$\boldsymbol{V}_1$ is formed by the corresponding $Q+1$ rows of $[\boldsymbol{d}^0, \ldots, \boldsymbol{d}^Q]$, and $\overline{\boldsymbol{D}}(\omega_0)$ is a diagonal matrix with those selected $Q+1$ entries from $\boldsymbol{D}(\omega_0)$ on its main diagonal. Since $\operatorname{rank}(\boldsymbol{D}(\overline{\boldsymbol{u}})\overline{\boldsymbol{D}}(\omega_0)) = Q+1$ and $\operatorname{rank}(\boldsymbol{V}_1) = Q+1$, we deduce that $\operatorname{rank}(\boldsymbol{\Phi}_e) = Q+1, \forall \boldsymbol{e} \neq \boldsymbol{0}$.

Next, we prove the "only if" part by contradiction. Suppose that for some $\boldsymbol{e}$, $\boldsymbol{u} = \boldsymbol{\Theta}\boldsymbol{e}$ has only $\overline{Q}+1 < Q+1$ nonzero corresponding entries, that we collect in $\overline{\boldsymbol{u}} = [u_{n_0}, \ldots, u_{n_{\overline{Q}}}]^T$. Then, similarly to the "if" part, we can group the nonzero rows in a matrix $\boldsymbol{D}(\overline{\boldsymbol{u}})\overline{\boldsymbol{D}}(\omega_0)\boldsymbol{V}_1$. Now this $\boldsymbol{V}_1$ is a $(\overline{Q}+1) \times (Q+1)$ matrix, while $\overline{\boldsymbol{D}}(\omega_0)$ is a $(\overline{Q}+1) \times (\overline{Q}+1)$ matrix. It follows immediately that

$$\operatorname{rank}(\boldsymbol{V}_1) = \overline{Q}+1 < Q+1$$

and, hence, $\operatorname{rank}(\boldsymbol{\Phi}_e) < Q+1$, which implies that the maximum diversity cannot be achieved. □

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On Continuous-Time Optimal Deterministic Traffic Regulation

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***Abstract*—In this correspondence, we study the continuous-time deterministic traffic regulation problem. We propose a regulation form shown to be the optimal deterministic traffic regulator in the sense that it outputs the most packets while satisfying the constraint on the output process. We further investigate the subtle relation between continuous-time and discrete-time optimal deterministic regulators, and reduce our general regulation form to the known discrete-time optimal deterministic regulator when restricting arrival (departure) instants to integers and packet size to unity. Therefore, by extending traffic-regulation theory to continuous time, our work provides a fundamental framework for future research regarding quality-of-service (QoS)-guaranteed network design/analysis in continuous-time.**

***Index Terms*—Constrained optimization, network flows, quality-of-service (QoS), traffic management.**

I. INTRODUCTION

Traffic regulation has been widely accepted as an indispensable technique to provide quality-of-service (QoS)-guaranteed multimedia services in packet-switching communication networks (e.g., TCP/IP networks [1], asynchronous transfer mode (ATM) [2]). According to [3], a traffic source conforms to a nondecreasing, nonnegative function f if $R[t-\tau, t) \leq f(\tau)$ for *all* $\tau, t \geq 0$, where $R[t-\tau, t)$ (bits) denotes the amount of information bits in packets arriving in time interval $[t-\tau, t)$. A deterministic traffic regulator with constraint function f is a filter shaping an arbitrary traffic input such that the output process conforms to f.

For *discrete-time* systems, Chang [4] studied discrete-time deterministic traffic regulators in great detail and developed a general filtering method for traffic regulation. For *continuous-time* traffic regulation, a number of results also have been proposed in the literature. A special traffic regulator called leaky bucket (σ, ρ) regulator was discussed in [5]–[8]. The $(\vec{\sigma}, \vec{\rho})$ regulators were investigated in [3], [9], [10]. However, regulators of the $(\vec{\sigma}, \vec{\rho})$ type are limited to those with concave constraint functions. In [11], regulators with *nonconcave* constraint functions can be realized with a cascade of leaky buckets with *state-dependent* token generation rates, but the detailed implementation was left unspecified. In [12] and [13], Le Boudec successfully applied the continuous-time "network calculus" [14], [4], [15], [16] to traffic shapers, and we can further improve Le Boudec's results in several aspects. First, the regulators Le Boudec considered are *bit-processing* devices which assume *fluid* input streams. In practical packet-switching networks, data arrivals are packets or cells, and thus this assumption is generally not true. Second, the *continuity property* of constraint functions is very important to the derivation of the optimal regulation formulas, but this issue

Manuscript received July 15, 2001; revised November 11, 2002. This work was supported by the Ministry of Education, Taiwan, ROC, under Contract 89E-FA06-2-4-7.

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Communicated by V. Anantharam, Associate Editor for Communication Networks.

