Generalized Eigenvectors and Fractionalization of Offset DFTs and DCTs

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***Abstract*—The offset discrete Fourier transform (DFT) is a discrete transform with kernel $\exp[-j2\pi(m-a)(n-b)/N]$. It is more generalized and flexible than the original DFT and has very close relations with the discrete cosine transform (DCT) of type 4 (DCT-IV), DCT-VIII, discrete sine transform (DST)-IV, DST-VIII, and discrete Hartley transform (DHT)-IV. In this paper, we derive the eigenvectors/eigenvalues of the offset DFT, especially for the case where $a+b$ is an integer. By convolution theorem, we can derive the close form eigenvector sets of the offset DFT when $a+b$ is an integer. We also show the general form of the eigenvectors in this case. Then, we use the eigenvectors/eigenvalues of the offset DFT to derive the eigenvectors/eigenvalues of the DCT-IV, DCT-VIII, DST-IV, DST-VIII, and DHT-IV. After the eigenvectors/eigenvalues are derived, we can use the eigenvectors-decomposition method to derive the fractional operations of the offset DFT, DCT-IV, DCT-VIII, DST-IV, DST-VIII, and DHT-IV. These fractional operations are more flexible than the original ones and can be used for filter design, data compression, encryption, and watermarking, etc.**

***Index Terms*—Discrete fractional Fourier transform, eigenvectors, offset discrete Fourier transform, offset discrete cosine transform.**

I. Introduction

THE discrete Fourier transform (DFT) is a very useful tool in digital signal processing. The *offset DFT*, however, is a generalization of the DFT. It is defined as

$$\begin{aligned} X_{a,b}[m] &= \mathrm{DFT}_{a,b}\left(x[n]\right) \\ &= \sqrt{N^{-1}} \cdot \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(m-a)(n-b)} x[n]. \end{aligned} \tag{1}$$

Its inverse operation, i.e., the *offset inverse discrete Fourier transform (IDFT)*, is defined as

$$\begin{aligned} x[n] &= \mathrm{IDFT}_{a,b}\left(X_{a,b}[m]\right) \\ &= \sqrt{N^{-1}} \cdot \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}(n-b)(m-a)} X_{a,b}[m]. \end{aligned} \tag{2}$$

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The offset DFT has two parameters a, b, which correspond to the offsets in the frequency domain and time domain, respectively. It was first introduced in [1]. When $a=b=1/2$, it is also called the odd-time odd-frequency DFT [2]. The offset DFT is more flexible than the original DFT and can solve some problems that cannot be solved well by the original DFT. Its applications to filter design, signal representation, and fast computation of DFT have been developed [3], [4]. Many applications of the original DFT are also the potential applications of the offset DFT. Besides, the offset DFT has very close relations with the discrete cosine/sine transforms (DCT/DST) of types 2$\sim$4 and 6$\sim$8, and the discrete Hartley transform (DHT) of types 2$\sim$4 (see Section V).

In this paper, we derive the eigenvectors/eigenvalues of the offset DFT, especially for the case where $a+b$ is an integer. In [5], Tseng has derived an eigenvectors set of the offset DFT when $a=b=1/2$ by the commutative matrix method. In Section II, we make a more complete discussion on the eigenvectors of the offset DFT. In the case where $a+b$ is an integer, we can derive the closed form of the eigenvectors/eigenvalues of offset DFTs successfully from the eigenfunctions/eigenvalues of continuous offset FTs [6]. We also show the general form of the eigenvectors of the offset DFTs when $a+b$ is an integer.

When $a+b$ is not an integer, although the close form eigenvectors/eigenvalues of the offset DFT are very hard to derive, however, they have some regularities. In Section III, we describe these regularities.

In Section IV, we describe some interesting properties of the eigenvectors of the offset DFT.

After the eigenvectors/eigenvalues of the offset DFT are derived, in Section V, we use the relations between the offset DFT and the DCT-IV, DCT-VIII, DST-IV, DST-VIII, and DHT-IV to derive the eigenvectors/eigenvalues of these transforms. We can use the eigenvectors/eigenvalues obtained in this paper to derive several fractional operations. The discrete fractional Fourier transform [8], [9], [21], the fractional DCT-I and DST-I [10], and the fractional DCT-II [11] have been derived already. In Section VI of this paper, we use the eigenvectors-decomposition method to derive the fractional offset DFT (FRODFT), the fractional DCT-IV, the fractional DCT-VIII, the fractional DST-IV, the fractional DST-VIII, and the fractional DHT-IV. These fractional operations are more flexible than the original ones. They can replace the original operations in some signal processing applications (such as filter design and data compression). They are also useful for encryption, watermarking, and computing the continuous offset Fourier transform.

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II. Eigenvectors/Eigenvalues of Offset DFTs when $a+b=$ Integer

A. Reviews of the Eigenfunctions/Eigenvalues of Continuous Offset FTs

Before discussing the eigenvectors/eigenvalues of the offset DFT, we first review the eigenfunctions/eigenvalues of its continuous counterpart, i.e., the continuous offset Fourier transform (offset FT). It is defined as

$$G_F^{\tau,\eta}(\omega) = FT_{\tau,\eta}\left[g(t)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-j(\omega-\tau)(t-\eta)\right]g(t)dt. \quad (3)$$

The continuous offset FT is a generalization of the Fourier transform (FT) and is a special case of the special affine Fourier transform [22], [23]. In [6], we derived the following lemma.

