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Convergency of the Fuzzy Vectors in the Pseudo Fuzzy Vector Space over $F_p^1(1)$

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Abstract : In [4], we consider the pseudo fuzzy vector space SFR over $F_p^1(1)$. Here we further discuss the convergency of the fuzzy vectors in SFR.

Keywords : Fuzzy convergence.

§1. Introduction

In this article, we discuss the convergency of the fuzzy space over $F_p^1(1)$ ([4]). In section 2, we stated the pseudo fuzzy vector spoce SFR over $F_p^1(1)$ as the following:

From two points $P = (x^{(1)}, x^{(2)}, \cdots, x^{(n)})$ and $Q = (y^{(1)}, y^{(2)}, \cdots, y^{(n)})$ on R^n, we have the crisp vector $\overrightarrow{PQ}= (y^{(1)} - x^{(1)}, y^{(2)} - x^{(2)}, \cdots\cdots, y^{(n)} - x^{(n)})$. And the two level 1 fuzzy points $\widetilde{P}, \widetilde{Q}$ be $\widetilde{P} = (x^{(1)}, x^{(2)}, \cdots, x^{(n)})$, $\widetilde{Q} = (y^{(1)}, y^{(2)}, \cdots, y^{(n)})$ in $F_p^n(1) = \{(a^{(1)}, a^{(2)}, \cdots, a^{(n)} | \forall (a^{(1)}, a^{(2)}, \cdots, a^{(n)}) \in R^n\}$.

There is an one-one onto mapping $P = (x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \longleftrightarrow \widetilde{P} = (x^{(1)}, x^{(2)}, \cdots, x^{(n)})_1$. Therefore for the crisp vector $\overrightarrow{PQ}= (y^{(1)} - x^{(1)}, y^{(2)} - x^{(2)}, \cdots\cdots y^{(n)} - x^{(n)})_1 = \widetilde{Q} \ominus \widetilde{P}$.

Let the family of the fuzzy sets on R^n satisfying the definitions of convex and normal be F_c. Obviously, $F_p^n(1) \subset F_c$. If $\widetilde{X}, \widetilde{Y} \in F_c$, define the fuzzy vector $\widetilde{X}\widetilde{Y} = \widetilde{Y} \ominus \widetilde{X}$. Let $SFR = \{\overrightarrow{\widetilde{X}\widetilde{Y}} \mid \forall \widetilde{X}, \widetilde{Y} \in F_c\}$. Then we have the pseudo fuzzy vector spoce over $F_p^n(1)(= a_1 \mid \forall a \in R\}$. In section 3, we shall discuss the convergency of the fuzzy vectors in SFR.

§2. Preparation.

In [4], we discussed the pseudo fuzzy vector space SFR over $F_p^1(1)$. In order to discuss the convergence of the fuzzy vectors in SFR, we need to know some definitions.

Definition 2.1. (1°) The fuzzy set $\widetilde{A}$ on $R = (-\infty, \infty)$ is convex iff every ordinary set $A(\alpha) = \{x \mid \mu_{\widetilde{A}}(x) \geq \alpha\} \, \forall \, \alpha \in [0,1]$ is convex. And hence $A(\alpha)$ is a closed interval of R.

(2°) The fuzzy set $\widetilde{A}$ on R is normal iff $\bigvee_{x \in R} \mu_{\widetilde{A}}(x) = 1$.

Next, we extend this definition to R^n by saying the membership function of the fuzzy set $\widetilde{D}$ on R^n is $\mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \in [0,1] \, \forall \, (x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \in R^n$.

Definition 2.2. The α - cut $(0 \leq \alpha \leq 1)$ of the fuzzy set $\widetilde{D}$ on R^n is defined by $D(\alpha) = \{(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \mid \mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \cdots x^{(n)}) \geq \alpha\}$

Definition 2.3. (1°) The fuzzy set $\widetilde{D}$ on R^n is convex iff every ordinary set $D(\alpha) = \{(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \mid \mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \cdots x^{(n)}) \geq \alpha\} \, \forall \, \alpha \in [0,1]$ is convex closed subset of R^n.

(2°) The fuzzy set $\widetilde{D}$ is normal iff $\bigvee_{(x^{(1)}, x^{(2)}, \cdots x^{(n)}) \in R^n} \mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \cdots x^{(n)}) = 1$

Let the family of the fuzzy sets on R^n satisfying the Definition 2.3 (1°), (2°) be F_c.

Definition 2.4. (Pu and Liu [3]) The fuzzy set $a_\alpha (0 \leq \alpha \leq 1)$ on R is called a level α fuzzy point on R if its membership function $\mu_{a_\alpha}(x)$ is

$$\mu_{a_\alpha}(x) = \begin{cases} \alpha, & x = a \\ 0, & x \neq a \end{cases} \tag{1}$$

Let the family of all level α fuzzy points on R be $F_p^{(1)}(\alpha) = \{a_\alpha \mid \forall a \in R\}, 0 \leq \alpha \leq 1$

Definion 2.5 We call the fuzzy set $a^{(1)}, a^{(2)}, \cdots, a^{(n)})_\alpha, (0 \leq \alpha \leq 1)$. A level α fuzzy point on R^n if its membership function is

$$\mu_{(a^{(1)},a^{(2)},\cdots,a^{(n)})_\alpha}(x^{(1)},x^{(2)},\cdots,x^{(n)})$$
$$= \begin{cases} \alpha, & \text{if } (x^{(1)},x^{(2)},\cdots,x^{(n)}) = (a^{(1)},a^{(2)},\cdots,a^{(n)}) \\ 0, & \text{elsewhere} \end{cases} \quad (2)$$

Let the family of all level α fuzzy points on R^n be

$$F_p^{(n)}(\alpha) = \{(a^{(1)},a^{(2)},\cdots a_\alpha^{(n)} \,|\, \forall (A^{(1)},a^{(2)},cdots,a^{(n)}) \in R^n\}, 0 \leq \alpha \leq 1$$

and $F_p^{(n)} = \bigcup_{0\leq\alpha\leq 1} F_p^{(n)}(\alpha)$

For each $a_\alpha \in F_p^1(\alpha)$, regard $a_\alpha = (a,a,\cdots,a)_\alpha$ as a special level α fuzzy point on R^n degenerated from a level α fuzzy point $(a^{(1)},a^{(2)},\cdots,a^{(n)})$ with $a(1) = a^{(2)} = \cdots = a^{(n)}$. Hence we have the following expression :

$$\mu_{(a,a,\cdots,a)_\alpha}(x^{(1)},x^{(2)},\cdots,x^{(n)}) = \begin{cases} \alpha, & (x^{(1)},x^{(2)},\cdots,x^{(n)}) = (a,a,\cdots,a) \\ 0, & (x^{(1)},x^{(2)},\cdots,x^{(n)}) \neq (a,a,\cdots,a) \end{cases}$$

$$\stackrel{\text{say}}{=} \mu_{a_\alpha}(x^{(1)},x^{(2)},\cdots,x^{(n)})$$

Definition 2.6. For $D \subset R^n$, call $D_\alpha, 0 \leq \alpha \leq 1$ a level α domain on R^n, if its membership function is

$$\mu_{D_\alpha}(x^{(1)},x^{(2)},\cdots,x^{(n)}) = \begin{cases} \alpha, & \text{if } (x^{(1)},x^{(2)},\cdots,x^{(n)}) \in D \\ 0, & \text{if } (x^{(1)},x^{(2)},\cdots,x^{(n)}) \neq D \end{cases} \quad (3)$$

Let the family of all the level α fuzzy domain be $FD^* = \{E_\alpha \,|\, \forall E \subset R^n\}$, and let the family od all subsets of R^n be $\mathcal{P}\ (R^n) = \{E \,|\, \forall E \subset R^n\}$.

