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A novel interpretation of the two-dimensional discrete Hartley transform

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Abstract

In this paper, a novel view of the two-dimensional discrete Hartley transform (2-D DHT) is proposed. We show that the 2-D DHT can be obtained by projecting the two-dimensional discrete Fourier transform (2-D DFT) from the extension field to the basefield. The conjugacy relation and the trace function are applied to perform the projection operator. It is quite different from the traditional treatment of the 2-D DHT, which is trigonometric decomposition based.

Zusammenfassung

In diesem Beitrag wird eine neue Sicht auf die zweidimensionale Hartley-Transformation (2-D DHT) vorgeschlagen. Wir zeigen, daß die 2-D DHT durch Projektion der zweidimensionalen diskreten Fourier-Transformation (2-D DFT) vom Erweiterungsfeld auf das Basisfeld erhalten werden kann. Die Konjugiert-Beziehung und die Spurfunktion werden angewendet, um den Projektionsoperator darzustellen. Dies ist völlig verschieden von der traditionellen Behandlung der 2-D DHT, welche auf einer trigonometrischen Zerlegung basiert.

Résumé

Nous proposons dans cet article une approche nouvelle de la transformation de Hartley discrète bi-dimensionnelle (2-D DHT). Nous montrons que la 2-D DHT peut être obtenue en projetant la transformation de Fourier discrète bi-dimensionnelle (2-D DFT) du champ étendu sur le champ de base. La relation de conjugaison et la fonction trace sont appliquées pour mettre en oeuvre l'opérateur de projection. Ceci est complètement différent du traitement traditionnel de la 2-D DHT, qui est basé sur une décomposition trigonométrique.

Keywords: Extension field; Projection; Discrete Fourier transform, Discrete Hartley transform

1. Introduction

In recent years, there has been a growing interest regarding the study of discrete transforms in the area of digital signal processing (DSP) [10]. This is primarily due to the analysis of a signal is often performed by transforming the signal into a domain in which the signal characteristics of interest are exhibited most prominently. A major challenge in modern DSP system design is discovering algorithms that are inherently capable of matching the advantages offered by the high speed

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digital computer and the rapid advances in VLSI technology.

Since the discrete Fourier transform (DFT) has been developed for a long time, its corresponding fast algorithms and architectures have been explored widely and maturelly. Nevertheless, due to the fact of the complex transform kernel of DFT, both the storage and the computational costs are introduced in computation of the DFT when the input sequence is real. A transform that directly maps a real-valued sequence to a real-valued spectrum while preserving most of the useful properties of the DFT is sometimes preferred. One such transform is the discrete Hartley transform, which was orginally proposed by Hartley [6] and reintroduced to the society in 1983 by Bracewell [1]. Because the DHT transform kernel is similar to that of the DFT, the two transforms are closed related. Specially, the odd and the even parts of the DHT correspond to the negative of the imaginary and the real parts of the DFT. Consequently, the DHT can be used to obtain the DFT, and vice versa.

In addition, the DHT has many properties similar to those of the DFT. As Bracewell pointed out [2], a variety of algorithms which traditionally utilize the DFT, such as convolution, correlation and spectral analysis, can just as effectively be carried out with the DHT. Moreover, the DHT has two advantages over the DFT; namely (i) the forward and the inverse transforms are the same, and (ii) the Hartley transformed outputs are real-valued rather than complex, as with the DFT. Also, the Fourier spectrum can be calculated via the Hartley transform. Hence it is very desirable and helpful to have a systematic way for developing fast algorithms of DHT.

Recently Hong et al. [7, 8] proposed a general framework to construct basefield transforms. They show that any fast algorithm for the DFT, there is an equivalent fast algorithm for the basefield transform. Moreover, they show that the one-dimensional (1-D) DHT can be obtained by projecting the equivalent 1-D DFT from the extension field to the basefield [3]. The projection operator makes use of the conjugacy relation and the trace function. This approach is quite different from the traditional treatment of DHT [2, 9, 14], which is trigonometric decomposition based. They called it basefield transform because the transformed coefficients as well as the transform operations are in the same field as the inputs. For the ease of explanation, our derivation is closely related to the work presented by [8]. In this paper, the approach is extended to derive the two-dimensional discrete Hartley transform (2-D DHT) as the projection of the two-dimensional discrete Fourier transform (2-D DFT) from the extension field to the basefield.