- If $E(t)$ is an eigenfunction of the original FT, and λ is its corresponding eigenvalue

$$FT\left[E(t)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-j\omega t)E(t)dt = \lambda E(\omega) \quad (4)$$

then the offset FT with parameters τ, η has the eigenfunction as in (4), and the corresponding eigenvalue is $\lambda\exp[j(\tau-\eta)^2/4]$:

$$E_{\tau,\eta}(t) = \exp\left[\frac{j}{2}(\eta-\tau)t\right]E\left(t-\frac{\tau+\eta}{2}\right) \quad (5)$$

$$F_{\tau,\eta}\left[E_{\tau,\eta}(t)\right] = \lambda\exp\left[\frac{j}{4}(\tau-\eta)^2\right]E_{\tau,\eta}(t). \quad (6)$$

Thus, we can obtain the eigenfunctions/eigenvalues of the continuous offset FT from those of the original FT. For example, since the functions as follows are the eigenfunctions of the original FT, and their corresponding eigenvalues are $(-j)^m$

$$\phi_m(t) = \exp\left(-\frac{t^2}{2}\right)H_m(t)$$
$$FT\left[\phi_m(t)\right] = (-j)^m\phi_m(\omega) \quad (7)$$
$$H_m(t): \text{the Hermite polynomial of order } m$$
$$m\in Z^+$$

we can conclude that the continuous offset FT has the eigenfunctions as in (8), and the corresponding eigenvalues are $(-j)^q\exp[j(\tau-\eta)^2/4]$:

$$\Phi_{q,\tau,\eta}(t) = \exp\left[j\frac{\eta-\tau}{2}t - \frac{(2t-\eta-\tau)^2}{8}\right] \times H_q\left(t-\frac{\eta+\tau}{2}\right), \quad q\in Z^+ \quad (8)$$

$$FT_{\tau,\eta}\left[\Phi_{q,\tau,\eta}(t)\right] = (-j)^q e^{j\frac{(\tau-\eta)^2}{4}}\Phi_{q,\tau,\eta}(t). \quad (9)$$

Besides Hermite functions, some other functions (such as follows) are also the eigenfunctions of the original FT [12]:

$$\text{a) } \sum_{p=-\infty}^{\infty}\delta(t-p\sqrt{2\pi})$$
$$\text{b) } \left|\frac{t}{\sqrt{2\pi}}\right|^{-\frac{1}{2}}$$
$$\text{c) } \sec h\left(t\sqrt{\frac{\pi}{2}}\right) \quad (10)$$

and we can substitute them into $E(t)$ in (5) to obtain other eigenfunctions of the continuous offset FT. There are also many other possible choices for $E(t)$ in (5). Thus, as the original FT, the continuous offset FT also has varieties of eigenfunctions. Nevertheless, since the original FT only has four eigenvalues $1, -j, -1, j$, the offset FT has only four possible eigenvalues as well.

$$\lambda = (-j)^q\exp\left[\frac{j(\tau-\eta)^2}{4}\right], \quad \text{where } q=0,1,2,3. \quad (11)$$

B. Close Form Hermite-Like Eigenvectors of the Offset DFT

In [13], Mehta used the continuous eigenfunctions of the original FT to derive the discrete eigenvectors of the DFT. In this subsection, we use the continuous eigenfunctions of the offset FT [6] to derive the discrete eigenvectors of the offset DFT when $a+b$ is an integer.

As the case of the original FT, for the offset FT, the sampling operation in the time domain also corresponds to the summation of replication in the frequency domain:

$$FT_{\tau,\eta}\left[g(t)\sum_{k=-\infty}^{\infty}\delta\left(\frac{t}{\Delta_t}-k\right)\right] = \sum_{q=-\infty}^{\infty}e^{j\eta\cdot\frac{2\pi}{\Delta_t}q}G_F^{\tau,\eta}\left(\omega-\frac{2\pi}{\Delta_t}q\right) \quad (12)$$

where ${G_F}^{\tau,\eta}(\omega)$ means the continuous offset FT [see (3)] of $g(t)$. If we choose $g(t)=E_{\tau,\eta}(t)$, where $E_{\tau,\eta}(t)$ is an eigenfunction of the continuous offset FT with parameters τ, η

$$FT_{\tau,\eta}\left(E_{\tau,\eta}(t)\right) = \lambda\cdot E_{\tau,\eta}(t) \quad (13)$$

then ${G_F}^{\tau,\eta}(\omega) = \lambda\cdot E_{\tau,\eta}(t)$, and (12) becomes

$$\frac{\Delta_t}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}e^{-j(\omega-\tau)(k\Delta_t-\eta)}E_{\tau,\eta}(k\Delta_t) = \lambda\sum_{q=-\infty}^{\infty}e^{j\eta\cdot\frac{2\pi}{\Delta_t}q}E_{\tau,\eta}\left(\omega-\frac{2\pi}{\Delta_t}q\right). \quad (14)$$

After setting $\Delta_t=\sqrt{2\pi/N}$ and $\omega=m\sqrt{2\pi/N}$, we obtain

$$\frac{1}{\sqrt{N}}\sum_{k=-\infty}^{\infty}e^{-j\frac{2\pi}{N}\left(m-\tau\sqrt{\frac{N}{2\pi}}\right)\left(k-\eta\sqrt{\frac{N}{2\pi}}\right)}E_{\tau,\eta}\left(k\sqrt{\frac{N}{2\pi}}\right) = \lambda\sum_{q=-\infty}^{\infty}e^{j\sqrt{2\pi N}\cdot\eta\cdot q}E_{\tau,\eta}\left((m-qN)\sqrt{\frac{2\pi}{N}}\right). \quad (15)$$

Then, we replace k as $n+pN$ and replace q as $-p$ and obtain

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(m-a)(n-b)} \times \sum_{p=-\infty}^{\infty} e^{j2\pi a\,p} E_{\tau,\eta}\left((n+pN)\sqrt{\frac{2\pi}{N}}\right) = \lambda \sum_{p=-\infty}^{\infty} e^{-j2\pi b\,p} E_{\tau,\eta}\left((m+pN)\sqrt{\frac{2\pi}{N}}\right)$$

$$\text{where } a=\tau\sqrt{\frac{N}{2\pi}},\quad b=\eta\sqrt{\frac{N}{2\pi}}. \tag{16}$$