Then there is an one to one mapping η between $\mathcal{P}\ (R^n)$ and FD^*

$$\begin{aligned} E \in \mathcal{P} \longleftrightarrow \eta(E) = E_\alpha \in FD^* \\ \eta^{(-1)}(E_\alpha) = E, \quad \alpha \in [0,1] \end{aligned} \quad (4)$$

Since $\tilde{D} \in F_c$, the α -cut $D(\alpha)(0 \leq \alpha \leq 1)$ of $\tilde{D}$ can be mapped to $D(\alpha)_\alpha$. Thus we have the following decomposition principle:

$$\text{For} \quad \tilde{\mathrm{D}} \in \mathrm{F_c}, \quad \tilde{\mathrm{D}} = \bigcup_{\alpha \in [0,1]} \mathrm{D}(\alpha)_\alpha \tag{5}$$

From Kaufmann and Gupta [2], we have for $D, E \subset R^n$, $k \in R$,

$$\begin{aligned} D(+)E =& \{(x^{(1)} + y^{(1)}, x^{(2)} + y^{(2)}, \cdots, x^{(n)} + y^{(n)} \mid \\ & \forall (x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \in D, (y^{(1)}, y^{(2)}, \cdots, y^{(n)}) \in E\} \end{aligned} \tag{6}$$

$$\begin{aligned} D(-)E =& \{(x^{(1)} - y^{(1)}, x^{(2)} - y^{(2)}, \cdots, x^{(n)} - y^{(n)} \mid \\ & \forall (x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \in D, (y^{(1)}, y^{(2)}, \cdots, y^{(n)}) \in E\} \end{aligned} \tag{7}$$

$$k(\cdot)D = \{(kx^{(1)}, kx^{(2)}, \cdots, kx^{(n)}) \mid \forall (x^{(1)}, x^{(2)}, \cdots, X^{(n)}) \in D\} \tag{8}$$

From (4) - (8), and the definition of the α - cut, we have :

$$\text{The } \alpha - \text{cut of } \tilde{\mathrm{D}}(+)\tilde{\mathrm{E}} \text{ is } \mathrm{D}(\alpha) + \mathrm{E}(\alpha), \quad \tilde{\mathrm{D}} \oplus \tilde{\mathrm{E}} = \bigcup_{0 \leq \alpha \leq 1} (\mathrm{D}(\alpha)(+)\mathrm{E}(\alpha))_\alpha \tag{9}$$

$$\text{The } \alpha - \text{cut of } \tilde{\mathrm{D}}(-)\tilde{\mathrm{E}} \text{ is } \mathrm{D}(\alpha) - \mathrm{E}(\alpha), \quad \tilde{\mathrm{D}} \ominus \tilde{\mathrm{E}} = \bigcup_{0 \leq \alpha \leq 1} (\mathrm{D}(\alpha)(-)\mathrm{E}(\alpha))_\alpha \tag{10}$$

$$\text{The } \alpha - \text{cut of } \mathrm{k_1} \odot \mathrm{wtD} \text{ is } \mathrm{k}(\cdot)\mathrm{D}(\alpha), \quad \mathrm{k_1} \odot \tilde{\mathrm{D}} = \bigcup_{0 \leq \alpha \leq 1} (\mathrm{k}(\cdot)\mathrm{D}(\alpha))_\alpha \tag{11}$$

In the crisp case on R^n, we can consider the n - dimensional vector space E^n over R.

Let $P = (P^{(1)}, p^{(2)}, \cdots, p^{(n)})$, $Q = (q^{(1)}, q^{(2)}, \cdots, q^{(n)})$, $A = (a^{(1)}, a^{(2)}, \cdots, a^{(n)})$, $B = (b^{(1)}, b^{(2)}, \cdots, b^{(n)}) \quad \in R^N$; $k \in R$.

Define the crisp vectors $\overrightarrow{PQ}$, $\overrightarrow{AB} + \overrightarrow{PQ}$ and $k \cdot \overrightarrow{PQ}$ as follows:

$$\overrightarrow{PQ} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \cdots, q^{(n)} - p^{(n)}) = Q(-)P \tag{12}$$

$$\begin{aligned} \overrightarrow{AB} + \overrightarrow{PQ} = & (b^{(1)} + q^{(1)} - a^{(1)} - p^{(1)}, b^{(2)} + q^{(2)} - a^{(2)} - p^{(2)}, \\ & \cdots\cdots, b^{(n)} + q^{(n)} - a^{(n)} - p^{(n)}) \end{aligned} \tag{13}$$

$$k\cdot \overrightarrow{PQ}= (kq^{(1)} - kq^{(1)}, kq^{(2)} - kq^{(2)}, \cdots\cdots, kq^{(n)} - kq^{(n)}) \tag{14}$$

Let $O = (0,0,\cdots,0) \in R^n$, $\overrightarrow{OP}= (p^{(1)}, p^{(2)}, \cdots, p^{(n)})$, $\overrightarrow{OO}= (0,0,\cdots,0)$. And let $E^n = \{q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \cdots, q^{(n)} - p^{(n)}, ||forallP, Q \in R^n\}$. This is the family of all level 1 fuzzy point on R. There is an one - one onto mapping between the point $a^{(1)}, a^{(2)}, \cdots, a^n)$ on R^n and the level 1 fuzzy point $(a^{(1)}, a^{(2)}, \cdots, a^n)_1$ on $F_p^n(1)$:

$$\begin{aligned}\rho : (a^{(1)}, a^{(2)}, \cdots, a^{(n)}) \in R^n \longleftrightarrow &\rho((a^{(1)}, a^{(2)}, \cdots, a^{(n)})) \\ &= (a^{(1)}, a^{(2)}, \cdots, a^{(n)})_1 \in F_p^n(1)\end{aligned} \tag{15}$$

Let $\widetilde{P} = (p^{(1)}, p^{(2)}, \cdots, p^{(n)})_1$, $\widetilde{Q} = (q^{(1)}, q^{(2)}, \cdots, q^{(n)})_1 \in F_p^n(1)$. From (12), (15) we have the following definition:

$$\overrightarrow{\widetilde{P}\widetilde{Q}}= (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \cdots, q^{(n)} - p^{(n)})_1 = \widetilde{Q} - \widetilde{P} \tag{16}$$

Let $FE^n = \{\overrightarrow{\widetilde{P}\widetilde{Q}} \ |\forall \widetilde{P}, \widetilde{Q} \in F_p^n(1)\}$. From (12) and (16), we have the one - one onto mapping:

$$\begin{aligned}&\overrightarrow{PQ}= (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \cdots, q^{(n)} - p^{(n)}) \\ &\longleftrightarrow \rho(\overrightarrow{PQ}= (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \cdots, q^{(n)} - p^{(n)}) \in FE^n\end{aligned}$$

and

$$\begin{aligned}&\overrightarrow{AB} - \overrightarrow{PQ} \\ &= (b^{(1)} + q^{(1)} - a^{(1)} - p^{(1)}, b^{(2)} + q^{(2)} - a^{(2)} - p^{(2)}, \cdots, b^{(n)} + q^{(n)} - a^{(n)} - p^{(n)}) \\ &\longleftrightarrow \\ &(b^{(1)} + q^{(1)} - a^{(1)} - p^{(1)}, b^{(2)} + q^{(2)} - a^{(2)} - p^{(2)}, \cdots, b^{(n)} + q^{(n)} - a^{(n)} - p^{(n)})_1 \\ &= \overrightarrow{AB} \oplus \overrightarrow{PQ} \\ &k\cdot \overrightarrow{PQ}= (kq^{(1)} - kp^{(1)}, kq^{(2)} - p^{(2)} \cdots, kq^{(n)} - p^{(n)}) \\ &\longleftrightarrow (kq^{(1)} - kp^{(1)}, kq^{(2)} - p^{(2)} \cdots, kq^{(n)} - p^{(n)})_1 = k_1 \odot \overrightarrow{\widetilde{P}\widetilde{Q}}\end{aligned}$$

Therefore $FE^n = \{\widetilde{P}\widetilde{Q} \mid \forall \widetilde{P}, \widetilde{Q} \in F_p^n(1)\}$ is a vector space over $F_p^n(1)$ in fuzzy sense.

In [4], we further extend FE^n:

For $\widetilde{X}, \widetilde{Y} \in F_c$, define $\overrightarrow{\widetilde{X}\widetilde{Y}} = \widetilde{Y} \ominus \widetilde{X}$ and call $\overrightarrow{\widetilde{X}\widetilde{Y}}$, a fuzzy vector. Let $SFR = \{\overrightarrow{\widetilde{X}\widetilde{Y}} \mid \forall \widetilde{X}, \widetilde{Y} \in F_c\}$. We proved the following properties hold in [4].

Property 2.1. For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{W}\widetilde{Z}} \in SFR$,
$\overrightarrow{\widetilde{X}\widetilde{Y}} = \overrightarrow{\widetilde{W}\widetilde{Z}}$ iff $\widetilde{Y} \ominus \widetilde{X} = \widetilde{Z} \ominus \widetilde{W}$.