The rest of this paper is organized as follows. In Section 2, the corresponding mathematical preliminaries for basefield transforms are described. In Section 3, the techniques described in Section 2 are used to derive the forward/inverse 2-D DHT as the projection of the forward/inverse 2-D DFT from the extension field (complex field) to the basefield (real field). In Section 4, the term "projection" is explained and showed why normal bases are the natural settings for expressing conjugacy relations. Finally, conclusion and discussions are given in Section 5.

2. Mathematical preliminaries

In this section, the mathematical preliminaries needed for the following derivations are set up with a review of fields and field extensions. Additionally, the expressions of normal bases, dual bases and traces, which have been indicated to be very useful analytic tools in general fields, will be briefly described. In the following, some terms of field theory terminology are also included to give a better understanding of normal bases expansion. These terminology terms, which are italicized in the first time they occur, can also be found in [4, 5, 11–13] as well.

If K is a subset of the field F and $(K, +, \cdot)$ is itself a field, then we say that K is a *subfield* or a *basefield* of F. Equivalently, a field F is said to be an *extension* (field) of K if F contains K. Clearly, the notions of subfields and extension fields are transitive. In the following, K denotes a given field and F an extension of K.

A polynomial in x with coefficients in K, or a polynomial in x over K, is an expression of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

where n is a nonnegative integer and each a_i is an element of K. The set of all polynomials in x over K is denoted by $K[x]$. Let α be an element in F. If there exists a nonzero polynomial $f(x) \in K[x]$ such that

$f(\alpha) = 0$, then α is said to be *algebraic* over K. Elements that are not algebraic over K are called *transcendental* over K. Assume $\alpha \in F$ be algebraic over K, then the unique monic polynomial of the least degree such that $f(\alpha) = 0$ is called the *minimal polynomial* for α over K. An element $\beta \in F$ is said to be a *conjugate* of α over K if β has the same minimal polynomial over K as α. The set of all elements that are conjugates of α over K is called the conjugacy class of α.

An *isomorphism* of a field onto itself is an *automorphism* of the field. If F is an extension of K, the set of all automorphisms of F which leave fixed each element of K forms a group called the *Galois group* or the automorphism group of F over K (notation: $\mathrm{Gal}(F/K)$ or $\mathrm{Aut}_K F$). If $\alpha \in F$ and $\mathrm{Gal}(F/K) = \{\phi_i\}$, then the *orbit* of α under $\mathrm{Gal}(F/K)$ is the set $\{\phi_i(\alpha)\}$.

Another viewpoint is to regard an extension F of the field K as a vector space over K. When considering the vector space structure of a field extension, we will use the notation F_K. The degree of the extension F of the field K, written as $[F : K]$, is the dimensionality of F_K. F is called a finite extension if $[F : K]$ is finite. Given a finite extension F of K of degree m, there exists m linearly independent bases elements: $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ in F such that $\{\alpha_i\}$ is a *basis* for the vector space F_K. A special useful basis $\{\alpha_i\}$ for F_K is called a *normal basis* in which the α_i's are conjugates of each other with respect to the basefield K. The corresponding *dual basis* of $\{\alpha_i\}$ is defined to be the unique set of basis $\{\beta_j\}$ such that

$$\mathrm{Tr}(\alpha_i \beta_j) = \delta_{ij}, \quad \forall i, j,$$

where δ_{ij} denotes the delta function, i.e., $\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ if $i = j$, and $\mathrm{Tr}(\cdot)$ is the so-called *trace function* which is a linear functional from F to K defined by

$$\mathrm{Tr}(\alpha) = \sum \{\text{All conjugates of } \alpha\}.$$

In other words, the bases $\{\beta_j\}$ and $\{\alpha_i\}$ are *trace-orthogonal*. Summarizing the characteristics of trace function relevant to later discussions, for all $\xi, \zeta \in F$, we have [11–13]:

(1) $\mathrm{Tr}(\xi) \in K$,
(2) $\mathrm{Tr}(\xi + \zeta) = \mathrm{Tr}(\xi) + \mathrm{Tr}(\zeta)$,
(3) $\mathrm{Tr}(\lambda\xi) = \lambda \mathrm{Tr}(\xi), \quad \forall \lambda \in K$,
(4) $\mathrm{Tr}(\xi) = \mathrm{Tr}(\xi^*)$, ξ^* is a conjugate of ξ,
(5) Tr maps F onto K.

