(G(Z, D_{m,k,s})) \le \lceil \frac{m+sk+1}{s+1} \rceil + 1$ when $m \ge (s + 1)k$. In this paper, by establishing bounds for the circular chromatic number $\chi_c(G(Z, D_{m,k,s}))$ of $G(Z, D_{m,k,s})$, we determine the values of $\chi(G(Z, D_{m,k,s}))$ for all positive integers m, k, s and $\chi_c(G(Z, D_{m,k,s}))$ for some positive integers m, k, s.

1. INTRODUCTION

Given a set D of positive integers, the *distance graph* $G(Z, D)$ has all integers as vertices, and two vertices are adjacent if and only if their difference is in D; that is, the vertex set is Z and the edge set is $\{uv : |u - v| \in D\}$. We call D the *distance set*. This paper studies chromatic and circular chromatic numbers of some distance graphs with certain distance sets.

The circular chromatic number of a graph is a natural generalization of the chromatic number of a graph, introduced by Vince [15] as the name "star chromatic number." Suppose p and q are positive integers such that $p \ge 2q$. Let G be a graph with at least one edge. A (p, q)-*coloring* of $G = (V, E)$ is a mapping c from V to $\{0, 1, \dots, p - 1\}$ such that $q \le |c(x) - c(y)| \le p - q$ for any edge xy in E. The *circular chromatic number* $\chi_c(G)$ of G is the infimum of the ratios p/q for which there exists a (p, q)-coloring of G.

Note that for $p \ge 2$, a $(p, 1)$-coloring of a graph G is simply an ordinary p-coloring of G. Therefore, $\chi_c(G) \le \chi(G)$ for any graph G. Let G be a graph which is not a null graph. On the other hand, it has been shown [15] that for all finite graphs G, we have $\chi(G) - 1 < \chi_c(G)$. Applying a result of de Bruijn and Erdős [6], this can be proved also for infinite graphs. Therefore, $\chi(G) = \lceil \chi_c(G) \rceil$ if $G \ne N_n$. In particular, two graphs with the same circular chromatic number also have the same chromatic number. However, two graphs with the same chromatic number may have different circular chromatic numbers. Thus $\chi_c(G)$ is a refinement of $\chi(G)$, and it contains more information about the structure of the graph. It is usually much more difficult to determine the circular chromatic number of a graph than to determine its chromatic number.

The fractional chromatic number of a graph is another well-known variation of the chromatic number. A *fractional coloring* of a graph G is a mapping c from $\mathcal{I}(G)$, the set of all independent sets of G, to the interval $[0, 1]$ such that $\sum_{x \in I \in \mathcal{I}(G)} c(I) \ge 1$ for all vertices x of G. The *fractional chromatic number* $\chi_f(G)$ of G is the infimum of the value $\sum_{I \in \mathcal{I}(G)} c(I)$ of a fractional coloring c of G.

For any graph G, it is well known that

$$\max\{\omega(G), |G|/\alpha(G)\} \le \chi_f(G) \le \chi_c(G) \le \lceil \chi_c(G) \rceil = \chi(G), \qquad (*)$$

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where $\omega(G)$ (respectively, $\alpha(G)$) is the *clique* (respectively, *independence*) *number* of G which is the maximum size of a pairwise adjacent (respectively, non-adjacent) vertex subset of $V(G)$.

For simplicity, let $\omega(S, D)$, $\alpha(S, D)$, $\chi_f(S, D)$, $\chi_c(S, D)$ and $\chi(S, D)$ denote the clique number, the independence number, the fractional chromatic number, the circular chromatic number and the chromatic number of the distance graph $G(S, D)$, respectively.

For different types of distance sets D, the problem of determining $\chi(Z, D)$ has been studied extensively, see Refs [4, 5, 7, 9, 10, 12, 16–19]. For instance, the case that D contains at most three integers were studied by Eggleton, Erdős and Skilton [9], Chen, Chang and Huang [5], Voigt [16], Deuber and Zhu [7], Zhu [18], and at last completely determined by Zhu [19].