Property 2.2. For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{W}\widetilde{Z}} \in SFR$, $k \in R$,

(1°) $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = \overrightarrow{\widetilde{A}\widetilde{B}}$, where $\widetilde{A} = \widetilde{X} \oplus \widetilde{W}, \widetilde{B} = \widetilde{Y} \oplus \widetilde{Z}$.

(2°) $k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}} = \overrightarrow{\widetilde{C}\widetilde{D}}$, where $\widetilde{C} = k_1 \odot \widetilde{X}, \widetilde{D} = k_1 \odot \widetilde{Y}$.

Property 2.3 For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{W}\widetilde{Z}}, \overrightarrow{\widetilde{U}\widetilde{V}} \in SFR$, $k, t \in R$

(1°) $\overrightarrow{\widetilde{X}\widetilde{Y}} \ominus \overrightarrow{\widetilde{W}\widetilde{Z}} = \overrightarrow{\widetilde{W}\widetilde{Z}} \oplus \overrightarrow{\widetilde{X}\widetilde{Y}}$

(2°) $(\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}}) \oplus \overrightarrow{\widetilde{U}\widetilde{V}} = \overrightarrow{\widetilde{X}\widetilde{Y}} \oplus (\overrightarrow{\widetilde{W}\widetilde{Z}} \oplus \overrightarrow{\widetilde{U}\widetilde{V}})$

(3°) $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{O}\widetilde{O}} = \overrightarrow{\widetilde{X}\widetilde{Y}}$

(4°) $k_1 \odot (t_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) = (kt)_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}$

(5°) $k_1 \odot (\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}}) = (k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) \oplus (k_1 \odot \overrightarrow{\widetilde{W}\widetilde{Z}})$

(6°) $1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}} = \overrightarrow{\widetilde{X}\widetilde{Y}}$ where $\overrightarrow{\widetilde{O}\widetilde{O}} = (0, 0, \cdots, 0)_1$

In SFR, the followings do not hold.

(7°) For $\overrightarrow{\widetilde{X}\widetilde{Y}} \in SFR$, and $\overrightarrow{\widetilde{X}\widetilde{Y}} \neq \overrightarrow{\widetilde{O}\widetilde{O}}$, there exists $\overrightarrow{\widetilde{W}\widetilde{Z}}$ $(\neq \overrightarrow{\widetilde{O}\widetilde{O}}) \in SFR$ such that $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = \overrightarrow{\widetilde{O}\widetilde{O}}$

(8°) $(k+t)_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) = (k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) \oplus (t_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}})$

From Property 2.3, we know that SFR satisfies all the conditions that vector space required except (7°) and (8°). Therefore in [4], we called SFR, a pseudo fuzzy vector space over $F_p^1(1)$.

Example 2.1. A moving station carrying a missle on it, This car left from point $P = (2,5)$ passing through point $Q = (4,6)$ arrived at $R = (8,9)$, and aiming at the target $Z = (100,200)$. As we can see the missle usually falls in the vicinity of Z, say $\widetilde{Z}$ instead of hitting at Z exactly.

Let the membership function of $\widetilde{Z}$ is

$$\mu_{\widetilde{Z}}(x^{(1)}, x^{(2)}) = \begin{cases} \frac{1}{25}(25 - (x^{(1)} - 100)^2 - (x^{(2)} - 200)^2), & \text{if } (x^{(1)} - 100)^2 + (x^{(2)} - 200)^2) \le 25 \\ 0, & \text{elsewhere} \end{cases}$$

Let the level 1 fuzzy points $\widetilde{P} = (2,5)_1$, $\widetilde{Q} = (4,6)_1$, $R = (8,9)_1$. We have the fuzzy routes

$$\widetilde{P} \longrightarrow \widetilde{Q} \longrightarrow \widetilde{R} \longrightarrow \widetilde{Z}$$

and hence the fuzzy vectors $\overrightarrow{\widetilde{P}\widetilde{Q}} = (2,1)_1$, $\overrightarrow{\widetilde{Q}\widetilde{R}} = (4,3)_1$, $\overrightarrow{\widetilde{R}\widetilde{Z}} = \widetilde{Z} \ominus \widetilde{R}$, $\overrightarrow{\widetilde{P}\widetilde{Z}} = \widetilde{Z} \ominus \widetilde{P}$. By extension theory, the membership function function of $\overrightarrow{\widetilde{R}\widetilde{Z}} = \widetilde{Z} \ominus \widetilde{R}$ is $\mu_{\overrightarrow{\widetilde{R}\widetilde{Z}}}(z^{(1)}, z^{(2)}) = \sup_{z^{(j)} = v^{(j)} - u^{(j)}, j=1,2} \mu_{\widetilde{R}}(u^{(1)}, u^{(2)}) \wedge \mu_{\widetilde{Z}}(v^{(1)}, v^{(2)})$

$$= \mu_{\widetilde{Z}}(z^{(1)} + 8, z^{(2)} + 9)$$

$$= \begin{cases} \frac{1}{25}(25 - (z^{(1)} - 92)^2 - (z^{(2)} - 191)^2), & \text{if } (z^{(1)} - 92)^2 + (z^{(2)} - 191)^2) \le 25 \\ 0, & \text{elsewhere} \end{cases}$$

Similarly,

$$\mu_{\overrightarrow{\widetilde{P}\widetilde{Z}}}(z^{(1)}, z^{(2)}) = \begin{cases} \frac{1}{25}(25 - (z^{(1)} - 98)^2 - (z^{(2)} - 195)^2), & \text{if } (z^{(1)} - 98)^2 + (z^{(2)} - 195)^2) \le 25 \\ 0, & \text{elsewhere} \end{cases} \tag{17}$$

Let $S = (98,202)$. It is clearly that $(98,202)$ is within the circle with center

(100, 200), radius 5. The crisp vector start with the point $P = (2,5)$, and end at $S = (98,202)$ is $\overrightarrow{\widetilde{P}\widetilde{S}} = (96,197)$. Its grade of membership in $\widetilde{P}\widetilde{Z}$ from (17) is $\mu_{\widetilde{P}\widetilde{Z}}(96,197) = \frac{1}{25}(25^2 - 2^2 - 2^2) = 0.68$. Let the aim is $T = (100,200)$. The crisp vector beginning at $P = (2,5)$ and aiming at $T = (100,200)$ is $\overrightarrow{PT} = (98,195)$. Its grade of membership in $\overrightarrow{\widetilde{P}\widetilde{Z}}$ again from (17) is $\mu_{\widetilde{P}\widetilde{Z}}(98,195) = \frac{1}{25}(25 - 0^2 - 0^2) = 1$.

Example 2.2. In a shooting practice, let $C = ((10,30), 1 + \frac{1}{m}) = \{(x,y) \mid (x-10)^2 + (y-30)^2 \leq (1+\frac{1}{m})^2\}$ Always shooting at $(1,2)$, and aiming at $Z = (10,30)$. The first time, bullet was falling in $C((10,30), 2(= 1+1))$. The second time was falling in $C((10,30), 1 + \frac{1}{2})$. The m-th time was falling in $C((10,30), 1 + \frac{1}{m})$. In other words, the bullet was more and more closed to $C((10,30),1)$, that is, more and more accuracy.

Let the fuzzy aim be $\widetilde{Z}_m$, its membership function is $\mu_{\widetilde{Z}_m}$

$$= \begin{cases} \frac{1}{(1+\frac{1}{m})^2}[(1+\frac{1}{m})^2 - (x-10)^2 - (y-30)^2], & \text{if } (x-10)^2 + (y-30)^2 \leq (1+\frac{1}{m})^2 \\ 0, & \text{elsewhere} \end{cases}$$

Thus we have the m-th fuzzy vector $\overrightarrow{\widetilde{Q}\widetilde{Z}_m}, m = 1, 2, \cdots$, where $\widetilde{Q} = (1,2)_1$ In the next section, we shall discuss the convergency of the fuzzy vectors in SFR and find out the limit fuzzy vector $\lim_{n\to\infty} \overrightarrow{\widetilde{Q}\widetilde{Z}_m}$.

§3. The convergency of the vectors in SFR.