Digital Object Identifier 10.1109/TIT.2003.813489

was not addressed in [12] and [13]. Third, although the regulators (or shapers) in [12] and [13] are defined in continuous-time setting, Le Boudec considered only systems for which there is a minimum time granularity. This restriction implies these systems are still of discrete-time type. Consequently, in packet-switching networks, how to construct continuous-time optimal deterministic regulators with *general* constraint functions is a very interesting problem that still needs further investigation. In this correspondence, we present continuous-time optimal deterministic traffic regulation formulas which specify the earliest possible departure time of each packet arrival. Furthermore, we also show that discrete-time optimal deterministic regulators discussed in [4] can be regarded as special cases of continuous-time optimal deterministic regulators. This fact demonstrates the subtle relation between continuous-time and discrete-time optimal deterministic regulators.

Throughout this correspondence, we denote the arrival time, the departure time, and the length of the nth packet by a_n, b_n, and L_n, respectively. In addition, we assume 1) $L_{\max}$ (bits) is the maximal size of the packets in an input source. 2) For any $t \geq 0$, there are only a finite number of packet arrivals in $[0, t]$. Thus, we have $\lim_{n\to\infty} a_n = \infty$. 3) $f \leq g$ means that the inequality holds pointwisely; $\sup f_\alpha$ and $\inf f_\alpha$ are taken in a per-point manner, e.g., $(\sup f_\alpha)(x) = \sup f_\alpha(x)$. We also denote by $T[t-\tau, t)$ (bits) the amount of information bits in packets departing from a regulator in interval $[t-\tau, t)$. The entire content L_n of a packet being released at time b_n is assumed to be released at time b_n.

The rest of this correspondence is organized as follows. In Section II, we introduce maximal embedded subadditive functions and give some properties regarding these functions. The continuous-time optimal regulation formulas are given in Section III along with the corresponding mathematical proofs. In Section IV, we investigate the subtle relation between continuous-time optimal regulators and the known discrete-time ones. We present a potential realization structure of the proposed optimal regulators in Section V, and, finally, conclude this correspondence in Section VI.

II. Maximal Embedded Subadditive Functions

For an arbitrary traffic source active in $[0, \varphi]$, the pointwisely smallest constraint function $\tilde{f}$ which this source conforms to can be constructed according to $\tilde{f}(\tau) \triangleq \sup\{T[t-\tau, t): t \in [0, \varphi]\}$. It is not difficult to see that $\tilde{f}$ is always subadditive, i.e.,

$$\tilde{f}(x+y) \leq \tilde{f}(x) + \tilde{f}(y), \qquad \text{for } x, y \geq 0.$$

This observation shows that subadditive functions play an important role in deterministic traffic regulation theory. To begin with, two important definitions are given as follows.

Definition 1: The collection of nonnegative, nondecreasing, and left-continuous functions $f: \mathbb{R}^+ \to \mathbb{R}^+$ with $f(0) = 0$ is denoted by $\mathcal{G}$.

Definition 2: Given $f \in \mathcal{G}$, we define $\overline{f} \triangleq \sup\{g: g \in \mathcal{G}, g \leq f, g$ is subadditive$\}$.

Given $f \in \mathcal{G}$, one can easily show by definition that the corresponding $\overline{f}$ is subadditive and $\overline{f} \in \mathcal{G}$. Consequently, we call the $\overline{f}$ corresponding to $f \in \mathcal{G}$ the *maximal subadditive function embedded in* f. The next theorem is of great importance for continuous-time deterministic regulators, and is actually a continuous-time extension of [17, Lemma 2.1].

Theorem 1: A traffic source conforms to $f \in \mathcal{G}$ if and only if it conforms to $\overline{f}$.

Proof: The necessary condition is clear. We only need to show the sufficient condition.

Suppose there is a traffic source that does not conform to $\overline{f}$. Then, for some $t \geq 0$ and $\tau > 0$, we have $T[t-\tau, t) > \overline{f}(\tau)$. Hence we have $\tilde{f}(\tau) > \overline{f}(\tau)$, where

$$\tilde{f}(\tau) \triangleq \sup\{T[t-\tau, t): t \in [0, \varphi]\}$$

where $[0, \varphi]$ is the time interval over which the source is active. But by its definition, $\tilde{f}$ is subadditive and is pointwise upper-bounded by f. This contradicts the definition of $\overline{f}$ as the pointwise largest subadditive function that is pointwise upper-bounded by f. □

If we wish a regulator's output to conform to $f \in \mathcal{G}$, then $f(0+)$ (where the "+" means the limit from the right) must be larger than or equal to $L_{\max}$. Otherwise, a packet of length $L_{\max}$ would never be allowed to pass the regulator. Therefore, f must satisfy $f(0+) \geq L_{\max}$. The next lemma shows the relation between $\overline{f}(0+)$ and $f(0+)$ for a given $f \in \mathcal{G}$.