Given positive integers m, k, s with $m > sk$, let $D_{m,k,s}$ denote the distance set $\{1, 2, \dots, m\}\setminus\{k, 2k, \dots, sk\}$. For $s = 1$, the chromatic number of $G(Z, D_{m,k,1})$ was first studied in [9, 12, 13] and finally completely determined by Chang, Liu and Zhu [4]. They also determined the fractional chromatic number of $G(Z, D_{m,k,1})$. The circular chromatic number of $G(Z, D_{m,k,1})$ was then determined by Chang, Huang and Zhu [2]. Recently, Liu and Zhu [14] determined the fractional chromatic number of $G(Z, D_{m,k,s})$ for a general s, which gives a lower bound of $\chi(Z, D_{m,k,s})$. Liu and Zhu [14], Deuber and Zhu [8] also studied $\chi(Z, D_{m,k,s})$ for $s = 2, 3$, prime $s + 1$, and obtained some results for general s. Moreover, Deuber and Zhu [8] showed that for any values of m, k, s with $m \ge (s+1)k$, $\lceil \frac{m+sk+1}{s+1} \rceil \le \chi(Z, D_{m,k,s}) \le \lceil \frac{m+sk+1}{s+1} \rceil + 1$. In this paper, by establishing bounds for $\chi_c(Z, D_{m,k,s})$, we determine the values of $\chi(Z, D_{m,k,s})$ for all positive integers m, k, s, and $\chi_c(Z, D_{m,k,s})$ for some positive integers m, k, s.

Note that it becomes an easy case if $m < (s+1)k$. Define a coloring f of $G(Z, D_{m,k,s})$ by: $f(x) = x \bmod k$ for any $x \in Z$. As $D_{m,k,s}$ contains no multiples of k, it can be easily verified that f is a proper coloring. Thus, $\chi(Z, D_{m,k,s}) \le k$. As any consecutive k vertices in $G(Z, D_{m,k,s})$ form a clique, $k \le \omega(Z, D_{m,k,s})$. This implies that all values in $(*)$ are equal to k for $G = G(Z, D_{m,k,s})$ if $m < (s+1)k$ (see Ref. [14]). Therefore, throughout the article, we assume $m \ge (s+1)k$.

The following table shows all results concerning the distance graph $G(Z, D_{m,k,s})$. Note that the value of $\chi_f(Z, D_{m,k,s})$ is determined in Ref. [14] and some value of $\chi(Z, D_{m,k,s})$ is determined in Refs [8, 14]. Also, all values of $\chi_c(Z, D_{m,k,s})$ are given in this paper, which also implies the results of $\chi(Z, D_{m,k,s})$. Let

$d = \gcd(k, m + sk + 1)$,
$a = (m + sk + 1) \bmod d(s + 1)$,
$b = (m + sk + 1) \bmod (s + 1)$.

Therefore,

$a = 0$ means $d(s + 1) \mid (m + sk + 1)$,
$a \ne 0$ means $d(s + 1) \nmid (m + sk + 1)$,
$b = 0$ means $(s + 1) \mid (m + sk + 1)$,
$b \ne 0$ means $(s + 1) \nmid (m + sk + 1)$.

Note that when $m \ge (s+1)k$ with $b \ne 0$ and $d > 1$, we only know that $\frac{m+sk+1}{s+1} \le \chi_c(Z, D_{m,k,s}) \le \frac{m+sk+2}{s+1}$, but still do not know the exact value of $\chi_c(Z, D_{m,k,s})$.

2. Main results

In the study of the chromatic number of the distance graphs $G(Z, D_{m,k,s})$ with distance sets $D_{m,k,s} = \{1, 2, \dots, m\}\setminus\{k, 2k, \dots, sk\}$, Liu and Zhu [14] obtained the following result on fractional chromatic numbers, which asserts a lower bound for the circular chromatic numbers and chromatic numbers (by $(*)$).

<table>
<tr><td colspan="3">Conditions of parameters</td><td>$\chi_f(Z, D_{m,k,s})$</td><td>$\chi_c(Z, D_{m,k,s})$</td><td>$\chi(Z, D_{m,k,s})$</td></tr>
<tr><td colspan="3">$m < (s+1)k$</td><td>k [14]</td><td>k [14]</td><td>k [14]</td></tr>
<tr><td rowspan="4">$m \geq (s+1)k$</td><td colspan="2">$a = 0$</td><td rowspan="4">$\frac{m+sk+1}{s+1}$ [14]</td><td rowspan="2">$\frac{m+sk+1}{s+1}$</td><td>$\frac{m+sk+1}{s+1}$ [14]</td></tr>
<tr><td rowspan="2">$b \neq 0$</td><td>$d = 1$</td><td rowspan="2">$\lceil \frac{m+sk+1}{s+1} \rceil$</td></tr>
<tr><td>$d > 1$</td><td>$\leq \frac{m+sk+2}{s+1}$</td></tr>
<tr><td colspan="2">$a \neq 0,\ b = 0$</td><td>$\frac{m+sk+2}{s+1}$</td><td>$\frac{m+sk+1}{s+1}+1$</td></tr>
</table>

THEOREM 1 ([14]). *For positive integers m, k and s with $m \geq (s+1)k$,*

$$\chi_f(Z, D_{m,k,s}) = \frac{m+sk+1}{s+1}.$$

Liu and Zhu [14] also gave an upper bound of $\chi(Z, D_{m,k,s})$ as follows.