Before we try to investigate the convergency of the fuzzy vectors in SFR, we first define the following open set in R^n and discuss some properties 3.1 - 3.10. Let

$$O((a^{(1,1)}, a^{(1,2)}), \cdots, (a^{(n,1)}, a^{(n,2)}))$$
$$= \{(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \mid a^{(j,1)} < x^{(j)} < a^{(j,2)}, j = 1, 2, \cdots, n\}$$

From (6) - (8), we have

$$
\begin{aligned}
&O((a^{(1,1)}, a^{(1,2)}), \cdots, (a^{(n,1)}, a^{(n,2)}))(+)O((b^{(1,1)}, b^{(1,2)}) \cdots, (b^{(n,1)}, b^{(n,2)})) \\
&= \{(z^{(1)}, z^{(2)}, \cdots, z^{(n)}) \mid z^{(j)} = x^{(j)} + y^{(j)},\ a^{(j,1)} < x^{(j)}, < a^{(j,2)}, \\
&\qquad b^{(j,1)} < y^{(j)} < b^{j,2)};\ j = 1, 2, \cdots, n\} \qquad (18) \\
&= O((a^{(1,1)} + b^{(1,1)}, a^{(1,2)} + b^{(1,2)}), \cdots\cdots, (a^{(n,1)} + b^{(n,1)}, a^{(n,2)} + b^{n,2)}))
\end{aligned}
$$

$$
\begin{aligned}
&O((a^{(1,1)}, a^{(1,2)}), \cdots, (a^{(n,1)}, a^{(n,2)}))(-)O((b^{(1,1)}, b^{(1,2)}) \cdots, (b^{(n,1)}, b^{(n,2)})) \\
&= \{(z^{(1)}, z^{(2)}, \cdots, z^{(n)}) \mid z^{(j)} = x^{(j)} - y^{(j)},\ a^{(j,1)} < x^{(j)}, < a^{(j,2)}, \\
&\qquad b^{(j,1)} < y^{(j)} < b^{j,2)};\ j = 1, 2, \cdots, n\} \qquad (19) \\
&= O((a^{(1,1)} - b^{(1,1)}, a^{(1,2)} - b^{(1,2)}), \cdots\cdots, (a^{(n,1)} - b^{(n,1)}, a^{(n,2)} - b^{n,2)}))
\end{aligned}
$$

If $k > 0$,

$$
\begin{aligned}
&k(\cdot)O((a^{(1,1)}, a^{(1,2)}), \cdots, (a^{(n,1)}, a^{(n,2)})) \\
&= \{(z^{(1)}, z^{(2)}, \cdots, z^{(n)}) \mid z^{(j)} = kx^{(j)},\ a^{(j,1)} < x^{(j)} < a^{(j,2)},\ j = 1, 2, \cdots, n\} \qquad (20) \\
&= O((ka^{(1,1)}, ka^{(1,2)}), \cdots\cdots, (ka^{(n,1)}, ka^{(n,2)}))
\end{aligned}
$$

If $k < 0$,

$$
\begin{aligned}
&k(\cdot)O((a^{(1,1)}, a^{(1,2)}), \cdots, (a^{(n,1)}, a^{(n,2)})) \\
&= \{(z^{(1)}, z^{(2)}, \cdots, z^{(n)}) \mid z^{(j)} = kx^{(j)},\ a^{(j,1)} < x^{(j)} < a^{(j,2)},\ j = 1, 2, \cdots, n\} \qquad (21) \\
&= O((ka^{(1,2)}, ka^{(1,1)}), \cdots\cdots, (ka^{(n,2)}, ka^{(n,1)}))
\end{aligned}
$$

Let $\mathcal{B} = \{O((a^{(1,1)}, a^{(1,2)}), \cdots\cdots, (a^{(n,1)}, a^{(n,2)}) \mid \forall a^{(j,1)} < a^{(j,2)},\ a^{(j,1)}, a^{(j,2)} \in R;\ j = 1, 2, \cdots, n;\ 0 \le \alpha \le 1\}$

Let $\mathcal{B}^*$ be the family of fuzzy sets in $\mathcal{B}$ or any arbitrary unions of these fuzzy sets.

Remark 3.1. Any intersection of two fuzzy sets in $\mathcal{B}$ belongs to $\mathcal{B}$. And when two fuzzy sets in $\mathcal{B}$ have no intersection, we called their intersection $\emptyset$.

Let $F = F_p^n \cup F_c \cup\ \mathcal{B}$. In order to consider the problem of convergency, we first consider the topology for F.

Definition 3.1. $\widetilde{Q} \in F$ is an open fuzzy set iff for each $(x^{(1)}, x^{(2)}, \cdots, x^{(n)})_\alpha \subset \widetilde{Q}$, there exists $\widetilde{O} \in \mathcal{B}$ such that $(x^{(1)}, x^{(2)}, \cdots, x^{(n)})_\alpha \subset \widetilde{O} \subset \widetilde{Q}$. Let T_F be the family of all open fuzzy sets satisfying Definition 3.1. Obviously, $\mathcal{B}^* \subset T_F$. Definition 3.2 (Chang [1]). T is a family of fuzzy sets in the space X satisfying the following :

(1°) $\emptyset, X \in T$.

(2°) $\widetilde{A}, \widetilde{B} \in T$, then $\widetilde{A} \cap \widetilde{B} \in T$.

(3°) $\widetilde{A}_j \in T, j \in I$ (any index set), then $\bigcup_{j \in I} \widetilde{A}_j \in T$. T is called fuzzy topology for X. And (X, T) is called fuzzy topological space (abbreviate as fts).

Property 3.1. T_F is a fuzzy topology for R^n, (R^n, T_F) is a fuzzy topological sets in R^n are restricted in F.

Proof.

(1°) It is obvious that $R^n \in T_F$. Definition 3.2 (1°) is fulfilled.

(2°) For $\widetilde{D}, \widetilde{E} \in T_F$, and $(x^{(1)}, x^{(2)}, \cdots, x^{(n)})_\alpha \subset \widetilde{D} \cap \widetilde{E}$, we have $(x^{(1)}, x^{(2)}, \cdots, x^{(n)})_\alpha \subset \widetilde{D}$, and $(x^{(1)}, x^{(2)}, \cdots, x^{(n)})_\alpha \subset \widetilde{E}$. From Definition 3.1 , there exist $\widetilde{I}, \widetilde{J} \in \mathcal{B}$ such that $(x^{(1)}, x^{(2)}, \cdots, x^{(n)})_\alpha \subset \widetilde{I} \subset \widetilde{D}$. and $(x^{(1)}, x^{(2)}, \cdots, x^{(n)})_\alpha \subset \widetilde{J} \subset \widetilde{E}$. Therefore $(x^{(1)}, x^{(2)}, \cdots, x^{(n)})_\alpha \subset \widetilde{I} \cap \widetilde{J}$. Hence $\widetilde{I} \cap \widetilde{J} \subset \widetilde{D} \cap \widetilde{E}$. Thus $\widetilde{D} \cap \widetilde{E} \in T_F$. Definition 3.2 (2°) is fulfilled.

(3°) For $\widetilde{D}_j \in T_F, j \in I$, and each $(x^{(1)}, x^{(2)}, \cdots, x^{(n)})_\alpha \subset \bigcup_{j \in I} \widetilde{D}_j$, there exists $m \in I$ such that $(x^{(1)}, x^{(2)}, \cdots, x^{(n)})_\alpha \subset \widetilde{D}_m$. By Definition 3.1, there is an $\widetilde{J} \in \mathcal{B}$ such that $(x^{(1)}, x^{(2)}, \cdots, x^{(n)})_\alpha \subset \widetilde{J} \subset \widetilde{D}_m \subset \bigcup_{j \in I} \widetilde{D}_j \subset T_F$. Thus Definition (3°) is is fulfilled.

Hence from Definition 3.2, T_F is a fuzzy topology for R^n, and (R^n, T_F) is a fuzzy topological space. i.e., If we set $X = R^n, T = T_F$ in Definition 3.2, then Definition

3.2 holds. Therefore Definition 3.3, Definition 3.4 and Property 3.2 can all be applied.

Definition 3.3 (Chang [1]), Definition 3.2) A fuzzy set $\widetilde{U}$ in a $fts\,(X,T)$ is a neighborhool of a fuzzy set $\widetilde{A}$ iff there exists fuzzy set $\widetilde{O} \in T$ suvh that $\widetilde{A} \subset \widetilde{O} \subset \widetilde{U}$.

Definition 3.4 (Chang [1], Definition 3). If a sequence of fuzzy sets $\{\widetilde{A}_n, n = 1, 2, \cdots\}$ is in a $fts\,(X,T)$, then we say this sequence converges to a fuzzy set $\widetilde{A}$ iff it is eventually contained in each neighborhood of $\widetilde{A}$ (i.e., if $\widetilde{B}$ is any neighborhood of $\widetilde{A}$ there is a positive integer m such that whenever $n \geq m$, $\widetilde{A}_n \subset \widetilde{B}$).