That is

$$\mathrm{DFT}_{a,b}\left[\sum_{p=-\infty}^{\infty} e^{j2\pi a\,p} E_{\tau,\eta}\left((n+pN)\sqrt{\frac{2\pi}{N}}\right)\right] = \lambda \sum_{p=-\infty}^{\infty} e^{-j2\pi bp} E_{\tau,\eta}\left((m+pN)\sqrt{\frac{2\pi}{N}}\right). \tag{17}$$

Besides, if $a+b = A$, where A is any integer, then $\exp(j2\pi ap) = \exp(-j2\pi bp)$. Therefore, we obtain the following conclusion:

• *The offset DFT with parameters* $\boldsymbol{a}$, $\boldsymbol{b}$, *where* $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{A}$ ($\boldsymbol{A}$ *is an integer*) has the *eigenvectors* as

$$G_{a,b}[n] = \sum_{p=-\infty}^{\infty} e^{j2\pi a\,p} \cdot E_{a\sqrt{\frac{2\pi}{N}},b\sqrt{\frac{2\pi}{N}}} \times \left((n+pN)\sqrt{\frac{2\pi}{N}}\right),\quad n\in[0,N-1] \tag{18}$$

where $E_{a\sqrt{2\pi/N},b\sqrt{2\pi/N}}(t)$ is the eigenfunction of the continuous offset FT (see (3)) with parameters $a\sqrt{2\pi/N}$, $b\sqrt{2\pi/N}$ (suppose that its corresponding eigenvalue is λ):

$$FT_{a\sqrt{\frac{2\pi}{N}},b\sqrt{\frac{2\pi}{N}}}\left(E_{a\sqrt{\frac{2\pi}{N}},b\sqrt{\frac{2\pi}{N}}}(t)\right) = \lambda\cdot E_{a\sqrt{\frac{2\pi}{N}},b\sqrt{\frac{2\pi}{N}}}(t).$$

The corresponding *eigenvalue* for $G_{a,b}[n]$ is also λ (the same as the above):

$$\mathrm{DFT}_{a,b}\left(G_{a,b}[n]\right) = \lambda\cdot G_{a,b}[n]. \tag{19}$$

Thus, we can obtain the eigenvectors/eigenvalues of the offset DFT easily from the eigenfunctions/eigenvalues of the continuous offset FT.

Since the continuous offset FT has infinite possible eigenfunctions, there are also infinite possible choices of $E_{a\sqrt{2\pi/N},b\sqrt{2\pi/N}}(t)$ in (18). As usual, the infinite summation in (18) converges. However, sometimes, it may not converge. Approximately, the condition that (18) converges is that

$$\int_{-\infty}^{\infty}\left|E_{a\sqrt{\frac{2\pi}{N}},b\sqrt{\frac{2\pi}{N}}}(t)\right| < \infty. \tag{20}$$

For example, if we choose $E_{a\sqrt{2\pi/N},b\sqrt{2\pi/N}}(t)$ as the Hermite–Gaussian function in (8), or the shifting-modulation version of the function in (10c), the series in (18) converges. However, if we choose it as the shifting-modulation version of the functions in (10a) and (10b), the series does not converge.

If we choose $E_{\tau,\eta}(t)$ as the Hermite–Gaussian function [see (8) and (9)], after substituting (8) into (18) ($\tau = a\sqrt{2\pi/N}$, $\eta = b\sqrt{2\pi/N}$), we obtain

$$V_{q,a,b}[n] = e^{j\pi\frac{b-a}{N}n}\sum_{p=-\infty}^{\infty}(-1)^{(a+b)p} e^{-\frac{\pi\left(n+pN-\frac{(a+b)}{2}\right)^2}{N}} \times H_q\left(\left(n+pN-\frac{a+b}{2}\right)\sqrt{\frac{2\pi}{N}}\right) \tag{21}$$

where $n\in[0,N-1]$. We use $V_{q,a,b}[n]$ to denote the eigenvectors obtained this way. Besides, from (9), we find that its corresponding eigenvalue is

$$\lambda_q = (-j)^q\cdot\exp\left[\frac{j\pi(a-b)^2}{2N}\right]. \tag{22}$$

Thus, when $a+b$ is an integer, we can obtain the close form eigenvectors of the offset DFT from the summation of the sampling values of Hermite functions with some extra phase terms. In (21), it seems that we should calculate the infinite summation. Nevertheless, from experiments

$$\exp\left(-\frac{x^2}{2}\right)H_q(x) < 10^{-5} \quad \text{when } |x| > \text{threshold} \approx 4.73 + 0.47\cdot q^{0.73}. \tag{23}$$