Lemma 1: Given $f \in \mathcal{G}$. Then $f(0+) = \overline{f}(0+)$.

Proof: Since $f \geq \overline{f}$, we have $f(0+) \geq \overline{f}(0+)$. Conversely, define $\hat{g}(x) = 0$ for $x = 0$ and $\hat{g}(x) = f(0+)$ for $x > 0$. It is not difficult to see that $\hat{g} \leq f$ and $\hat{g}$ is subadditive. Hence, by definition we know $\overline{f} \geq \hat{g}$ and

$$\lim_{x\to 0} \overline{f}(x) \geq \lim_{x\to 0} \hat{g}(x) = \lim_{x\to 0} f(0+) = f(0+).$$

Thus, we know $\overline{f}(0+) \geq f(0+)$ and the lemma is proved. □

Consequently, we know that $\overline{f}(0+) \geq L_{\max}$ provided $f(0+) \geq L_{\max}$. In summary, Theorem 1 and Lemma 1 tell us that constructing an optimal deterministic regulator with constraint function $f \in \mathcal{G}$ with $f(0+) \geq L_{\max}$ is equivalent to constructing one with a pointwisely smaller, subadditive constraint function $\overline{f} \in \mathcal{G}$.

In addition, continuous-time inverse functions are also defined as follows.

Definition 3: For a nonnegative, nondecreasing function $f: \mathbb{R}^+ \to \mathbb{R}^+$, we define $f^{-1}: \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f^{-1}(x) \triangleq \inf\{s \geq 0: x \leq f(s)\}.$$

III. Optimal Deterministic Regulation Formulas

To produce the output process of a regulator with a *subadditive* constraint function $\overline{f} \in \mathcal{G}$ with $\overline{f}(0+) \geq L_{\max}$, we assert that the packet departure times be determined by the following rules:

$$b_1 = a_1, \quad \text{and} \quad b_n = \max\{a_n, {b_n}'\}, \qquad \forall n \geq 2, \tag{1}$$

where

$$b_n' = \max\left\{b_i + \overline{f}^{-1}\left(\sum_{k=i}^{n} L_k\right): 1 \leq i \leq n-1\right\}. \tag{2}$$

In the following, we show the output process determined by (1) conforms to $\overline{f}$.

Theorem 2: For a given subadditive function $\overline{f} \in \mathcal{G}$ with $\overline{f}(0+) \geq L_{\max}$, the following three conditions are equivalent:

1) $T[t-\tau, t) \leq \overline{f}(\tau)$, for all $\tau, t \geq 0$;

2) for a fixed $\nu > 0$ and $\forall n \in \mathbb{N}$

$$T[t-\tau, t) \leq \overline{f}(\tau), \qquad \text{for } 0 \leq \tau, t \leq b_n + \nu;$$

3) For a fixed $\nu > 0$ and $\forall\, n \in \mathbb{N}$

$$T^n[t-\tau, t) \le \overline{f}(\tau), \qquad \text{for } 0 \le \tau,\, t \le b_n + \nu$$

where $T^n[t-\tau, t)$ is defined using only the first n packet departures.

Proof: Since $T[t-\tau, t) = T[0, t)$ and $T^n[t-\tau, t) = T^n[0, t)$ for $\tau > t$, without loss of generality we may assume $\tau \le t$.

Condition 1) obviously implies condition 2). On the other hand, suppose condition 2) is true. Since $\lim_{n\to\infty} b_n = \infty$, for an arbitrary t it follows that

$$T[t-\tau, t) \le \overline{f}(\tau), \qquad \text{for all } 0 \le \tau \le t.$$

To prove that condition 2) and condition 3) are equivalent, note that

$$T^n[t-\tau, t) \le T[t-\tau, t), \qquad \text{for all } 0 \le \tau \le t$$

since $T^n[t-\tau, t)$ results from only the first n packet departures. Hence, condition 2) implies condition 3). Conversely, suppose condition 3) is true. Fixing $n \in \mathbb{N}$, we can find $m \in \mathbb{N}$ large enough such that $b_m \ge b_n + \nu$. Then condition 3) implies that

$$T^m[t-\tau, t) \le \overline{f}(\tau), \qquad \forall\, 0 \le \tau \le t \le b_m + \nu.$$

Since $b_k \ge b_n + \nu$ for $k > m$, it follows that

$$T[t-\tau, t) = T^m[t-\tau, t), \qquad \forall\, 0 \le \tau \le t \le b_n + \nu$$

and, consequently, we have

$$T[t-\tau, t) \le \overline{f}(\tau), \qquad \forall\, 0 \le \tau \le t \le b_n + \nu.$$

Since n is arbitrary, it follows condition 3) implies condition 2). Thus, we proved conditions 2) and 3) are equivalent. □

Theorem 3: Given a subadditive function $\overline{f} \in \mathcal{G}$ with $\overline{f}(0+) \ge L_{\max}$. The output process determined by (1) satisfies condition 3) of Theorem 2. In particular, the output process determined by (1) conforms to $\overline{f}$.