If the basis $\{\alpha_i\}$ consists of the orbit under $\mathrm{Gal}(F/K)$ of a single element $\alpha \in F$, i.e., $\{\alpha_i\} = \{\phi_i(\alpha) \mid \phi_i \in \mathrm{Gal}(F/K)\}$, then $\{\phi_i(\alpha)\}$ is called a normal basis and α is called the *generator* of the basis, written as $\langle\alpha\rangle$. The dual basis of a normal basis is a normal basis as well [11]. Therefore, if $\alpha \in F$ is a generator of a normal basis for F_K, then there exists an element $\beta \in F$ such that β is the generator of the normal basis dual to $\langle\alpha\rangle$. The above introduction of mathematical preliminaries is developed for the general fields. If we restrict the extension field F to be the complex field and the basefield K a real field, it certainly works as well. From the viewpoint of vector space, there exists two linearly independent vectors α_0, α_1, such that $\{\alpha_0, \alpha_1\}$ forms a basis for F over K. The bases of the form $\{\alpha, \alpha^*\}$, i.e., the basis elements are *complex conjugates* of each other are called normal bases. According to the definition of trace function, it follows that

$$\mathrm{Tr}(\xi) = \xi + \xi^*, \quad \forall \xi \in F.$$

In the following derivations, F and K are defined to be a complex field and a real field, respectively.

3. The two-dimensional DHT basefield transform

Hong et al. [8] proposed a general framework for constructing basefield transforms. They show that the 1-D DHT can be obtained by projecting [3] the equivalent 1-D DFT from the extension field to the basefield. The projection operator makes use of the conjugacy relation and the trace function. In this section, the approach is extended to derive the 2-D DHT as the projection of 2-D DFT from the extension field to the basefield.

3.1. The forward two-dimensional discrete Hartley transform

In the following, we adopt the definition of 2-D DHT proposed by Bracewell [2], i.e.

$$\tilde{X}(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) \mathrm{cas}\left(\frac{2\pi n_1 k_1}{N_1} + \frac{2\pi n_2 k_2}{N_2}\right), \quad (1)$$

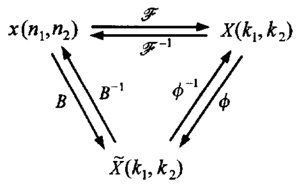

Fig. 1. The relationship among $x(n_1,n_2)$, DFT$\{x(n_1,n_2)\}$ (i.e., $X(k_1,k_2)$) and DHT$\{x(n_1,n_2)\}$ (i.e., $\tilde{X}(k_1,k_2)$).

where $\operatorname{cas}\theta = \cos\theta + \sin\theta$ and $k_i = 0,1,\ldots,N_i-1$, $i = 1,2$. The 2-D DFT of input sequence $x(n_1,n_2)$ is defined as

$$X(k_1,k_2) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x(n_1,n_2)\omega_{N_1}^{n_1k_1}\omega_{N_2}^{n_2k_2}, \tag{2}$$