The error of the above approximation is very small when $q < 25$. Therefore, in (21), we only have to calculate the summation of

$$V_{q,a,b}[n] = e^{j\pi\frac{b-a}{N}n}\sum_{p=P_1}^{P_2}(-1)^{(a+b)p} e^{-\frac{\pi\left(n+pN-\frac{(a+b)}{2}\right)^2}{N}} \times H_q\left(\left(n+pN-\frac{a+b}{2}\right)\sqrt{\frac{2\pi}{N}}\right) \tag{24}$$

where

$$P_1 = k\left[-(4.73+0.47q^{0.73})\sqrt{\frac{1}{2\pi N}} - \frac{n'}{N}\right]_G$$
$$P_2 = \left[(4.73+0.47q^{0.73})\sqrt{\frac{1}{2\pi N}} - \frac{n'}{N}\right]_G$$
$$n' = n - \frac{(a+b)}{2} \tag{25}$$

and $[x]_G$ means the largest integer no more than x. Thus, there are at most

$$\left[2(4.73+0.47\cdot q^{0.73})\sqrt{\frac{1}{2\pi N}}\right]_G + 1 \text{ terms} \tag{26}$$

that should be considered. For example, if $q=5$ and $N=8$, then $2(4.73+0.47\cdot q^{0.73}\ \sqrt{1/2\pi N}) = 1.766$. In this case, only two terms ($p=-1$, 0) should be considered. When

$$(4.73+0.47\cdot q^{0.73})\sqrt{\frac{1}{2\pi N}} < 0.5 \tag{27}$$

TABLE I
DISTRIBUTION OF THE EIGENVALUES OF N-POINT OFFSET DFTs WHEN $a+b$ IS AN INTEGER. WHEN $a+b$ IS EVEN, $e=1$ AND $o=0$. WHEN $a+b$ IS ODD, $e=0$ AND $o=1$

Number of points	Multiplicities of the eigenvalues ($\phi=\pi(a-b)^2/2N$)			
	$\exp(j\phi)$	$-j\exp(j\phi)$	$-\exp(j\phi)$	$j\exp(j\phi)$
$N=4m$	$m+e$	m	m	$m-1+o$
$N=4m+1$	$m+e$	$m+o$	m	m
$N=4m+2$	$m+1$	$m+o$	$m+e$	m
$N=4m+3$	$m+1$	$m+1$	$m+e$	$m+o$

only one term has to be considered. In this case, $V_{q,a,b}[n]$ can be expressed as

$$V_{q,a,b}[n] \approx e^{j\pi\frac{b-a}{N}n}(-1)^{p_n(a+b)}e^{-\frac{\pi\left(n+p_nN-\frac{(a+b)}{2}\right)^2}{N}} \times H_q\left(\left(n+p_nN-\frac{a+b}{2}\right)\sqrt{\frac{2\pi}{N}}\right) \quad (28)$$

where

$$\begin{aligned} p_n &= 0 \text{ when } 0 \le \left(\left(n-\frac{(a+b)}{2}\right)\right)_N < \frac{N}{2} \\ &\quad ((\;)) \text{ means modulus operation} \\ p_n &= -1 \text{ when } \frac{N}{2} < \left(\left(n-\frac{(a+b)}{2}\right)\right)_N < N. \end{aligned} \quad (29)$$

That is, if (27) is satisfied, then $V_{q,a,b}[n]$ is quite similar to the continuous Hermite function, except for some modulation, and the negative part of the continuous Hermite function is mapped to the locations of $N/2 < ((n-(a+b)/2))_N < N$. The constraint in (27) is satisfied if

$$\text{① } N \text{ is larger} \quad \text{② } q \text{ is small.} \quad (30)$$

Thus, we can conclude that if the number of points increase, $V_{q,a,b}[n]$ converges to the continuous Hermite function. The rate of convergence is faster when the order q is smaller.

In (21), q can be any non-negative integer. Nevertheless, for the N-point offset DFT, there are at most N independent eigenvectors. We then try to select N of the eigenvectors to construct a complete and independent eigenvectors set. We can first observe the eigenvectors of the offset DFT. From the "eig" command of MATLAB, we obtain the results in Table I:

From (22), we find that $\lambda_{q+4L} = (-j)^q \cdot \exp(j\phi)$ for any integer L. Therefore, the multiplicities of $(-j)^q \cdot \exp(j\phi)$ $(q = 0,1,2,3)$ are just the rank of $\text{Span}\{V_{q,a,b}[n],\ V_{q+4,a,b}[n],\ V_{q+8,a,b}[n], V_{q+12,a,b}[n],\ldots\ldots\}$. We also find the following theorem.

Theorem 1: If the multiplicities of $(-j)^q\cdot\exp(j\phi)$ are k, then $\{V_{q+4h,a,b}[n] | h=0,1,2,\ldots\ldots,k-1\}$ form a complete independent eigenvectors set of the eigenspace of $(-j)^q\cdot\exp(j\phi)$.

The proof of Theorem 1 is in the Appendix. Besides, the eigenvectors belonging to different eigenspaces must be independent. We can use the above results to choose the complete and independent eigenvectors set among (21).

For example, when $N=7$, $a=0.3$, and $b=0.7$, from Table I, we find that the multiplicities of $\exp(j\phi)$, $-j\exp(j\phi)$, $-\exp(j\phi)$, and $j\exp(j\phi)$ are 2, 2, 1, and 2, respectively. Thus, the complete independent eigenvectors set for each eigenspace are

$$\begin{aligned} \exp(j\phi) &: V_{0,a,b}[n], \quad V_{4,a,b}[n] \\ -j\exp(j\phi) &: V_{1,a,b}[n], \quad V_{5,a,b}[n] \\ -\exp(j\phi) &: V_{2,a,b}[n] \\ j\exp(j\phi) &: V_{3,a,b}[n], \quad V_{7,a,b}[n]. \end{aligned} \quad (31)$$

Thus, the complete independent eigenvectors in this case are $\{V_{q,a,b}[n] | q=0,1,2,3,4,5,7\}$. We can use the similar way to obtain the complete independent eigenvectors set among (21) in other cases. The general results are as follows:

•*The complete, independent, and close form Hermite-like eigenvector set of the **N**-point offset DFT when **a** + **b** is an integer* [the corresponding eigenvalues are as (22)]

$$V_{q,a,b}[n] = e^{j\pi\frac{b-a}{N}n}\sum_{p=-\infty}^{\infty}(-1)^{(a+b)p}e^{-\frac{\pi\left(n+pN-\frac{(a+b)}{2}\right)^2}{N}} \times H_q\left(\left(n+pN-\frac{a+b}{2}\right)\sqrt{\frac{2\pi}{N}}\right) \quad (32)$$

where

$$\begin{aligned} n &\in [0, N-1] \\ q &= 0,1,2,\ldots\ldots,N-2,N_1 \\ N_1 &= N \quad \text{when } N+a+b \text{ is even} \\ N_1 &= N-1 \text{ when } N+a+b \text{ is odd.} \end{aligned} \quad (33)$$

C. Eigenvectors of the Offset DFT Obtained by Commutative Matrix and Linear Combination Methods

Except for the sampling method introduced in Section II-B, we can also use other methods to obtain the eigenvectors of the offset DFT when $a+b$ is an integer.