Proof: By (1), we know $b_1 = a_1$. Clearly, for $0 \le \tau,\, t \le b_1 + \nu$ we have $T^1[t-\tau, t) \le L_1 \le \overline{f}(\tau)$. Thus, the conclusion holds for $n = 1$. Suppose

$$T^{n-1}[t-\tau, t) \le \overline{f}(\tau), \qquad \text{for } 0 \le \tau,\, t \le b_{n-1} + \nu. \tag{3}$$

Now we consider $T^n[t-\tau, t)$ for $0 \le \tau,\, t \le b_n + \nu$.

Given departure times $b_1, \ldots, b_n$, consider those $\tau \in [0, b_n + \nu]$ such that $\tau = b_j - b_i$ for some $i, j \in \{1, \ldots, n\}$ and $i < j$, it can be seen there are only a finite number of such points. Arrange those τ in an increasing order, and denote them by $\tau_1, \ldots, \tau_l$ and also let $\tau_0 = 0$, $\tau_{l+1} = b_n + \nu$.

Note that for $0 \le t \le b_n + \nu$, we can write

$$T^n[t-\tau, t) = \sum_{i=1}^{n} L_i \cdot I_{\tau, i}(t)$$

where $I_{\tau, i}(t) = 1$ for $t \in (b_i, b_i + \tau]$ and $I_{\tau, i}(t) = 0$ otherwise. With the above decomposition, it can be seen that

$$h_n(\tau) \triangleq \sup\{T^n[t-\tau, t) \colon 0 \le t \le b_n + \nu\}$$

is a nondecreasing, left-continuous *step* function in $[0, b_n + \nu]$. That is, $[0, b_n + \nu]$ can be divided into a finite number of subintervals $I_1, \ldots, I_m$ such that $h_n(\tau) = c_i$ for t interior to I_i. In addition, the only possible discontinuous points of $h_n(\tau)$ are those τ_k, $k = 1, \ldots, l$. Consequently, to check whether $T^n[t-\tau, t) \le \overline{f}(\tau)$ for $0 \le \tau,\, t \le b_n + \nu$, we only need to check $h_n(\tau_k+) \le \overline{f}(\tau_k+)$ for $k = 1, \ldots, l$.

By (1), we know

$$b_n \ge b_i + \overline{f}^{-1}\left(\sum_{k=i}^{n} L_k\right), \qquad \forall\, i = 1, \ldots, n-1$$

which implies that for all $i = 1, \ldots, n-1$

$$\sum_{k=i}^{n} L_k \le \overline{f}\left(\overline{f}^{-1}\left(\sum_{k=i}^{n} L_k\right)+\right) \le \overline{f}((b_n - b_i)+). \tag{4}$$

Now, with (3) and (4), one can use mathematical induction to show that

$$h_n(\tau_k+) \le \overline{f}(\tau_k+), \qquad \text{for } k = 1, \ldots, l$$

and, consequently, $T^n[t-\tau, t) \le \overline{f}(\tau)$, for $0 \le \tau,\, t \le b_n + \nu$.

By mathematical induction, we conclude that the output process determined by (1) satisfies the third condition of Theorem 2. In particular, by Theorem 2, the output process determined by (1) conforms to $\overline{f}$. □

Having developed continuous-time deterministic regulators with *subadditive* constraint functions, now we are ready to define continuous-time deterministic regulators with *general* constraint functions $f \in \mathcal{G}$ with $f(0+) \ge L_{\max}$.

Definition 4: Suppose the departure time of the nth packet from a continuous-time deterministic regulator $f \in \mathcal{G}$ with $f(0+) \ge L_{\max}$ is denoted by b_n. Then we set $b_1 = a_1$ and $b_n = \max\{a_n, b_n{}'\}\ \forall\, n \ge 2$, where

$$b_n' = \max\left\{b_i + \overline{f}^{-1}\left(\sum_{k=i}^{n} L_k\right) : 1 \le i \le n-1\right\}. \tag{5}$$

The next theorem shows the optimality of the proposed continuous-time deterministic regulators.

Theorem 4: Given an input process $\{(a_n, L_n)\}$. Then for any regulator output process $\{(c_n, L_n)\}$ conforming to $f \in \mathcal{G}$ with $f(0+) \ge L_{\max}$ and $c_n \ge a_n$ for all $n \in \mathbb{N}$, we must have $c_n \ge b_n$, where b_n is the departure time calculated from Definition 4.