LEMMA 2 ([14]). *For positive integers m, k, s with $m \geq (s+1)k$ and $d = \gcd(k, m+sk+1)$,*

$$\chi(Z, D_{m,k,s}) \leq d \left\lceil \frac{m+sk+1}{d(s+1)} \right\rceil.$$

By Lemma 2, it is clear that if $d(s+1) \mid (m+sk+1)$, then $\chi(Z, D_{m,k,s}) \leq \frac{m+sk+1}{s+1}$. Hence we have

THEOREM 3 ([14]). *For positive integers m, k, s with $m \geq (s+1)k$ and $d = \gcd(k, m+sk+1)$, if $d(s+1) \mid (m+sk+1)$, then*

$$\chi_c(Z, D_{m,k,s}) = \chi(Z, D_{m,k,s}) = \frac{m+sk+1}{s+1}.$$

Note that Theorem 3 only gives the values of $\chi_c(Z, D_{m,k,s})$ and $\chi(Z, D_{m,k,s})$ under the condition $d(s+1) \mid (m+sk+1)$, although for $\frac{m+sk+1}{s+1}$ to be an integer we only need $(s+1) \mid (m+sk+1)$.

Next, we show that if $s+1$ divides $m+sk+1$ but $d(s+1)$ does not, then $\chi(Z, D_{m,k,s}) > \frac{m+sk+1}{s+1}$. Let $G[i, j]$ denote the subgraph of $G(Z, D_{m,k,s})$ induced by $V[i, j] = \{i, i+1, \ldots, j\}$ for any integers $i \leq j$.

LEMMA 4. *For positive integers m, k, s with $m \geq (s+1)k$ and $d = \gcd(k, m+sk+1)$, if $(s+1) \mid (m+sk+1)$ but $d(s+1) \nmid (m+sk+1)$, then*

$$\chi_c(G[0, m+sk+k-1]) > \frac{m+sk+1}{s+1} \text{ and } \chi(Z, D_{m,k,s}) > \frac{m+sk+1}{s+1}.$$

PROOF. Since $\chi_c(G[0, m+sk+k-1]) > \chi(G[0, m+sk+k-1]) - 1$ and $\frac{m+sk+1}{s+1}$ is an integer, it suffices to show that $\chi(G[0, m+sk+k-1]) > \frac{m+sk+1}{s+1}$. Suppose $\chi(G[0, m+$

$sk + k - 1]) \le \frac{m+sk+1}{s+1}$; that is, $G[0, m + sk + k - 1]$ has an $\frac{m+sk+1}{s+1}$-coloring f. For any integer $0 \le i \le k-1$, the subgraph $G[i, m+sk+i]$ has $m+sk+1$ vertices and independence number $s + 1$. Since f is an $\frac{m+sk+1}{s+1}$-coloring, each color class of f consists of exactly $s + 1$ vertices of $G[i, m + sk + i]$. It follows that $f(i) = f(m + sk + i + 1)$ for any integer $0 \le i \le k - 2$. Now, consider the color classes of f for the graph $G[0, m + sk]$. For each color class $C = \{x_1, x_2, \ldots, x_{s+1}\}$, where $x_1 < x_2 < \ldots < x_{s+1}$, the difference $x_{i+1} - x_i$ of two consecutive vertices in C is called a *gap*. Note that there is at most one gap greater than m and all other gaps are equal to k. Suppose there is a gap greater than $m + 1$ and all other $s - 1$ gaps are equal to k. Let the first vertex $x_1 = i$ and the last vertex $x_{s+1} = j$. Then $i \le k-2$ and $j \ge m + (s-1)k + i + 2$, which imply that $f(i) = f(j) = f(m + sk + i + 1)$, contradicting $1 \le (m + sk + i + 1) - j \le k - 1$. Therefore, all gaps are equal to k or exactly one gap is equal to $m + 1$ with the others equal to k. Then C is of the form $\{i, i + k, i + 2k, \ldots, i + sk\}$ (where each number is calculated modulo $m + sk + 1$).