Commutative Matrix Method: In [14] and [21], the authors developed a novel commutative matrix method to obtain the eigenvectors of the DFT. The commutative matrix has a close relation with the continuous differential equation [21]. Recently, in [5], Tseng has used the similar commutative matrix method to derive the eigenvectors of the offset DFT when $a = b = 1/2$. In fact, as long as $a+b$ is an integer, we can use the commutative matrix method to derive the eigenvectors of the offset DFT.

If $\mathbf{S_{a,b}}$ is the matrix commutative with the offset DFT, it satisfies

$$\mathbf{S_{a,b}F_{a,b}} = \mathbf{F_{a,b}S_{a,b}} \tag{34}$$

where $\mathbf{F_{a,b}}$ is the offset DFT transform matrix:

$$\mathbf{F_{a,b}}[m,n] = \sqrt{N^{-1}}e^{-j\frac{2\pi}{N}(m-a)(n-b)}, \quad m,n \in [0, N-1]. \tag{35}$$

There are many possible choices for $\mathbf{S_{a,b}}$ when $a+b$ is an integer. Here, we choose $\mathbf{S_{a,b}}$ as

$$\mathbf{S_{a,b}} = \begin{bmatrix} D_0 & C_2 & 0 & 0 & \cdots & 0 & C_3 \\ C_1 & D_1 & C_2 & 0 & \cdots & 0 & 0 \\ 0 & C_1 & D_2 & C_2 & \ddots & \vdots & \vdots \\ 0 & 0 & C_1 & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & C_2 & 0 \\ 0 & \vdots & \vdots & \cdots & C_1 & D_{N-2} & C_2 \\ C_4 & 0 & 0 & \cdots & 0 & C_1 & D_{N-1} \end{bmatrix} \tag{36}$$

where

$$\begin{aligned} D_m &= 2\cos\left[\frac{2\pi}{N}\left(m - \frac{a+b}{2}\right)\right] \\ C_1 &= e^{j\frac{\pi}{N}(b-a)} \\ C_2 &= e^{j\frac{\pi}{N}(a-b)} \\ C_3 &= e^{j\frac{\pi}{N}(b-a)}e^{-j2\pi b} \\ C_4 &= e^{j\frac{\pi}{N}(a-b)}e^{-j2\pi a}. \end{aligned} \tag{37}$$

We can show that if $a+b$ is an integer, then (34) is satisfied. There are some facts to be noticed.

1) $\mathbf{S_{a,b}}$ is a Hermitian (conjugated symmetric) matrix

$$\mathbf{S_{a,b}} = \overline{\mathbf{S_{a,b}}}^{\mathbf{T}}. \tag{38}$$

Besides, except for some exceptional cases, all the eigenvalues of $\mathbf{S_{a,b}}$ are different. The two exceptional cases are ① $a+b$ is even, and $N = 4m$ (m is an integer), ② $a+b$ is odd, and $N = 4m+2$. In these cases, the multiplicities of $\lambda = 0$ are two.

2) From the theorems of linear algebra, if $\mathbf{e_1}$, $\mathbf{e_2}$ are two eigenvectors of the Hermitian matrix $\mathbf{S_{a,b}}$ with different eigenvalues λ_1, λ_2, then $\mathbf{e_1}$ and $\mathbf{e_2}$ are orthogonal:

$$\langle \mathbf{e_1}, \mathbf{e_2} \rangle = \sum_{n=0}^{N-1} \mathbf{e_1}[n]\overline{\mathbf{e_2}[n]} = 0$$
$$\text{if } \mathbf{S_{a,b}e_1} = \lambda_1\mathbf{e_1}, \quad \mathbf{S_{a,b}e_2} = \lambda_2\mathbf{e_2}, \quad \lambda_1 \neq \lambda_2. \tag{39}$$

Since all the eigenvalues of $\mathbf{S_{a,b}}$ are different, the eigenvectors of $\mathbf{S_{a,b}}$ form an orthogonal eigenvectors set.

3) If $\mathbf{e}$ is an eigenvector of $\mathbf{S_{a,b}}$, and the corresponding eigenvalue is λ, then since

$$\mathbf{S_{a,b}F_{a,b}e} = \mathbf{F_{a,b}S_{a,b}e} = \lambda\mathbf{F_{a,b}e} \tag{40}$$

$\mathbf{F_{a,b}} \cdot \mathbf{e}$ is also the eigenvector of $\mathbf{S_{a,b}}$, and its eigenvalue is also λ. Since all the eigenvalues of $\mathbf{S_{a,b}}$ are different, if $\mathbf{F_{a,b}} \cdot \mathbf{e}$ and $\mathbf{e}$ are the eigenvectors of $\mathbf{S_{a,b}}$ belonging to the same eigenvalue λ, then $\mathbf{F_{a,b}} \cdot \mathbf{e}$ must be the constant multiplication of $\mathbf{e}$:

$$\mathbf{F_{a,b}e} = \rho\mathbf{e}, \quad \text{where } \rho \text{ is some constant.} \tag{41}$$

That is, if $\mathbf{e}$ is the eigenvector of $\mathbf{S_{a,b}}$, then it is also the eigenvector of $\mathbf{F_{a,b}}$.