Proof: From Definition 4 we know $c_1 \ge a_1 = b_1$. Suppose $c_j \ge b_j$ holds for all $1 \le j \le n-1$. If

$$c_n < c_i + \overline{f}^{-1}\left(\sum_{k=i}^{n} L_k\right)$$

for some $i \in \{1, \ldots, n-1\}$, for

$$0 < \epsilon < c_i + \overline{f}^{-1}\left(\sum_{k=i}^{n} L_k\right) - c_n$$

we would have

$$\begin{aligned} \overline{f}(c_n - c_i + \epsilon) &< \overline{f}\left(\overline{f}^{-1}\left(\sum_{k=i}^{n} L_k\right)\right) \quad \left(\text{by definition of } \overline{f}^{-1}\right) \\ &\le \sum_{k=i}^{n} L_k \quad \left(\text{by the left-continuity of } \overline{f}\right) \\ &\le T[c_i, c_n + \epsilon) \end{aligned}$$

which implies $\{(c_n, L_n)\}$ did not conform to f (by Theorem 1). However, since $\{(c_n, L_n)\}$ conforms to f, we must have

$$c_n \ge \max\left\{c_i + \overline{f}^{-1}\left(\sum_{j=i}^{n} L_j\right) : 1 \le i \le n-1\right\}.$$

Hence, the induction hypothesis implies

$$\begin{aligned} c_n &\ge \max\left\{c_i + \overline{f}^{-1}\left(\sum_{j=i}^{n} L_j\right) : 1 \le i \le n-1\right\} \\ &\ge \max\left\{b_i + \overline{f}^{-1}\left(\sum_{j=i}^{n} L_j\right) : 1 \le i \le n-1\right\} \\ &\triangleq b_n. \end{aligned}$$

By induction, the theorem is proved. □

Note that the condition $c_n \geq a_n$ is referred to as the causal condition in [4] since the departure time cannot be less than the arrival time.

IV. Special Form in the Discrete-Time Case

In this section, we consider a special case where packet arrivals and departures occur only at $t \in \mathbb{Z}^+$, and all packets are with unit length. Note that the constraint sequence f must satisfy $f(1) \geq 1$.

Similar to the continuous-time case, we denote by $\mathcal{F}$ the collection of nonnegative, nondecreasing sequences $f: \mathbb{Z}^+ \to \mathbb{Z}^+$ with $f(0) = 0$. The maximal subadditive sequence $\overline{f}$ embedded in $f \in \mathcal{F}$ can be defined similarly to Definition 2.

Definition 5: Given $f \in \mathcal{F}$, we define $\overline{f} \stackrel{\Delta}{=} \sup\{g: g \in \mathcal{F},\, g \leq f,\, g$ is sub-additive$\}$.

According to [4, Lemma 2.2], $\overline{f}$ (the maximal subadditive sequence embedded in $f \in \mathcal{F}$) is pointwisely identical to f^* (the subadditive closure of f). Consequently, given $f \in \mathcal{F}$, $\overline{f}$ has the following properties: 1) $\overline{f}$ is subadditive, 2) $\overline{f} \in \mathcal{F}$, 3) f is subadditive if and only if $f = \overline{f}$, and 4) a discrete-time traffic source conforms to f if and only if it conforms to $\overline{f}$.

In addition, discrete-time inverse sequences are also defined as follows.

Definition 6: For a nonnegative, nondecreasing function $f: \mathbb{Z}^+ \to \mathbb{Z}^+$. We define $f^{-1}: \mathbb{Z}^+ \to \mathbb{Z}$ by $f^{-1}(x) \stackrel{\Delta}{=} \min\{s \geq 0: x \leq f(s)\} - 1$.

The constant "-1" in the definition of $f^{-1}(\cdot)$ is used to make discrete-time regulation formulas identical to their continuous-time counterparts. Otherwise, the right-hand side of (6) in Definition 7 would need to append a constant "-1." Also, by definition, one can show the following inequalities

$$f(f^{-1}(x)) < x \leq f(f^{-1}(x) + 1)$$

and

$$1 + f^{-1}(f(x)) \leq x \leq f^{-1}(f(x) + 1).$$

Analogous to the continuous-time case, we define discrete-time optimal traffic regulators in a parallel form as follows.

Definition 7: Suppose the departure time of the nth packet from a discrete-time deterministic regulator $f \in \mathcal{F}$ with $f(1) \geq 1$ is denoted by b_n. Then, we set $b_1 = a_1$ and $b_n = \max\{a_n, {b_n}'\}\ \forall n \geq 2$, where

$$b_n' = \max\left\{b_i + \overline{f}^{-1}(n - i + 1): 1 \leq i \leq n - 1\right\}. \qquad (6)$$

The following theorems justify Definition 7, and their proofs can be obtained by slightly modifying those of Theorems 2, 3, and 4, respectively.