Property 3.2. $\{\widetilde{A}_n\}$ is an increasing fuzzy sets, $\widetilde{A}_1 \subset \widetilde{A}_2 \subset \cdots\cdots \subset \widetilde{A}$ and $\lim_{n\to\infty} \mu_{\widetilde{A}_n}(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) = \mu_{\widetilde{A}}(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$, $\forall (x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \in R^n$. Then the sequence $\widetilde{A}_n, n = 1, 2, \cdots$ converges to $\widetilde{A}$, denoted by $\lim_{n\to\infty} \widetilde{A}_n = \widetilde{A}$.

Proof. It follows from Definition 3.4 easily.

Definition 3.5. $\bigcup_{\alpha\in[0,1]} O((a^{(1,1)}(\alpha), a^{(1,2)}(\alpha)), \cdots (a^{(n,1)}(\alpha), a^{(n,2)}(\alpha)))_\alpha\ (\in T_F)$ is a neighborhood of $\widetilde{d} \in F_c$ iff for each $\alpha \in [0,1]$, there exixts $O((a^{(1,1)}(\alpha), a^{(1,2)}(\alpha)), \cdots (a^{(n,1)}(\alpha), a^{(n,2)}(\alpha)))_\alpha \in \mathcal{B}$ such that $D(\alpha)_\alpha \subset O((a^{(1,1)}(\alpha), a^{(1,2)}(\alpha)), \cdots (a^{(n,1)}(\alpha), a^{(n,2)}(\alpha)))_\alpha$.

Remark 3.1. From the Decomposition theory we can have $\widetilde{D} = \bigcup_{\alpha\in[0,1]} D(\alpha)_\alpha \subset O((a^{(1,1)}(\alpha), a^{(1,2)}(\alpha)), \cdots (a^{(n,1)}(\alpha), a^{(n,2)}(\alpha)))_\alpha \in T_F$. Hence we know Definition 3.5 holds by Definition 3.3.

Definition 3.6. In F_c, the sequence of fuzzy sets $\widetilde{D}_k = \bigcup_{\alpha\in[0,1]} D_k(\alpha)_\alpha,\ k = 1, 2, \cdots$ converges to $\widetilde{D} = \bigcup_{\alpha\in[0,1]} D(\alpha)_\alpha,\ k = 1, 2, \cdots\ (\in F_\alpha)$ iff for each neighborhood $\widetilde{D} = \bigcup_{\alpha\in[0,1]} O((a^{(1,1)}(\alpha), a^{(1,2)}(\alpha)), \cdots (a^{(n,1)}(\alpha), a^{(n,2)}(\alpha)))_\alpha$, there exists a natural number m such that whenever $k \geq m$, $D_k(\alpha)_\alpha \subset O((a^{(1,1)}(\alpha), a^{(1,2)}(\alpha)), \cdots (a^{(n,1)}(\alpha),$

$a^{(n,2)}(\alpha)))_\alpha$, denoted by $\lim_{k\to\infty} \widetilde{D}_k = \widetilde{D}$.

Since $D \subset R^n$, and $D_\alpha(\in FD^*)$ is one to one onto mapping , from Definition 3.6, we can get the following:

Property 3.3. In F_c, the sequence of fuzzy sets $\widetilde{D}_k = \bigcup_{\alpha\in[0,1]} D_k(\alpha)_\alpha$, $k = 1, 2, \cdots$, converges to $\widetilde{D} = \bigcup_{\alpha\in[0,1]} D(\alpha)_\alpha$ iff for each $\alpha \in [0,1]$ and every neighborhood $O((a^{(1,1)}, a^{(1,2)}), \cdots, (a^{(n,1)}, a^{(n,2)}))_\alpha$ of $D(\alpha)_\alpha$, there exists natural number m such that whenever $k \geq m$, $D_k(\alpha)_\alpha \subset O((a^{(1,1)}, a^{(1,2)}), \cdots, (a^{(n,1)}, a^{(n,2)}))_\alpha$ iff for each $\alpha \in [0,1]$ and every neighborhood $O((a^{(1,1)}, a^{(1,2)}), \cdots, (a^{(n,1)}, a^{(n,2)}))_\alpha$ there exists m such that whenever $k \geq m$, $D_k(\alpha) \subset O((a^{(1,1)}, a^{(1,2)}), \cdots, (a^{(n,1)}, a^{(n,2)}))_\alpha$

The convergency of fuzzy vectors need the following:

Property 3.4. For each $\alpha \in [0,1]$, the α -cut $D_k(\alpha), E_k(\alpha), k = 1, 2, \cdots, m$ of $\widetilde{D}_k, \widetilde{E}_k$ in F_c satisfies the following:

(1°) $(D_k(\alpha)(+)E_k(\alpha))_\alpha = D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha$

(2°) $(D_k(\alpha)(-)E_k(\alpha))_\alpha = D_k(\alpha)_\alpha \ominus E_k(\alpha)_\alpha$

(3°) For each α - cut of $\bigcup_{k=1}^m [\widetilde{D}_k \ominus \widetilde{E}_k]$ is $\bigcup_{k=1}^m [\widetilde{D}_k(+)\widetilde{E}_k]$

(3° − 1) $(\bigcup_{k=1}^m (D_k(\alpha)(+)E_k(\alpha)))_\alpha = \bigcup_{k=1}^m (D_k(\alpha)(+)E_k(\alpha))_\alpha$
$= \bigcup_{k=1}^m (D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha) = (\bigcup_{k=1}^m D_k(\alpha)_\alpha) \oplus (\bigcup_{k=1}^m D_k(\alpha)_\alpha)$

(3° − 2) $\bigcup_{k=1}^m (\widetilde{D}_k \oplus \widetilde{E}_k) = (\bigcup_{k=1}^m D_k) \oplus (\bigcup_{k=1}^m E_k)$

(4°) The α - cut of $\bigcup_{k=1}^m (\widetilde{D}_k \ominus \widetilde{E}_k)_\alpha$ is $\bigcup_{k=1}^m [\widetilde{D}_k(-)\widetilde{E}_k]$

(4° − 1) $(\bigcup_{k=1}^m (D_k(\alpha)(-)E_k(\alpha)))_\alpha = \bigcup_{k=1}^m (D_k(\alpha)(-)E_k(\alpha))_\alpha$
$= \bigcup_{k=1}^m (D_k(\alpha)_\alpha \ominus E_k(\alpha)_\alpha) = (\bigcup_{k=1}^m D_k(\alpha)_\alpha) \ominus (\bigcup_{k=1}^m D_k(\alpha)_\alpha)$

$(4° - 2)$ $\bigcup_{k=1}^{m} (\widetilde{D}_k \ominus \widetilde{E}_k) = (\bigcup_{k=1}^{m} D_k) \ominus (\bigcup_{k=1}^{m} E_k)$

Proof.

(1°)

$$\mu_{D_k(\alpha)_\alpha \ominus E_k(\alpha)_\alpha}(z^{(1)}, z^{(2)}, \cdots, z^{(n)})$$

$$= \sup_{z^{(j)} = x^{(j)} + y^{(j)}); j=1,2,\cdots,n} \mu_{D_k(\alpha)_\alpha}(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$$

$$\wedge \mu_{E_k(\alpha)_\alpha}(y^{(1)}, y^{(2)}, \cdots, y^{(n)})$$

$$= \sup_{x^{(1)}, x^{(2)}, \cdots, x^{(n)}} \mu_{D_k(\alpha)_\alpha}(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$$

$$\wedge \mu_{E_k(\alpha)_\alpha}(z^{(1)} - x^{(1)}, z^{(2)} - x^{(2)}, \cdots, z^{(n)} - x^{(n)})$$

$$= \alpha, \quad \text{if} \quad (\mathrm{x}^{(1)}, \mathrm{x}^{(2)}, \cdots, \mathrm{x}^{(n)}) \in \mathrm{D_k}(\alpha) \quad \text{and}$$

$$(z^{(1)} - x^{(1)}, z^{(2)} - x^{(2)}, \cdots, z^{(n)} - x^{(n)}) \in E_k(\alpha)$$

$$= \alpha, \quad \text{if} \quad (\mathrm{z}^{(1)}, \mathrm{z}^{(2)}, \cdots, \mathrm{z}^{(n)}) \in \mathrm{D_k}(\alpha)(+)\mathrm{E_k}(\alpha)$$

$$= \mu_{(D_k(\alpha)+E_k(\alpha))_\alpha}(z^{(1)}, z^{(2)}, \cdots, z^{(n)}), \quad \forall (x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \in R^n$$

Q.E.D.