where $\omega_{N_i} = \mathrm{e}^{-\mathrm{j}2\pi/N_i}, i = 1,2$, and $\mathrm{j} = \sqrt{-1}$. In order to have a better understanding, the relations among $x(n_1,n_2)$, $X(k_1,k_2)$ and $\tilde{X}(k_1,k_2)$ are illustrated in Fig. 1. In the figure, $x(n_1,n_2)$ is the input sequence. The 2-D DFT of $x(n_1,n_2)$ is denoted by $X(k_1,k_2)$, and the corresponding basefield transform (i.e. the 2-D DHT) by $\tilde{X}(k_1,k_2)$. Note that $x(n_1,n_2)$ and $\tilde{X}(k_1,k_2)$ reside in the basefield K while $X(k_1,k_2)$ is in the extension field F of K. The function $\mathscr{F}$ between $x(n_1,n_2)$ and $X(k_1,k_2)$ is the conventional 2-D DFT mapping. The function B is the basefield transform that we are seeking for. An intermediate map ϕ is a projection operator which projects a complex number $X(k_1,k_2)$ to a real number $\tilde{X}(k_1,k_2)$. Since $\mathscr{F}$ and B (if it exists) are bijections, clearly the basefield transform exists if and only if the intermediate map ϕ exists. If the map ϕ from $X(k_1,k_2)$ to $\tilde{X}(k_1,k_2)$ can be constructed, then the composition of $\mathscr{F}$ and ϕ will yield a basefield transform. The construction of ϕ can easily be achieved by the fact that ϕ is a linear functional from F to K if and only if there exists an $\alpha \in F$ such that [8]

$$\phi(\xi) = \mathrm{Tr}(\alpha\xi), \quad \forall \xi \in F. \tag{3}$$

Applying Eq. (3) to the Fourier coefficients $X(k_1,k_2)$, we obtain

$$\phi : X(k_1,k_2) \mapsto \tilde{X}(k_1,k_2) = \mathrm{Tr}(\alpha X(k_1,k_2)). \tag{4}$$

The map ϕ defines a one-to-one correspondence between $X(k_1,k_2)$ and $\tilde{X}(k_1,k_2)$. Since $X(k_1,k_2)$ is the Fourier transform of a real sequence, it satisfies the conjugacy relation, i.e., $X(k_1,k_2) = X^*(-k_1,-k_2)$. Considering the action of ϕ on a conjugacy class $\{X(k_1,k_2), X(-k_1,-k_2)\}$, we will have

$$\begin{aligned}\tilde{X}(k_1,k_2) &= \phi(X(k_1,k_2)) \\ &= \mathrm{Tr}(\alpha X(k_1,k_2)) \\ &= \alpha X(k_1,k_2) + \alpha^* X^*(k_1,k_2),\end{aligned} \tag{5}$$

$$\begin{aligned}\tilde{X}(-k_1,-k_2) &= \phi(X(-k_1,-k_2)) \\ &= \mathrm{Tr}(\alpha X(-k_1,-k_2)) \\ &= \alpha^* X(k_1,k_2) + \alpha X^*(k_1,k_2).\end{aligned} \tag{6}$$

Notice that Eqs. (5) and (6) show that the 2-D DHT coefficients $\tilde{X}(k_1,k_2)$ and $\tilde{X}(-k_1,-k_2)$ can be interpreted as the linear combination of the conjugacy class of the 2-D DFT with respect to the normal basis generated by α. The corresponding matrix M for the projection of conjugacy class can then be obtained: Eqs. (5) and (6) can be written in matrix form as

$$\begin{aligned}\begin{bmatrix}\tilde{X}(k_1,k_2)\\ \tilde{X}(-k_1,-k_2)\end{bmatrix} &= \begin{bmatrix}\phi(X(k_1,k_2))\\ \phi(X(-k_1,-k_2))\end{bmatrix} \\ &= \begin{bmatrix}\alpha & \alpha^*\\ \alpha^* & \alpha\end{bmatrix}\begin{bmatrix}X(k_1,k_2)\\ X^*(k_1,k_2)\end{bmatrix} \\ &\triangleq M\begin{bmatrix}X(k_1,k_2)\\ X^*(k_1,k_2)\end{bmatrix}.\end{aligned} \tag{7}$$

Eq. (7) shows that the relationship between 2-D DFT and 2-D DHT is determined by the ϕ map. By selecting α to be $(1+\mathrm{i})/2$, $\mathrm{i} = \sqrt{-1}$, it can be verified that the basefield transform $\tilde{X}(k_1,k_2)$ is equal to the 2-D DHT of $x(n_1,n_2)$. That is, since $B = \phi \circ \mathscr{F}$, direct composing the two functions yields the desired basefield transform:

$$\begin{aligned}\tilde{X}(k_1,k_2) &= \mathrm{Tr}(\alpha X(k_1,k_2)) \\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x(n_1,n_2)\mathrm{Tr}(\alpha\omega_{N_1}^{n_1k_1}\omega_{N_2}^{n_2k_2}) \\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x(n_1,n_2) \\ &\quad\times \operatorname{cas}\left(\frac{2\pi n_1k_1}{N_1} + \frac{2\pi n_2k_2}{N_2}\right).\end{aligned} \tag{8}$$

Eq. (8) shows that the 2-D DHT coefficients can be obtained as a projection of the 2-D DFT coefficients from the extension field to the basefield.