Theorem 5: For a given subadditive sequence $\overline{f} \in \mathcal{F}$ with $\overline{f}(1) \geq 1$, the following three conditions are equivalent:

1) $T[t - \tau, t) \leq \overline{f}(\tau)$, for all $\tau, t \in \mathbb{Z}^+$;
2) for a fixed $\nu \in \mathbb{N}$ and $\forall n \in \mathbb{N}$, $T[t - \tau, t) \leq \overline{f}(\tau)$, for $\tau, t \in [0, \ldots, b_n + \nu]$;
3) for a fixed $\nu \in \mathbb{N}$ and $\forall n \in \mathbb{N}$, $T^n[t - \tau, t) \leq \overline{f}(\tau)$, for $\tau, t \in [0, \ldots, b_n + \nu]$, where $T^n[t - \tau, t)$ results from only the first n packet departures.

Theorem 6: Given a subadditive sequence $\overline{f} \in \mathcal{F}$ with $\overline{f}(1) \geq 1$, the output process determined by Definition 7 satisfies condition 3) of Theorem 5. In particular, the output process determined by Definition 7 conforms to $\overline{f}$.

Theorem 7: Given an input process $\{(a_n, L_n)\}$, then for any regulator output process $\{(c_n, L_n)\}$ conforming to $f \in \mathcal{F}$ with $f(1) \geq 1$ and $c_n \geq a_n$, for all $n \in \mathbb{N}$, we must have $c_n \geq b_n$, where b_n is the departure time calculated from Definition 7.

We will show later that the discrete-time optimal deterministic regulators in Definition 7 are identical to those discussed in [4].

Lemma 2: Given subadditive $f \in \mathcal{F}$ with $f(1) \geq 1$. For $x, y \geq 1$ we have $f^{-1}(x) + f^{-1}(y) \leq f^{-1}(x + y - 1)$.

Proof: Let q, r be $f^{-1}(x), f^{-1}(y)$, respectively ($q, r \geq 0$ since $x, y \geq 1$). By definition, it follows that

$$f(q + r) \leq f(q) + f(r) \leq x + y - 2$$

and

$$f(f^{-1}(x + y - 1) + 1) \geq x + y - 1.$$

Hence, we know $f(q + r) < f(f^{-1}(x + y - 1) + 1)$, which implies

$$q + r < f^{-1}(x + y - 1) + 1$$

and

$$q + r \leq f^{-1}(x + y - 1).$$

Hence, the result is proved. □

With Lemma 2, we can show the following theorem.

Theorem 8: For $n \geq 2$, (6) is identical to

$$b_n'' = \max\left\{a_i + \overline{f}^{-1}(n - i + 1): 1 \leq i \leq n - 1\right\}.$$

Proof: We prove this by induction. If $m = 2$, it follows that ${b_2}' = a_1 + \overline{f}^{-1}(2)$. By hypothesis, we know that $b_1 = a_1$. Hence, the conclusion holds for $m = 2$. Suppose the conclusion also hold for $m \in \{2, \ldots, n - 1\}$. Now let $m = n$. It is easy to see that ${b_n}' \geq {b_n}''$ since $a_i \leq b_i$. For those $i \in \{1, \ldots, n - 1\}$ such that $a_i = b_i$, we have

$$b_i + \overline{f}^{-1}(n - i + 1) \leq a_i + \overline{f}^{-1}(n - i + 1).$$

For those $i \in \{1, \ldots, n - 1\}$ such that $a_i < b_i$, there must exist a $j \in \{1, \ldots, i - 1\}$ such that $b_i = a_j + \overline{f}^{-1}(i - j + 1)$. However, by Lemma 2, it follows that

$$\begin{aligned} b_i + \overline{f}^{-1}(n - i + 1) &= a_j + \overline{f}^{-1}(i - j + 1) + \overline{f}^{-1}(n - i + 1) \\ &\leq a_j + \overline{f}^{-1}(n - j + 1). \end{aligned}$$

Therefore, we must have ${b_n}' \leq {b_n}''$, and have proved the case for $m = n$. □

According to Definition 7, $b_n = \max\{a_n, {b_n}'\}$, and note that $a_n = a_n + \overline{f}^{-1}(n - n + 1)$. By Theorem 8, we then have

$$b_n = \max\left\{a_i + \overline{f}^{-1}(n - i + 1): 1 \leq i \leq n\right\}. \qquad (7)$$

After determining b_n, the departure time of the nth packet, we know $B[0, k)$, the amount of departure in $[0, k)$, can be expressed as

$$B[0, k) = \min\{n - 1: b_n \geq k,\, n \in \mathbb{N}\}. \qquad (8)$$

Discrete-time optimal deterministic regulators have been discussed in [4], and we quote the definition here.