Let $u = \frac{m+sk+1}{d}$. Divide the vertex set of $G[0, m + sk]$ into d subsets of the form $\{i, i + k, i + 2k, \ldots, i + (u-1)k\} \pmod{m + sk + 1}$, each of size u. Then each of these d subsets is the union of some color classes of size $s + 1$, so $s + 1$ divides u, i.e., $d(s + 1) \mid (m + sk + 1)$, a contradiction. Hence $\chi(G[0, m + sk + k - 1]) > \frac{m+sk+1}{s+1}$. □

We then show that $\chi_c(Z, D_{m,k,s}) \le \frac{m+sk+2}{s+1}$ for any positive integers m, k, s with $m \ge (s+1)k$. It follows that $\chi(Z, D_{m,k,s}) \le \lceil \frac{m+sk+2}{s+1} \rceil$ by (∗). Hence, $\chi(Z, D_{m,k,s}) = \lceil \frac{m+sk+1}{s+1} \rceil$ when $(s + 1) \nmid (m + sk + 1)$, and $\chi(Z, D_{m,k,s}) = \frac{m+sk+1}{s+1} + 1$ when $(s + 1) \mid (m + sk + 1)$ but $d(s+1) \nmid (m+sk+1)$. These, together with Theorem 3, give all values of the chromatic numbers $\chi(Z, D_{m,k,s})$.

To calculate the upper bound of $\chi_c(Z, D_{m,k,s})$, we first give an $(m+sk+2, s+1)$-coloring c of the subgraph $G[0, m + sk]$ and then extend it to an $(m + sk + 2, s + 1)$-coloring of $G(Z, D_{m,k,s})$. Intuitively, the coloring is the mapping c from $V[0, m + sk]$ to $\{0, 1, \ldots, m + sk\}$ given in the following algorithm, although we in fact define it directly in the proof of Lemma 5.

```
Algorithm.
begin
    for j := 0 to m + sk do c(j) := −1;
    i := d − 1;
    c(i) := 0;
    repeat
        j := (i + k) mod (m + sk + 1);
        if c(j) ≠ −1 then j := j − 1;
        c(j) := c(i) + 1;
        i := j;
    until c(i) = m + sk
end
```

LEMMA 5. *For positive integers m, k, s with $m \ge (s+1)k$, there exists an $(m+sk+2, s+1)$-coloring c of $G[0, m + sk]$ such that $c(x) = c(x - k) + 1$ for $k \le x \le m + sk$.*

PROOF. Suppose $k = dk'$ and $m + sk + 1 = dm'$, where $d = \gcd(k, m + sk + 1)$. Since $\gcd(k', m') = 1$, there exists an integer n such that $k'n \equiv 1 \pmod{m'}$. Let $a_i = (in) \bmod m'$

for $0 \le i \le m' - 1$. Consider the mapping c from $V[0, m + sk]$ to $\{0, 1, \dots, m + sk\}$, where $m + sk = dm' - 1$, defined by

$$c(x) = a_{i_x} + (d - 1 - j_x)m' \ \text{ for } \ x = i_x d + j_x,$$

where $0 \le i_x \le m' - 1$ and $0 \le j_x \le d - 1$. Note that $i_x = \lfloor x/d \rfloor$ and $j_x = x \bmod d$. It is straightforward to check that c is a one-to-one and hence onto mapping.

First, note that for $k \le x \le m + sk$, $i_x = i_{x-k} + k'$ and $j_x = j_{x-k}$. Therefore, $a_{i_x} = (i_x n) \bmod m' = (i_{x-k} n + k'n) \bmod m' = a_{i_{x-k}} + 1$ as $i_x \ne 0$, and so $c(x) = c(x - k) + 1$.

Next, we show that $s + 1 \le |c(x) - c(y)| \le (m + sk + 2) - (s + 1)$ for any edge xy in $G[0, m + sk]$. Let $x = i_x d + j_x$ and $y = i_y d + j_y$, where $0 \le i_x, i_y \le m' - 1$ and $0 \le j_x, j_y \le d - 1$. Without loss of generality, we may assume that $c(x) > c(y)$.