3.2. The inverse two-dimensional discrete Hartley transform

To have an inverse basefield transform it must have to verify that the above matrix M is invertible. Fortunately, it is easy to verify that

$$M^{-1} = \begin{bmatrix} \beta & \beta^* \\ \beta^* & \beta \end{bmatrix},$$

where $\{\beta, \beta^*\}$ is the dual basis of $\{\alpha, \alpha^*\}$. That is, if $\alpha = (1+\mathrm{i})/2$ then it can be verified that $\beta = \alpha^* = (1-\mathrm{i})/2$. Now, Eq. (7) can be rewritten as

$$\begin{bmatrix} X(k_1,k_2) \\ X(-k_1,-k_2) \end{bmatrix} = \begin{bmatrix} \beta & \beta^* \\ \beta^* & \beta \end{bmatrix} \begin{bmatrix} \tilde{X}(k_1,k_2) \\ \tilde{X}(-k_1,-k_2) \end{bmatrix}. \quad (9)$$

According to Eq. (9),

$$\begin{aligned} \phi^{-1} : \tilde{X}(k_1,k_2) \mapsto X(k_1,k_2) \\ = \beta\tilde{X}(k_1,k_2) + \beta^*\tilde{X}(-k_1,-k_2). \end{aligned} \quad (10)$$

And the corresponding inverse relation can be obtained by applying B^{-1} $(= \mathscr{F}^{-1} \circ \phi^{-1})$ to $X(k_1,k_2)$, i.e.

$$\begin{aligned} X(k_1,k_2) &= \frac{1}{N_1N_2}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}(\beta\tilde{X}(n_1,n_2) \\ &\quad + \beta^*\tilde{X}(-n_1,-n_2))\omega_{N_1}^{-n_1k_1}\omega_{N_2}^{-n_2k_2} \\ &= \frac{1}{N_1N_2}\Bigg(\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\beta\tilde{X}(n_1,n_2)\omega_{N_1}^{-n_1k_1}\omega_{N_2}^{-n_2k_2} \\ &\quad + \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\beta^*\tilde{X}(-n_1,-n_2)\omega_{N_1}^{-n_1k_1}\omega_{N_2}^{-n_2k_2}\Bigg) \\ &= \frac{1}{N_1N_2}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\tilde{X}(n_1,n_2)(\beta\omega_{N_1}^{-n_1k_1}\omega_{N_2}^{-n_2k_2} \\ &\quad + \beta^*\omega_{N_1}^{n_1k_1}\omega_{N_2}^{n_2k_2}) \\ &= \frac{1}{N_1N_2}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\tilde{X}(n_1,n_2)\mathrm{Tr}(\beta\omega_{N_1}^{-n_1k_1}\omega_{N_2}^{-n_2k_2}) \\ &= \frac{1}{N_1N_2}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\tilde{X}(n_1,n_2) \\ &\quad \times \mathrm{cas}\left(\frac{2\pi n_1k_1}{N_1} + \frac{2\pi n_2k_2}{N_2}\right). \end{aligned} \quad (11)$$

Eq. (11) shows that the inverse 2-D DHT can be obtained by applying the same technique as that of the forward 2-D DHT. Also, this result is the same as the one given in [2].