Definition 8: Suppose that each packet has unit length. Let $B(i)$, $i \in \mathbb{Z}^+$, represent the amount of departure from the maximal deterministic regulator in $[0, i)$. If $f \in \mathcal{F}$ with $f(1) \geq 1$, then $B(i)$ is constructed by

$$B(i) = \min_{0 \leq j \leq i} \{R[0, j) + f^*(i-j)\} \tag{9}$$

where f^* is the subadditive closure of f.

Our primary goal here is to show that the discrete-time regulators defined by (7) are precisely the discrete-time optimal deterministic regulators in Definition 8.

Theorem 9: For each $k \in \mathbb{Z}^+$, we have

$$B[0, k) = \min\{R[0, s) + \overline{f}(k-s) : 0 \leq s \leq k\}.$$

Proof: By definition, we know $B[0, k) \leq R[0, k)$. Also, for $s \in \{0, \ldots, k-1\}$, it can be seen that

$$B[0, k) = B[0, s) + B[s, k) \leq R[0, s) + \overline{f}(k-s).$$

Consequently, we have

$$B[0, k) \leq \min\{R[0, s) + \overline{f}(k-s) \colon 0 \leq s \leq k\}.$$

Conversely, if $R[0, k) = 0$, then

$$\begin{aligned} \min\{R[0, s) + \overline{f}(k-s) \colon 0 \leq s \leq k\} &= R[0, k) + \overline{f}(0) \\ &= 0 \leq B[0, k). \end{aligned}$$

So we have proved

$$B[0, k) = \min\{R[0, s) + \overline{f}(k-s) \colon 0 \leq s \leq k\}.$$

Therefore, we may assume $R[0, k) > 0$ and thus $a_1 < k$. Suppose $\hat{s}$ is the argument achieving the minimum of

$$\min\{R[0, s) + \overline{f}(k-s) \colon 0 \leq s \leq k\}.$$

By definition, we know that

$$\begin{aligned} b_{R[0, \hat{s})+\overline{f}(k-\hat{s})} \triangleq \max\{&a_i + \overline{f}^{-1}(R[0, \hat{s}) + \overline{f}(k-\hat{s}) - i + 1) \colon \\ &1 \leq i \leq R[0, \hat{s}) + \overline{f}(k-\hat{s})\}. \end{aligned}$$

Considering $a_1 = l, 0 \leq l < k$, we partition $[1, R[0, \hat{s}) + \overline{f}(k-\hat{s})]$ into subintervals

$$\begin{aligned} &[1, R[0, l+1)], [R[0, l+1)+1, R[0, l+2)], \ldots, \\ &\qquad [R[0, k-1)+1, R[0, \hat{s}) + \overline{f}(k-\hat{s})]. \end{aligned}$$

Then

$$\begin{aligned} &b_{R[0, \hat{s})+\overline{f}(k-\hat{s})} \\ &\quad = \max\Big\{a_i + \overline{f}^{-1}\left(R[0, \hat{s}) + \overline{f}(k-\hat{s}) - i + 1\right) \colon \\ &\qquad\qquad i \text{ is the left boundary point of a nonempty sub-interval}\Big\}. \end{aligned}$$

For $i = 1 = R[0, l)+1$, since $R[0, \hat{s}) + \overline{f}(k-\hat{s}) \leq R[0, l) + \overline{f}(k-l)$, it can be seen that

$$\begin{aligned} a_1 + \overline{f}^{-1}(R[0, \hat{s}) + \overline{f}(k-\hat{s}) - R[0, l)) &\leq l + \overline{f}^{-1}(\overline{f}(k-l)) \\ &\leq l + k - l - 1 = k - 1. \end{aligned}$$

For $i = R[0, j) + 1$, it follows that

$$\begin{aligned} &a_{R[0, j)+1} + \overline{f}^{-1}(R[0, \hat{s}) + \overline{f}(k-\hat{s}) - R[0, j)) \\ &\quad \leq j + \overline{f}^{-1}(\overline{f}(k-j)) \\ &\quad \leq j + k - j - 1 = k - 1 \end{aligned}$$

which implies $b_{R[0, \hat{s})+\overline{f}(k-\hat{s})} \leq k - 1$.

However, according to (8), we know that

$$R[0, \hat{s}) + \overline{f}(k-\hat{s}) < B[0, k) + 1$$

which implies

$$R[0, \hat{s}) + \overline{f}(k-\hat{s}) \leq B[0, k)$$

and

$$B[0, k) \geq \min\{R[0, s) + \overline{f}(k-s) \colon 0 \leq s \leq k\}.$$

From the inequality proved in the first paragraph and the fact that $k \in \mathbb{Z}^+$ is arbitrary, the theorem is proved. □