Suppose $0 < c(x) - c(y) \le s$. Since $m \ge (s + 1)k$, we have $m' \ge (s + 1)k' > sk' > s$. It follows that either (1) $j_x = j_y$ and $0 < a_{i_x} - a_{i_y} \le s$, or (2) $j_y = j_x + 1$ and $m' - s \le a_{i_y} - a_{i_x} < m'$. In case (1), we have $0 < (i_x - i_y)n \bmod m' \le s$. Hence $i_x - i_y \equiv k', 2k', \dots$, or $sk' \pmod{m'}$. It follows that $x - y \equiv k, 2k, \dots$, or $sk \pmod{m + sk + 1}$, contradicting $|x - y| \in D_{m,k,s}$. In case (2), we have $0 \le a_{i_x} \le s - 1$ and $m' - s \le a_{i_y} < m'$. It follows that $i_x = 0, k', 2k', \dots$, or $(s - 1)k'$, and $i_y = m' - k', m' - 2k', \dots$, or $m' - sk'$. Hence $i_y - i_x = m' - k', m' - 2k', \dots$, or $m' - sk'$ by $m' - s \le a_{i_y} - a_{i_x} < m'$, which implies that $y - x = (m + sk + 2) - k, (m + sk + 2) - 2k, \dots$, or $(m + sk + 2) - sk$, a contradiction to $y - x \in D_{m,k,s}$. Therefore $s + 1 \le c(x) - c(y)$.

Suppose $m + sk + 2 - s \le c(x) - c(y) \le m + sk$ (note that $m + sk$ is the largest color, also $s \ge 2$). Since $m' > s$ and $m + sk + 1 = dm'$, we have that $c(x) - c(y) \ge (d - 1)m' + 2$ and so $j_y - j_x = d - 1$, i.e., $j_x = 0$ and $j_y = d - 1$. Then $m' - (s - 1) \le a_{i_x} - a_{i_y} \le m' - 1$. Hence $0 \le a_{i_y} \le s - 2$ and $m' - (s - 1) \le a_{i_x} \le m' - 1$. It follows that $i_y = 0, k', 2k', \dots$, or $(s - 2)k'$, and $i_x = m' - k', m' - 2k', \dots$, or $m' - (s - 1)k'$. Hence $i_x - i_y = m' - k', m' - 2k', \dots$, or $m' - (s - 1)k'$ by $m' - (s - 1) \le a_{i_x} - a_{i_y} \le m' - 1$, which implies $x - y = (m + sk + 1) - k - (d - 1), (m + sk + 1) - 2k - (d - 1), \dots$, or $(m + sk + 1) - (s - 1)k - (d - 1)$ that is an integer larger than $m + 1$, contradicting $|x - y| \in D_{m,k,s}$. Therefore, $c(x) - c(y) \le (m + sk + 2) - (s + 1)$.

Thus, c is an $(m + sk + 2, s + 1)$-coloring of $G[0, m + sk]$. □

THEOREM 6. *For positive integers m, k, s with $m \ge (s + 1)k$,*

$$\chi_c(Z, D_{m,k,s}) \le \frac{m + sk + 2}{s + 1}.$$

PROOF. Let c be the $(m + sk + 2, s + 1)$-coloring of $G[0, m + sk]$ given in Lemma 5. Consider the mapping $c' : Z \to \{0, 1, \dots, m + sk + 1\}$ defined by

$$c'(x) = \begin{cases} c(x), & \text{for } 0 \le x \le m + sk, \\ (c'(x - k) + 1) \bmod (m + sk + 2), & \text{for } x \ge m + sk + 1, \\ (c'(x + k) - 1) \bmod (m + sk + 2), & \text{for } x < 0. \end{cases}$$

We show that c' is a proper $(m+sk+2, s+1)$-coloring of $G(Z, D_{m,k,s})$ by induction. According to Lemma 5, c' is proper in the subgraph $G[0, m+sk]$. Suppose c' is proper in $G[0, x-1]$ for $x \geq m+sk+1$. Let xy be any edge of $G[0, x]$, i.e., $y = x - i$ for some $i \in D_{m,k,s}$. Since $x-k$ is adjacent to $y-k$ in $G[0, x-1]$, by the induction hypothesis, $s+1 \leq |c'(x-k) - c'(y-k)| \leq (m+sk+2)-(s+1)$. It follows that $s+1 \leq |c'(x) - c'(y)| \leq (m+sk+2)-(s+1)$. Hence c' is proper in $G(Z^+, D_{m,k,s})$ by induction. A similar argument works for negative vertices. Therefore, c' is a proper $(m+sk+2, s+1)$-coloring of $G(Z, D_{m,k,s})$. □

According to $(*)$, Lemma 4, Theorems 1 and 6, we have the following values of the chromatic numbers of the graphs $G(Z, D_{m,k,s})$ when $d(s+1) \nmid (m+sk+1)$.