4) Since $\mathbf{S_{a,b}}$ and $\mathbf{F_{a,b}}$ have the same eigenvectors, and the eigenvectors of $\mathbf{S_{a,b}}$ are orthogonal, we can use the commutative matrix $\mathbf{S_{a,b}}$ to find the orthogonal eigenvector set of the offset DFT. Although the eigenvector set obtained from the commutative matrix method has no closed form, it is an orthogonal eigenvectors set.

5) When $N = 4m$ and $a+b$ is even, or when $N = 4m+2$ and $a+b$ is odd, since the multiplicities of $\lambda = 0$ are two, the above conclusions should be modified a little. In this case, the two eigenvectors of $\mathbf{S_{a,b}}$ belonging to zero eigenvalue may not be the eigenvectors of the offset DFT. However, we can still derive the eigenvectors of the offset DFT from them. Suppose that

$$\begin{aligned} &\mathbf{S_{a,b}} \cdot \mathbf{e}_1 = \mathbf{S_{a,b}} \cdot \mathbf{e}_2 = 0 \\ &\qquad N = 4m \text{ and } a+b \text{ is even} \\ &\quad \text{or } N = 4m+2 \text{ and } a+b \text{ is odd.} \end{aligned} \tag{42}$$

We first define the $2N$-length vector $\widetilde{\mathbf{e}}_\mathbf{1}[n]$ as

$$\begin{aligned} &\widetilde{\mathbf{e}}_\mathbf{1}[n] = \mathbf{e_1}[n], \quad \text{when } n = 0 \sim N-1 \\ &\widetilde{\mathbf{e}}_\mathbf{1}[n] = \mathbf{e_1}[n-N], \quad \text{when } n = N \sim 2N-1 \\ &\qquad N = 4m \text{ and } a+b \text{ is even} \\ &\widetilde{\mathbf{e}}_\mathbf{1}[n] = -\mathbf{e_1}[n-N], \quad \text{when } n = N \sim 2N-1 \\ &\qquad N = 4m+2 \text{ and } a+b \text{ is odd.} \end{aligned} \tag{43}$$

Then, we define the operations $O_E(\ )$ and $O_O(\ )$ as in (44), shown at the bottom of the next page, where $((\ ))$ means modulus operation. In fact, $O_E(\ )$ and $O_O(\ )$ are similar to the operations of taking even part and taking odd parts, respectively. Then, we can show that $O_E(\mathbf{e_1}[n])$ and $O_O(\mathbf{e_1}[n])$ are eigenvectors of the offset DFT:

$$\begin{aligned} &\mathrm{DFT}_{a,b}\left[O_E\left(\mathbf{e_1}[n]\right)\right] = \lambda_E O_E\left(\mathbf{e_1}[n]\right) \\ &\mathrm{DFT}_{a,b}\left[O_O\left(\mathbf{e_1}[n]\right)\right] = \lambda_O O_O\left(\mathbf{e_1}[n]\right) \\ &\qquad \lambda_E = \exp(j\phi)\left(\frac{\phi = \pi(a-b)^2}{2N}\right) \\ &\qquad \lambda_O = j\exp(j\phi), \quad \text{when } N = 8m \\ &\qquad \lambda_E = -\exp(j\phi) \\ &\qquad \lambda_O = -j\exp(j\phi) \\ &\quad \text{when } N = 4(2k+1)m. \end{aligned} \tag{45}$$

Besides, we can also show that

$$\begin{aligned} &O_E(\mathbf{e_1}) = C_1 O_E(\mathbf{e_2}) \\ &O_O(\mathbf{e_1}) = C_2 O_O(\mathbf{e_2}), \quad C_1, C_2 \text{ are some constants.} \end{aligned} \tag{46}$$

Linear Combination Method: In [7], the authors used a simpler way to find the eigenvectors of the original DFT. That is, for any N-point vector $g[n]$, if $g_s[n] = g[n] + j^s G[n] + (-1)^s g[-n] + (-j)^s G[-n]$, where $G[n] = \mathrm{DFT}(g[n])$, then $g_s[n]$ is the eigenvector of the N-point DFT. We can also use the method similar as above to derive the eigenvectors of the offset DFT when $a+b$ is an integer.

For any N-point vector $g[n]$, if

$$\begin{aligned} g_S[n] =& g[n] + j^s e^{-j\pi\frac{(a-b)^2}{2N}} \mathrm{DFT}_{a,b}(g[n]) \\ &+ (-1)^s e^{-j\pi\frac{(a-b)^2}{N}} \mathrm{DFT}_{a,b}(\mathrm{DFT}_{a,b}(g[n])) \\ &+ (-j)^s e^{-j3\pi\frac{(a-b)^2}{2N}} \mathrm{DFT}_{a,b} \\ &\times (\mathrm{DFT}_{a,b}(\mathrm{DFT}_{a,b}(g[n]))) \\ =& g[n] + j^s e^{-j\pi\frac{(a-b)^2}{2N}} G_{a,b}[n] \\ &+ (-1)^s g[-n] + (-j)^s e^{-j\pi\frac{(a-b)^2}{2N}} G_{a,b}[-n] \end{aligned} \tag{47}$$

where $G_{a,b}[n] = \mathrm{DFT}_{a,b}(g[n])$, then when $a+b$ is an integer, four times the offset DFT is just some constant phase multiplication of the identity operation:

$$\begin{aligned} &\mathrm{DFT}_{a,b}(\mathrm{DFT}_{a,b}(\mathrm{DFT}_{a,b}(\mathrm{DFT}_{a,b}(g[n])))) \\ &\quad = e^{j2\pi\frac{(a-b)^2}{N}} g[n] \quad \text{when } a+b \text{ is an integer.} \end{aligned} \tag{48}$$