V. A Realization of the Optimal Regulation Formulas

The optimal deterministic regulator defined in Definition 4 is not directly realizable since determining $\{b_n\}_{n=1}^{\infty}$ needs infinite computation steps. In this section, we present a potential realization of the proposed optimal regulators. According to Definition 4, for $n \geq 2$, we can rewrite (5) as

$$b_n{}' = \max\left\{\inf\left\{b_i + t \colon t \geq 0, \overline{f}(t) \geq \sum_{j=i}^{n} L_j\right\} \colon 1 \leq i \leq n-1\right\}.$$

Since the set $\{b_i + t \colon \overline{f}(t) \geq \sum_{j=i}^{n} L_j\}$ must be of the form of interval $[a, \infty)$ or (a, ∞), it can be seen that for $n \geq 2$ (this can be shown by mathematical induction)

$$\begin{aligned} b_n{}' &= \inf \bigcap_{i=1}^{n-1} \left\{b_i + t \colon t \geq 0, \overline{f}(t) \geq \sum_{j=1}^{n} L_j - \sum_{j=1}^{i-1} L_j\right\} \\ &= \inf\left\{t \geq 0 \colon \min\left\{\sum_{j=1}^{i-1} L_j + \overline{f}(t-b_i) \colon 1 \leq i \leq n-1\right\}\right. \\ &\qquad \left. \geq \sum_{j=1}^{n} L_j\right\} \end{aligned}$$

which implies

$$b_n{}' = B_n^{-1}\left(\sum_{j=1}^{n} L_j\right)$$

where

$$B_n(t) \triangleq \min\left\{\sum_{j=1}^{i-1} L_j + \overline{f}(t-b_i) \colon 1 \leq i \leq n-1\right\}. \tag{10}$$

With a subroutine $\min\{\cdot, \cdot\}$ to find the minimum of two functions, we can set $B_2(t) = \overline{f}(t-b_1) = \overline{f}(t-a_1)$ and recursively update $B_n(t)$ by

$$B_{n+1}(t) = \min\left\{B_n(t), \sum_{j=1}^{n-1} L_j + \overline{f}(t-b_n)\right\}. \tag{11}$$

In summary, we sketch the complete realization structure in Fig. 1. Note the dashed line implies the update of $B_n(t)$ occurs only after b_n

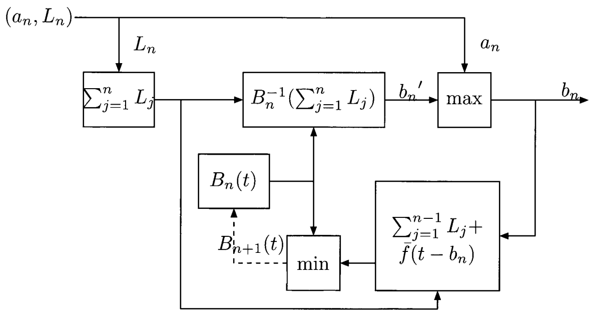

Fig. 1. The complete block diagram of the proposed realization.

has been determined. When the nth packet with length L_n arrives at $t = a_n$, b_n is determined and $B_n(t)$ is updated according to (11).

VI. Conclusion

Discrete-time traffic regulation problem has been systematically solved in [4]. However, to the best of the authors' knowledge, how to optimally regulate a traffic source in continuous-time setting has remained open till now. In this correspondence, we successfully determined the regulation formulas of continuous-time optimal deterministic regulators. Theorem 4 shows that for all causal output processes conforming to a given constraint function f, the nth departure time of the continuous-time optimal deterministic regulator is earliest for all $n \in \mathbb{N}$.

When comparing the continuous-time regulation formula (5) and its discrete-time counterpart (6), one may find they are actually identical (packet sizes are all unity in the discrete-time case). However, the continuous-time output accumulation function $T[0, t)$ cannot be written in a form similar to (9). The critical point for this subtle distinction is Lemma 2, whose continuous-time counterpart is not true. Consequently, discrete-time optimal deterministic regulators can be regarded as special cases of general continuous-time optimal deterministic regulators.

One important issue we did not discuss in this correspondence is the implementation complexity of the realization structure mentioned in Section V. Without some more carefully designed algorithms, the current structure may be too complex to be realized. For example, how to efficiently represent and recursively update $B_n(t)$ in Fig. 1 is very critical to the feasibility of these optimal regulators. In addition, a fast inverse function computation ($B_n^{-1}(\cdot)$) is also an important component. These implementation issues will be studied in our future work.

Traffic regulation has been widely accepted as an indispensable part of QoS-guaranteed multimedia networks. Therefore, by extending traffic-regulation theory to continuous time, our work provides a fundamental framework for future research regarding QoS-guaranteed network design/analysis in continuous time.

Acknowledgment

The authors wish to thank Prof. Cheng-Shang Chang, who is with the National Tsing Hua University, Hsinchu, Taiwan, ROC, and the Associate Editor, Prof. V. Anantharam, for their helpful comments and suggestions that have improved this correspondence.

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