(2°) Similar as (1°)

(3°) Let $\widetilde{S}_k = \widetilde{D}_k \oplus \widetilde{E}_k$, from (9), we have $\bigcup_{k=1}$

$= \bigcup_{k=1}^{m} \bigcup_{\alpha \in [0,1]} (D_k(\alpha)(+)E_k(\alpha))_\alpha$

$= \bigcup_{\alpha \in [0,1]} \bigcup_{k=1}^{m} (D_k(\alpha)(+)E_k(\alpha))_\alpha$

Therefore, the α - cut of $\bigcup_{k=1}^{m} (\widetilde{D}_k(\alpha) \oplus \widetilde{E}_k) = \bigcup_{k=1}^{m} S_k$ is

$\bigcup_{k=1}^{m} S_k(\alpha) = \bigcup_{k=1}^{m} (D_k(\alpha)(+)E_k(\alpha)$

$(3° - 1)$ For each $\alpha \in [0, 1]$, the subset $\bigcup_{k=1}^{m} S_k(\alpha)$ of R^n corresponds to the fuzzy set

$\bigcup_{k=1}^{m} S_k(\alpha)_\alpha = (\bigcup_{k=1}^{m} (D_k(\alpha)(+)E_k(\alpha))_\alpha$

We first prove

$$(\bigcup_{k=1}^{m} S_k(\alpha))_\alpha = (\bigcup_{k=1}^{m} S_k(\alpha)_\alpha \tag{22}$$

$$\mu_{\underset{k=1}{\cup}^m S_k(\alpha)_\alpha}(z^{(1)}, z^{(2)}, \cdots, z^{(n)}) = \underset{k=1}{\vee}^m \mu_{S_k(\alpha)_\alpha}(z^{(1)}, z^{(2)}, \cdots, z^{(n)})$$
$$= \alpha, \quad \text{if} \quad (z^{(1)}, z^{(2)}, \cdots, z^{(n)}) \in S_k(\alpha) \quad \text{for some} \quad k \in \{1, 2, \cdots, m\}$$
$$= \alpha, \quad \text{if} \quad (z^{(1)}, z^{(2)}, \cdots, z^{(n)}) \in \underset{k=1}{\cup}^m S_k(\alpha)$$
$$= \mu_{(\underset{k=1}{\cup}^m S_k(\alpha))_\alpha}(z^{(1)}, z^{(2)}, \cdots, z^{(n)}) \quad \forall\ (z^{(1)}, z^{(2)}, \cdots, z^{(n)}) \in R^n$$

Therefore $(\underset{k=1}{\cup}^m S_k(\alpha))_\alpha = \underset{k=1}{\cup}^m S_k(\alpha)_\alpha$

Hence $(\underset{k=1}{\cup}^m (D_k(\alpha)(+)E_k(\alpha)))_\alpha = \underset{k=1}{\cup}^m (D_k(\alpha)(+)E_k(\alpha))_\alpha$

For each $\alpha \in [0, 1]$, and each k, $(1°)$ holds. Therefore

$$\underset{k=1}{\cup}^m (D_k(\alpha)(+)E_k(\alpha))_\alpha = \underset{k=1}{\cup}^m (D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha)$$

Finally, we shall prove

$$\underset{k=1}{\cup}^m (D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha) = \underset{k=1}{\cup}^m (D_k(\alpha)_\alpha) \oplus \underset{k=1}{\cup}^m (E_k(\alpha)_\alpha)$$

$$\mu_{\underset{k=1}{\cup}^m (D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha)}(z^{(1)}, z^{(2)}, \cdots, z^{(n)})$$
$$= \underset{k=1}{\vee}^m \mu_{D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha}(z^{(1)}, z^{(2)}, \cdots, z^{(n)})$$
$$= \underset{k=1}{\vee}^m \sup_{z^{(j)} = x^{(j)} + y^{(j)}; j=1,2,\cdots,n} \mu_{D_k(\alpha)_\alpha}(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \wedge \mu_{E_k(\alpha)_\alpha}(y^{(1)}, y^{(2)}, \cdots, y^{(n)})$$
$$= \underset{k=1}{\vee}^m \underset{y^{(1)}, y^{(2)}, \cdots, y^{(n)}}{\vee} [\mu_{D_k(\alpha)_\alpha}(z^{(1)} - y^{(1)}, z^{(2)} - y^{(2)}, \cdots, z^{(n)} - y^{(n)}) \wedge \mu_{E_k(\alpha)_\alpha}(y^{(1)}, y^{(2)}, \cdots, y^{(n)})]$$
$$= \underset{y^{(1)}, y^{(2)}, \cdots, y^{(n)}}{\vee} [\mu_{\underset{k=1}{\cup}^m D_k(\alpha)_\alpha}(z^{(1)} - y^{(1)}, z^{(2)} - y^{(2)}, \cdots, z^{(n)} - y^{(n)}) \wedge \mu_{\underset{k=1}{\cup}^m E_k(\alpha)_\alpha}(y^{(1)}, y^{(2)}, \cdots, y^{(n)})$$
$$= \mu_{(\underset{k=1}{\cup}^m D_k(\alpha)_\alpha) \oplus (\underset{k=1}{\cup}^m E_k(\alpha)_\alpha)}(z^{(1)}, z^{(2)}, \cdots, z^{(n)}) \forall (z^{(1)}, z^{(2)}, \cdots, z^{(n)}) \in R^n$$

$(3° - 2)$ By Decomposition Principle and $(3° - 1)$, we have

$$\underset{k=1}{\cup}^m (\widetilde{D}_k \oplus \widetilde{E}_k) = \underset{k=1}{\cup}^m \underset{\alpha \in [0,1]}{\cup} D_k(\alpha)(+)E_k(\alpha))_\alpha$$
$$= \underset{\alpha \in [0,1]}{\cup} \underset{k=1}{\cup}^m D_k(\alpha)(+)E_k(\alpha))_\alpha \tag{23}$$
$$= \underset{\alpha \in [0,1]}{\cup} [(\underset{k=1}{\cup}^m D_k(\alpha)_\alpha \oplus (\underset{k=1}{\cup}^m E_k(\alpha_+\alpha)]$$

Let $\widetilde{A} = \underset{k=1}{\cup}^m \widetilde{D}_k$, $\widetilde{B} = \underset{k=1}{\cup}^m \widetilde{E}_k$.

Feom (22), $A(\alpha)_\alpha = \underset{k=1}{\cup}^m \widetilde{D}_k(\alpha)_\alpha$, $B(\alpha)_\alpha = \underset{k=1}{\cup}^m \widetilde{E}_k(\alpha)_\alpha$, $\forall \alpha \in [0,1]$

$$\begin{aligned}\widetilde{A} \oplus \widetilde{B} &= \underset{\alpha\in[0,1]}{\cup} [A(\alpha)(+)B(\alpha)]_\alpha = \underset{\alpha\in[0,1]}{\cup} [A(\alpha)_\alpha \oplus B(\alpha)_\alpha] \\ &= \underset{\alpha\in[0,1]}{\cup} [(\underset{k=1}{\cup}^m \widetilde{D}_k(\alpha)_\alpha) \ominus (\underset{k=1}{\cup}^m \widetilde{E}_k(\alpha)_\alpha)\end{aligned} \tag{24}$$

From (23), (24), we have

$$\begin{aligned}(\underset{k=1}{\cup}^m (\widetilde{D}_k \oplus \widetilde{E}_k) &= \underset{\alpha\in[0,1]}{\cup} [(\underset{k=1}{\cup}^m D_k(\alpha)_\alpha) \oplus (\underset{k=1}{\cup}^m (E_k(\alpha)_\alpha)] \\ &= (\underset{k=1}{\cup}^m \widetilde{D}_k) \oplus (\underset{k=1}{\cup}^m \widetilde{E}_k)\end{aligned}$$

(4°), $(4^\circ - 1)$, $(4^\circ - 2)$ can be proved similarly as (3°), $(3^\circ - 1)$, $(3^\circ - 2)$.