Therefore

$$\begin{aligned} \mathrm{DFT}_{a,b}(g_S[n]) =& \mathrm{DFT}_{a,b}(g[n]) \\ &+ j^s e^{-j\pi\frac{(a-b)^2}{2N}} \mathrm{DFT}_{a,b}(\mathrm{DFT}_{a,b}(g[n])) \\ &+ (-1)^s e^{-j\pi\frac{(a-b)^2}{N}} \mathrm{DFT}_{a,b} \\ &\times (\mathrm{DFT}_{a,b}(\mathrm{DFT}_{a,b}(g[n]))) \\ &+ (-j)^s e^{-j3\pi\frac{(a-b)^2}{2N}} e^{j2\pi\frac{(a-b)^2}{N}} g[n] \\ =& (-j)^s e^{j\pi\frac{(a-b)^2}{2N}} g_S[n]. \end{aligned} \tag{49}$$

That is, $g_S[n]$ is the eigenvector of the offset DFT belonging to the eigenvalue $(-j) \cdot \exp[j\pi(a-b)^2/2N]$. Therefore, we can obtain the eigenvector of the N-point offset DFT from any N-point vector easily by (47) when $a+b$ is an integer.

D. General Form of the Eigenvectors of the Offset DFT

In Sections II-B and C, we illustrated that when $a+b$ is an integer, we can use at least three methods to obtain the eigenvectors of the offset DFT.

① Sampling the eigenfunctions of the continuous offset FT Close from eigenvectors set is available, and the eigenvectors have very clear relations with the eigenfunctions of the continuous offset FT.

② Commutative matrix method (the orthogonal eigenvectors set is available).

③ Linear combination method (the simplest way to obtain the eigenvectors).

The eigenvector sets obtained from these methods are different. In fact, when $a+b$ is an integer, the offset FT has infinite possible eigenvectors.

From the theory of linear algebra, if $\mathbf{e_1}, \mathbf{e_2}, \ldots,$ and $\mathbf{e_k}$ are the eigenvectors of an matrix $\mathbf{M}$ belonging to the same eigenvalue λ, the linear combination of them is also the eigenvector of $\mathbf{M}$ belonging to the eigenvalue λ. Since in (32), $V_{q,a,b}[n]$ and $V_{q+4,a,b}[n]$ always have same eigenvalue, we can conclude the following.

- *The general form of the eigenvector of the offset DFT when* $\boldsymbol{a+b}$ *is an integer*

$$\begin{aligned} E_{d,a,b}[n] =& \sum_{L=0}^{L_d} c_L V_{d+4L,a,b}[n] \\ =& e^{j\pi\frac{b-a}{N}n} \sum_{p=-\infty}^{\infty} (-1)^{(a+b)p} e^{-\frac{\pi\left(\frac{n+pN-(a+b)}{2}\right)^2}{N}} \\ &\times \sum_{L=0}^{L_d} c_L H_{d+4L}\left(\left(n+pN-\frac{a+b}{2}\right)\sqrt{\frac{2\pi}{N}}\right) \end{aligned} \tag{50}$$

where

$$\begin{aligned} d &= 0,1,2,3, \quad c_L\text{'s are free to choose} \\ L_d &= \mathrm{Max}\{L \mid d+4L \in 0,1,2,\ldots\ldots,N-2,N\} \\ &\qquad \text{when } N+a+b \text{ is even} \\ L_d &= \mathrm{Max}\{L \mid d+4L \in 0,1,2,\ldots\ldots,N-1\} \\ &\qquad \text{when } N+a+b \text{ is odd.} \end{aligned}$$

The corresponding *eigenvalues* for $E_{d,a,b}[n]$ are

$$\lambda_d = (-j)^d \cdot e^{j\pi\frac{(a-b)^2}{2N}}. \tag{51}$$

The eigenvectors derived from Sections II-B, C are all the special cases of (50).

III. Eigenvectors/Eigenvalues of Offset DFTs when $a+b \neq$ Integer

In Section II, we use three methods to derive the eigenvectors/eigenvalues of the offset DFT when $a+b$ is an integer. However, when $a+b$ is not an integer, these methods cannot be

$$\begin{aligned} O_E(\mathbf{e_1}[n]) &= \frac{e^{j\pi\frac{b-a}{N}n}\left(e^{j\pi\frac{a-b}{N}n}\widetilde{\mathbf{e}}_\mathbf{1}[n] + e^{j\pi\frac{b-a}{N}((a+b-n))_N}\widetilde{\mathbf{e}}_\mathbf{1}\left[((a+b-n))_{2N}\right]\right)}{2} \\ O_O(\mathbf{e_1}[n]) &= \frac{e^{j\pi\frac{b-a}{N}n}\left(e^{j\pi\frac{a-b}{N}n}\widetilde{\mathbf{e}}_\mathbf{1}[n] - e^{j\pi\frac{b-a}{N}((a+b-n))_N}\widetilde{\mathbf{e}}_\mathbf{1}\left[((a+b-n))_{2N}\right]\right)}{2} \end{aligned} \tag{44}$$

applied, and we can only explore the eigenvectors/eigenvalues by experiments. In this case, we find that the eigenvectors/eigenvalues of the offset DFT when $a+b\neq$ integer are rather complicated. Nevertheless, there still exists some regularity. We describe the regularity as follows.