THEOREM 7. *Suppose m, k, s are positive integers with $m \geq (s+1)k$ and $d = \gcd(k, m+sk+1)$. If $(s+1) \nmid (m+sk+1)$, then*

$$\chi(Z, D_{m,k,s}) = \left\lceil \frac{m+sk+1}{s+1} \right\rceil.$$

If $(s+1) \mid (m+sk+1)$ and $d(s+1) \nmid (m+sk+1)$, then

$$\chi(Z, D_{m,k,s}) = \frac{m+sk+1}{s+1} + 1.$$

The following lemma is useful in determining the circular chromatic numbers of the distance graphs $G(Z, D_{m,k,s})$.

LEMMA 8 ([15]). *If $\chi_c(G) = p/q$ for any graph G, where p and q are relatively prime, then $p \leq |V(G)|$ and any (p, q)-coloring of G is an onto mapping.*

THEOREM 9. *For positive integers m, k, s with $m \geq (s+1)k$ and $d = \gcd(k, m+sk+1)$, if $(s+1) \mid (m+sk+1)$ and $d(s+1) \nmid (m+sk+1)$, then*

$$\chi_c(Z, D_{m,k,s}) = \frac{m+sk+2}{s+1}.$$

PROOF. Suppose $\chi_c(G[0, m+sk+k-1]) = p/q$, where p and q are relatively prime. By Lemma 4, $\frac{p}{q} \geq \frac{m+sk+1}{s+1} + \frac{1}{q}$ since $(s+1) \mid (m+sk+1)$; and, by Lemma 8, $p \leq |V[0, m+sk+k-1]| = m+(s+1)k$. If $\frac{p}{q} < \frac{m+sk+2}{s+1}$, then $\frac{1}{q} < \frac{1}{s+1}$. Therefore $q > s+1$, which implies that $p > \frac{q}{s+1}(m+sk+1) \geq \frac{s+2}{s+1}(m+sk+1) > m+(s+1)k$ since $m \geq (s+1)k$, a contradiction. Hence, $\chi_c(Z, D_{m,k,s}) \geq \frac{p}{q} \geq \frac{m+sk+2}{s+1}$. By Theorem 6, we have $\chi_c(Z, D_{m,k,s}) = \frac{m+sk+2}{s+1}$. □

The next theorem determines the circular chromatic number of the distance graph $G(Z, D_{m,k,s})$ when k is relatively prime to $m+sk+1$.

THEOREM 10. *For positive integers m, k, s with $m \geq (s+1)k$, if k is relatively prime to $m+sk+1$, then*

$$\chi_c(Z, D_{m,k,s}) = \frac{m+sk+1}{s+1}.$$

PROOF. By Theorem 1 and $(*)$, it suffices to show that $\chi_c(Z, D_{m,k,s}) \leq \frac{m+sk+1}{s+1}$, that is, $G(Z, D_{m,k,s})$ has an $(m+sk+1, s+1)$-coloring. Since k is relatively prime to $m+sk+1$, there exists an integer n such that $nk \equiv 1 \pmod{m+sk+1}$. Consider the mapping c defined by $c(i) = (in) \bmod (m+sk+1)$ for all $i \in Z$. Choose any edge ij of $G(Z, D_{m,k,s})$. If $0 \leq |c(i) - c(j)| \leq s$ or $(m+sk+1) - s \leq |c(i) - c(j)| \leq m+sk$, then $c(i) - c(j) \equiv 0, 1, \ldots, s, -1, -2, \ldots,$ or $-s \pmod{m+sk+1}$. It implies that $i - j \equiv 0, k, \ldots, sk, -k, -2k, \ldots,$ or $-sk \pmod{m+sk+1}$, contradicting $|i-j| \in D_{m,k,s}$. Thus, c is an $(m+sk+1, s+1)$-coloring of $G(Z, D_{m,k,s})$.□

We conclude that all values $\chi_c(Z, D_{m,k,s})$ are determined except for the case when $(s+1) \not| (m+sk+1)$ and $\gcd(k, m+sk+1) > 1$.

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Lingling Huang
Department of Hospital and Health Care Administration,
Chungtai Institute of Health Science and Technology,
Taichung, Taiwan
E-mail: lhuang@chtai.ctc.edu.tw
and

Gerard J. Chang
Department of Applied Mathematics,
National Chiao Tung University,
Hsinchu 300,
Taiwan
E-mail: gjchang@math.nctu.edu.tw