4. The basefield transform as a projection

The aforementioned frameworks provide a novel interpretation of the 2-D DHT as the projection of the 2-D DFT from the extension field to the basefield. In the following, the reason why we call it a projection is described. Let $\{\gamma_0, \gamma_1\}$ be an arbitrary basis of F_K. Then for every k,

$$X(k_1,k_2) = X^{(0)}(k_1,k_2)\gamma_0 + X^{(1)}(k_1,k_2)\gamma_1, \quad (12)$$

for some $X^{(0)}(k_1,k_2)$, $X^{(1)}(k_1,k_2) \in K$. As mentioned previously,

$$\begin{aligned} \tilde{X}(k_1,k_2) &= \phi(X(k_1,k_2)) \\ &= \mathrm{Tr}(\alpha X(k_1,k_2)) \\ &= X^{(0)}(k_1,k_2)\mathrm{Tr}(\alpha\gamma_0) \\ &\quad + X^{(1)}(k_1,k_2)\mathrm{Tr}(\alpha\gamma_1). \end{aligned} \quad (13)$$

Consider what happens when $\{\gamma_0, \gamma_1\}$ is chosen to be the unique dual basis $\{\beta, \beta^*\}$ of $\{\alpha, \alpha^*\}$. Then

$$\begin{aligned} \tilde{X}(k_1,k_2) &= X^{(0)}(k_1,k_2)\mathrm{Tr}(\alpha\beta) + X^{(1)}(k_1,k_2)\mathrm{Tr}(\alpha\beta^*) \\ &= X^{(0)}(k_1,k_2). \end{aligned} \quad (14)$$

Similarly,

$$\begin{aligned} \tilde{X}(-k_1,-k_2) &= \phi(X(-k_1,-k_2)) \\ &= \mathrm{Tr}(\alpha X(-k_1,-k_2)) \\ &= \mathrm{Tr}(\alpha^* X^*(-k_1,-k_2)) \\ &= \mathrm{Tr}(\alpha^* X(k_1,k_2)) \\ &= X^{(0)}(k_1,k_2)\mathrm{Tr}(\alpha^*\beta) \\ &\quad + X^{(1)}(k_1,k_2)\mathrm{Tr}(\alpha^*\beta^*) \\ &= X^{(1)}(k_1,k_2). \end{aligned} \quad (15)$$

Combining Eqs. (13)–(15), we see that

$$X(k_1,k_2) = \tilde{X}(k_1,k_2)\beta + \tilde{X}(-k_1,-k_2)\beta^*, \quad \forall k. \quad (16)$$

The function ϕ thus picks out the β-component of the expansion, that is why we call ϕ a 'projection' although, as pointed out in [8], it is not a real projection operator. We conclude this section with a discussion why the normal bases are nature to express conjugacy relations. Consider the conjugacy class $\{X(k_1,k_2), X(-k_1,-k_2)\}$. From the aforementioned discussions, we obtain

$$X(k_1,k_2) = \tilde{X}(k_1,k_2)\beta + \tilde{X}(-k_1,-k_2)\beta^*,$$

$$X(-k_1,-k_2) = \tilde{X}(-k_1,-k_2)\beta + \tilde{X}(k_1,k_2)\beta^*.$$

Notice that with respect to a normal basis, the basis components of the elements of a conjugacy class are permutations of each other. In extracting the β-component of the conjugates, we thus have all the information required to reconstruct them. The result is also true when the extension field is a more general field. For this reason, normal bases are particularly useful for expressing conjugacy relations.

5. Conclusion

In this paper, a novel interpretation of 2-D DHT via projection framework is presented. We show that the 2-D DHT can be obtained by projecting the 2-D DFT from the extension field to the basefield. The projection operator makes use of the concept of field extension and the trace function. It is quite different from the traditional treatment of the 2-D DHT, which is trigonometric decomposition based. Applying the same technique, the projection operator discloses the closed relationship between the 2-D DHT and the 2-D DFT. The technique can be extended to derive the closed relationship between the multi-dimensional DHT and the multi-dimensional DFT. The relationship between DHT and DFT can also be investigated from the trigonometric relationship between their transform kernels. However, the normal basis expansion based approach is more systematic and more powerful; especially in developing fast algorithms [8]. Moreover, the mapping between basefield and extension field, constructed by the projection framework, is general enough to be applied to other DHT applications. A novel DHT-based real number code [15], which is shown to be effective in dealing with the fading channel effects, is under development based on the prescribed framework.

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