Property 3.5. $\widetilde{D}_k \in F_c, k = 1, 2, \cdots, m; q \neq 0$

(1°) The α - cut of $\underset{k=1}{\cup}^m (q_1 \odot \widetilde{D}_k)$ is $\underset{k=1}{\cup}^m (q(\cdot)D_k(\alpha))$

(2°) $\underset{k=1}{\cup}^m (q(\odot)D_k(\alpha)_\alpha = q_1 \odot (\underset{k=1}{\cup}^m D_k(\alpha)_\alpha)$

(3°) $\underset{k=1}{\cup}^m (q_1 \odot \widetilde{D}_k) = q_1 \odot (\underset{k=1}{\cup}^m \widetilde{D}_k)$.

Same way as the proof of Property 3.4.

Property 3.6. $\widetilde{D}_m, \widetilde{E}_m, \widetilde{D}, \widetilde{E} \in F_c, m = 1, 2, \cdot$, and $\lim_{m\to\infty} \widetilde{D}_m = \widetilde{D}$, $\lim_{m\to\infty} \widetilde{E}_m = \widetilde{E}$, then

(1°) $\lim_{m\to\infty} (\widetilde{D}_m \ominus \widetilde{E}_m) = \widetilde{D} \oplus \widetilde{E} = (\lim_{m\to\infty} (\widetilde{D}_m) \oplus \lim_{m\to\infty} (\widetilde{E}_m)$

(2°) $\lim_{m\to\infty} (\widetilde{D}_m \ominus \widetilde{E}_m) = \widetilde{D} \ominus \widetilde{E} = (\lim_{m\to\infty} (\widetilde{D}_m) \ominus \lim_{m\to\infty} (\widetilde{E}_m)$

(3°) $\lim_{m\to\infty} (k_1 \odot \widetilde{D}_m) = k_1 \odot \widetilde{D} = k_1 \odot (\lim_{m\to\infty} (\widetilde{D}_m))$, $k \neq 0$

Proof.

(1°) Since $\lim_{m\to\infty} \widetilde{D}_m = \widetilde{D}$, $\lim_{m\to\infty} \widetilde{E}_m = \widetilde{E}$, by Property 3.3, for each $\alpha \in [0,1]$ and every neighborhood $O((a^{(1,1)}, a^{(1,2)}), \cdots, (a^{(n,1)}, a^{(n,2)}))$ of $D(\alpha)$, there exists a natural number $m^{(1)}$ such that when $k \geq m^{(1)}$, $D_k(\alpha) \subset O((a^{(1,1)}, a^{(1,2)}), \cdots, (a^{(n,1)}, a^{(n,2)}))$. Also every neighborhood $O((b^{(1,1)}, b^{(1,2)}), \cdots (b^{(n,1)}, b^{(n,2)}))$ of $E(\alpha)$, there exists a natural number $m^{(2)}$ such that when $k \geq m^{(2)}$, $E_k(\alpha) \subset O((b^{(1,1)}, b^{(1,2)}), \cdots, (b^{(n,1)}, b^{(n,2)}))$.

Let $m = \max(m^{(1)}, m^{(2)})$. Then for each $\alpha \in [0, 1]$, when $k \geq m$, by (18) we have $D_k(\alpha)(+)E_k(\alpha) \subset O((a^{(1,1)} + b^{(1,1)}, a^{(1,2)} + b^{(1,2)}), \cdots\cdots,$
$(a^{(n,1)} + b^{(n,1)}, a^{(n,2)} + b^{(n,2)}))(\in T_F)$ and
$O((a^{(1,1)} + b^{(1,1)}, a^{(1,2)} + b^{(1,2)}), \cdots\cdots, (a^{(n,1)} + b^{(n,1)}, a^{(n,2)} + b^{(n,2)}))$ is the neighborhood of $D(\alpha)(+)E_(\alpha)$. By Decomposition Principle,

$$\widetilde{D}_k \oplus \widetilde{E}_k = \bigcup_{\alpha \in [0,1]} [D_k(\alpha) + E_k(\alpha))_\alpha$$

$$\widetilde{D} \ominus \widetilde{E} = \bigcup_{\alpha \in [0,1]} [D(\alpha) + E(\alpha))_\alpha$$

Hence by Property 3.3, we have $\lim_{m \to \infty} \widetilde{D}_m \oplus \widetilde{E}_m = \widetilde{D} \oplus \widetilde{E}$.

(2°) and (3°) can be proved the same way as 1°).

Property 3.7. $\widetilde{D}_k, \widetilde{E}_k, \widetilde{D}, \widetilde{E} \in F_c$, $k = 1, 2, \cdots$ and $\lim_{m \to \infty} \mu_{\bigcup_{k=1}^m \widetilde{D}_k}(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) = \mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$; $\lim_{m \to \infty} \mu_{\bigcup_{k=1}^m \widetilde{E}_k}(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) = \mu_{\widetilde{E}}(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$; $\forall (x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \in R^n$; $\mu_{\bigcup_{k=1}^m \widetilde{D}_k} \subset \widetilde{D}, \mu_{\bigcup_{k=1}^m \widetilde{E}_k} \subset \widetilde{E}$ $\forall m = 1, 2, \cdots$ then

(1°) $\lim_{m \to \infty} \bigcup_{k=1}^m (\widetilde{D}_k \oplus \widetilde{E}_k) = \widetilde{D} \ominus \widetilde{E} = (\lim_{m \to \infty} \bigcup_{k=1}^m \widetilde{D}_k) \oplus (\lim_{m \to \infty} \bigcup_{k=1}^m \widetilde{E}_k))$

(2°) $\lim_{m \to \infty} \bigcup_{k=1}^m (\widetilde{D}_k \ominus \widetilde{E}_k) = \widetilde{D} \ominus \widetilde{E} = (\lim_{m \to \infty} \bigcup_{k=1}^m \widetilde{D}_k) \ominus (\lim_{m \to \infty} \bigcup_{k=1}^m \widetilde{E}_k))$

(3°) When $q \neq 0$, $\lim_{m \to \infty} \bigcup_{k=1}^m (q_1 \odot \widetilde{D}_k) = q_1 \odot \widetilde{D}$.

Proof.

(1°) Since $\widetilde{D}_1 \subset \widetilde{D}_1 \cup \widetilde{D}_2 \subset \cdots \subset \bigcup_{k=1}^m \widetilde{D}_k \subset \ldots \subseteq \widetilde{D}$ and
$\lim_{m \to \infty} \mu_{\bigcup_{k=1}^m \widetilde{D}_k}(x^{(1)}, x^{(2)}, \cdots x^{(n)}) = \mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \cdots x^{(n)})$ $\forall (x^{(1)}, x^{(2)}, \cdots x^{(n)}) \in R^n$ Hence by Property 3.2, we have $\lim_{m \to \infty} \bigcup_{k=1}^m \widetilde{D}_k = \widetilde{D}$.

Similarly, $\lim_{m\to\infty} \cup_{k=1}^{m} \widetilde{E}_k = \widetilde{E}$

By Property 3.4 $(3^\circ - 2)$,

$\cup_{k=1}^{m} (\widetilde{D} \oplus \widetilde{E}_k) = (\cup_{k=1}^{m} \widetilde{D}_k) \oplus (\cup_{k=1}^{m} \widetilde{E}_k)$.

From Property 3.6 (1°),

$\lim_{m\to\infty} \cup_{k=1}^{m} (\widetilde{D}_k \oplus \widetilde{E}_k) = (\lim_{m\to\infty} \cup_{k=1}^{m} (\widetilde{D}_k) \ominus (\lim_{m\to\infty} \cup_{k=1}^{m} (\widetilde{E}_k)$

(2°), (3°) can be proved as (1°).

Next, we shall discuss the convergency of the fuzzy vectors in SFR.

Property 3.8. For $\widetilde{D}_m, \widetilde{E}_m, \widetilde{D}, \widetilde{E} \in F_c$; $m = 1, 2, \cdots$, $\lim_{m\to\infty} \widetilde{D}_m = \widetilde{D}$, $\lim_{m\to\infty} \widetilde{E}_m = \widetilde{E}$. Then the fuzzy vectors $\overrightarrow{\widetilde{E}_m\widetilde{D}_m}, m = 1, 2, \ldots$, converges to the fuzzy vectors $\overrightarrow{\widetilde{E}\widetilde{D}}$.

Proof. Since $\overrightarrow{\widetilde{E}_m\widetilde{D}_m} = \widetilde{D}_m \ominus \widetilde{E}_m$, $\overrightarrow{\widetilde{E}\widetilde{D}} = \widetilde{D} \ominus \widetilde{E}$. Then by Property 3.6 (2°) . $\lim_{m\to\infty} \overrightarrow{\widetilde{E}_m\widetilde{D}_m} = \widetilde{D} \ominus \widetilde{E}\ \overrightarrow{\widetilde{E}\widetilde{D}}$.

Property 3.9. $\widetilde{D}_k\widetilde{E}_k, \widetilde{D}, \widetilde{E} \in F_c$; $k = 1, 2, \cdots$; Let $\widetilde{Q}_m = \cup_{k=1}^{m} \widetilde{D}_k$, $\widetilde{S}_m = \cup_{k=1}^{m} \widetilde{E}_k$

$$\lim_{m\to\infty} \mu_{\widetilde{Q}_m}(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) = \mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$$

$$\lim_{m\to\infty} \mu_{\widetilde{S}_m}(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) = \mu_{\widetilde{E}}(x^{(1)}, x^{(2)}, \cdots, x^{(n)})\, \forall (x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \in R^n$$

$\widetilde{Q}_m \subset \widetilde{D}$, $\widetilde{S}_m \subset \widetilde{E}$. The sequence of fuzzy vectors $\overrightarrow{\widetilde{S}_m\widetilde{Q}_m}$, $m = 1, 2, \cdots$ converges to the fuzzy vextor $\overrightarrow{\widetilde{E}\widetilde{D}}$.

Proof. Similar as Property 3.7, we have

$\lim_{m\to\infty} \cup_{k=1}^{m} \widetilde{D}_k = \widetilde{D}$, and

$\lim_{m\to\infty} \cup_{k=1}^{m} \widetilde{E}_k = \widetilde{E}$ By Property 3.6 (2°)

$\lim_{m\to\infty} \overrightarrow{\widetilde{S}_m\widetilde{Q}_m} = (\lim_{m\to\infty} \cup_{k=1}^{m} \widetilde{D}_k) \ominus (\lim_{m\to\infty} \cup_{k=1}^{m} \widetilde{E}_k) = \widetilde{D} \ominus \widetilde{E} = \overrightarrow{\widetilde{E}\widetilde{D}}$

Property 3.10. $\widetilde{D}_{m,k}, \widetilde{E}_{m,k}, \widetilde{D}_k, \widetilde{E}_k \in F_c$; $m = 1, 2, \cdots,$; $k = 1, 2, \cdots r$ and for each $k \in \{1, 2, \cdots, r\}$. $\lim_{m\to\infty} \widetilde{D}_{k,m} = \widetilde{D}_k$, $\lim_{m\to\infty} \widetilde{D}_{k,m} = \widetilde{D}_k$, $q^{(k)} \neq 0, k = 1, 2, \cdots, r$. The sequence of the fuzzy vectors $\sum_{k=1}^{r} \ominus(q_1^{(k)}\odot \overrightarrow{\widetilde{E}_{m,k}\widetilde{D}_{m,k}})$, $m = 1, 2, \cdots$, converges to the fuzzy vector $\sum_{k=1}^{r} \oplus(q_1^{(k)}\odot \overrightarrow{\widetilde{E}_k\widetilde{D}_k})$

Proof. Since $\sum_{k=1}^{r} \ominus(q_1^{(k)}\odot \widetilde{E}_{m,k}\widetilde{D}_{m,k}) = \sum_{k=1}^{r} \oplus(q_1^{(k)}\odot (\widetilde{D}_{m,k}\ominus\widetilde{E}_{m,k}))$; $m = 1, 2, \cdots$.
For each k, by Property 3.6 (2°), $\lim_{m\to\infty} \widetilde{D}_{m,k}\ominus\widetilde{E}_{m,k} = \widetilde{D}_k\ominus\widetilde{E}_k$. By Property 3.6 (1°), (3°), we have

$$\lim_{m\to\infty} \sum_{k=1}^{r} \ominus(q_1^{(k)}\odot (\widetilde{D}_{m,k}\ominus\widetilde{E}_{m,k})) = \sum_{k=1}^{r} \oplus(q_1^{(k)}\odot (\widetilde{D}_k\ominus\widetilde{E}_k))$$
$$= \sum_{k=1}^{r} \ominus(q_1^{(k)}\odot \overrightarrow{\widetilde{E}_k\widetilde{D}_k})$$

Example 3.1. Consider the fuzzy vector $\lim_{m\to\infty} \overrightarrow{\widetilde{Q}\widetilde{Z}_m}$ in Example 2.2.

Let $\mu_{\widetilde{Z}}(x,y) = \begin{cases} 1-(x-10)^2-(y-30)^2, & \text{if } (x-10)^2+(y-30)^2 \le 1 \\ 0, & \text{elsewhwew} \end{cases}$

We shall prove $\lim_{m\to\infty} \widetilde{Z}_m = \widetilde{Z}$.
Since $C((10,30), 1+\frac{1}{m}) \subset C((10,30), 1+\frac{1}{m-1})$ and for any $(x,y)\in R^2$, the following holds.

$$\frac{1}{(1+\frac{1}{m})^2}[(1+\frac{1}{m})^2-(x-10)^2-(y-30)^2] \le \frac{1}{(1+\frac{1}{m-1})^2}[(1+\frac{1}{m-1})^2-(x-10)^2-(y-30)^2]$$

Therefore $\mu_{\widetilde{Z}_m}(x,y) \le \mu_{\widetilde{Z}_{m-1}}(x,y)\ \forall (x,y)\in R^2$ and hence $\widetilde{Z}_1 \supset \widetilde{Z}_2 \supset \cdots \widetilde{Z}_m \supset \cdots \supset \widetilde{Z}$. And obviously, $\lim_{m\to\infty} \mu_{\widetilde{Z}_m}(x,y) = \mu_{\widetilde{Z}(x,y)}\ \forall (x,y)\in R^2$. Let $\widetilde{Z}'_m, \widetilde{Z}'$ be the complement fuzzy sets of $\widetilde{Z}_m, \widetilde{Z}$ respectively. We have $\lim_{m\to\infty} \mu_{\widetilde{Z}'_m}(x,y) = \mu_{\widetilde{Z}'}(x,y)\ \forall (x,y) \in R^2$ and $\widetilde{Z}'_1 \subset \widetilde{Z}'_2 \subset \cdots \subset \widetilde{Z}'_m \subset \cdots \subset \widetilde{Z}'$. By Property 3.2, $\lim_{m\to\infty} \widetilde{Z} + m' = \widetilde{Z}'$. Thus $\lim_{m\to\infty} \widetilde{Z}_m = \widetilde{Z}$. Therefore from Propertu=y 3.8, $\lim_{m\to\infty} \overrightarrow{|wtQ\widetilde{Z}_m} = \overrightarrow{\widetilde{Q}\widetilde{Z}}$. The membership function of $\overrightarrow{\widetilde{Q}\widetilde{Z}}$ is

$$\mu_{\overrightarrow{\widetilde{Q}\widetilde{Z}}}(x,y) = \mu_{\widetilde{Z}\ominus\widetilde{Q}}(x,y) = \sup_{x=x^{(1)}-y^{(1)},\, y=x^{(2)}-y^{(2)}} \mu_{\widetilde{Z}}(x^{(1)},x^{(2)}) \wedge \mu_{\widetilde{Q}}(y^{(1)},y^{(2)})$$
$$= \mu_{\widetilde{Z}}(x+1, y+2) = \begin{cases} 1-(x-9)^2-(y-28)^2, & \text{if } (x-9)^2+(y-28)^2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

In the crisp case, starting from $Q=(1,2)$, aiming at $Z-(10,30)$, we could have the vector $\overrightarrow{QP}=(9,28)$. The grade of membership of $\overrightarrow{QP}$ belongs to the fuzzy vector $\overrightarrow{\widetilde{Q}\widetilde{Z}}$ is $\mu_{\overrightarrow{\widetilde{Q}\widetilde{Z}}}(9,28)=1$. And the point $R=(9.5,29.5)$ is in the circle with center $(9,28)$ and radius 1. The crisp vector of Q to R, $\overrightarrow{QR}=(8.5,27.5)$. The grade of membership function of $\overrightarrow{\widetilde{Q}\widetilde{Z}}$ is $\mu_{\overrightarrow{\widetilde{Q}\widetilde{Z}}}(8.5,27.5)=0.5$.

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