A. Regularity of Eigenvectors

When $a+b$ is not an integer, the vectors $V_{q,a,b}[\mathrm{n}]$ defined as (32) are no longer the eigenvectors of the offset DFT. Nevertheless, by approximation, we find that

$$\mathrm{DFT}_{a,b}\left(V_{q,a,b}[n]\right) \approx \begin{cases} e^{j\frac{\pi}{2}\frac{(a-b)^2}{N}}e^{-j\frac{\pi q}{2}} \\ \quad \times V_{q,a,b}[n], & \text{when } 0\leq n'\leq \frac{N}{2}-\Delta \\ e^{j\frac{\pi}{2}\frac{(a-b)^2}{N}}e^{j2\pi(a+b)} \\ \quad \times e^{-j\frac{\pi q}{2}}V_{q,a,b}[n], & \text{when } \frac{N}{2}+\Delta\leq n'\leq N-1. \end{cases} \tag{52}$$

where $n'=((n-(a+b)/2))_N$, and Δ is the smallest positive integer that satisfied

$$\exp(-2\pi\Delta)\frac{\left(\frac{2\pi\left(\frac{N}{2}+\Delta\right)^2}{N}\right)^q}{\left(\frac{2\pi\left(\frac{N}{2}-\Delta\right)^2}{N}\right)^q} < \text{threshold}. \tag{53}$$

Equation (53) can be proven from the fact that the Hermite function of order q is dominated by the term of $\exp(-x^2/2)\cdot x^q$. Equation (52) can be proved as follows.

The transformation relation in (17) is satisfied even when $a+b$ is not an integer. We can also choose $E_{\tau,\eta}(t)$ as the Hermite–Gaussian function defined in (8). Then

$$\begin{aligned} &\mathrm{DFT}_{a,b}\left[e^{j\pi\frac{b-a}{N}n}\sum_{p=-\infty}^{\infty}e^{j\pi p(a+b)}e^{-\frac{\pi\left(n+pN-\frac{(a+b)}{2}\right)^2}{N}}\right.\\ &\qquad\left.\times H_q\left(\left(n+pN-\frac{a+b}{2}\right)\sqrt{\frac{2\pi}{N}}\right)\right]\\ &=(-j)^q e^{j\pi\frac{(a-b)^2}{2N}}e^{j\pi\frac{b-a}{N}n}\\ &\quad\times\sum_{p=-\infty}^{\infty}e^{-j\pi p(a+b)}e^{-\frac{\pi\left(n+pN-\frac{(a+b)}{2}\right)^2}{N}}\\ &\quad\times H_q\left(\left(n+pN-\frac{a+b}{2}\right)\sqrt{\frac{2\pi}{N}}\right). \end{aligned} \tag{54}$$

If the constraint of (27) is satisfied, the infinite summation can be simplified into only one term:

$$\begin{aligned} &\mathrm{DFT}_{a,b}\left[e^{j\pi\frac{b-a}{N}n}e^{j\pi p_n(a+b)}e^{-\frac{\pi(n'+p_nN)^2}{N}}\right.\\ &\qquad\left.\times H_q\left((n'+p_nN)\sqrt{\frac{2\pi}{N}}\right)\right]\\ &\approx(-j)^q e^{j\pi\frac{(a-b)^2}{2N}}e^{j\pi\frac{b-a}{N}n}e^{-j\pi p_n(a+b)}e^{-\frac{\pi(n'+p_nN)^2}{N}}\\ &\quad\times H_q\left((n'+p_nN)\sqrt{\frac{2\pi}{N}}\right) \end{aligned} \tag{55}$$

where $n'=((n-(a+b)/2))_N$, $p_n=0$ when $0\leq n'<N/2$, and $p_n=-1$ when $N/2<n'<N$. After applying (28), the above equation can be rewritten as

$$\mathrm{DFT}_{a,b}\left\lfloor V_{q,a,b}[n]\right\rfloor \approx \lambda_n V_{q,a,b}[n] \tag{56}$$

where $V_{q,a,b}[n]$ is defined as (21) and can be approximated by (28) if (27) is satisfied. The value of λ_n depends on whether $n'=((n-(a+b)/2))_N$ is greater than $N/2$ or not.

$$\begin{aligned} \lambda_n &= (-j)^q e^{j\frac{\pi(a-b)^2}{2N}}, \quad \text{when } n'<\frac{N}{2}\\ \lambda_n &= (-j)^q e^{j\left[\frac{\pi(a-b)^2}{2N}+2\pi(a+b)\right]}, \quad \text{when } n'>\frac{N}{2}. \end{aligned} \tag{57}$$

This matches the conclusion in (52).

B. Regularity of Eigenvalues

1) Strictly speaking, when $a+b$ is not an integer, all the eigenvalues of the offset DFT are different. Nevertheless, we find that if N (the number of points) is sufficient large, most of the eigenvalues can still be approximated as

$$\lambda_q \approx \exp\left[j\left(-\frac{\pi q}{2}+\theta\right)\right]$$
$$\text{where}\quad \theta=\pi(a+b)+\frac{\pi}{2}\frac{(a-b)^2}{N}\quad q \text{ is any integer.} \tag{58}$$

For example, when $a=0.1$, $b=0.3$, and $N=32$, from Matlab experimentation, the phases of eigenvalues of the offset DFT are as in (59), shown at the bottom of the page. In comparison, the phases obtained from (58) are $\pi q/2+\pi(a-b)^2/2N+\pi(a+b)=1.2586, 2.8294, -1.8830$, and -0.3122. There are 24 eigenvalue phases in (59) that can be approximated well by the four values, and only eight eigenvalue phases cannot. Thus, using (58), we can approximate most of the eigenvalues of the offset DFT even when $a+b\neq$ integer.

Notice that in (59), many of the eigenvalues seem to be the same. However, they in fact differ only very slightly. We make

$$\begin{matrix} -2.4809 & -1.8830 & -1.8830 & -1.8830 & -1.8830 & -1.8830 & -1.8830 & -1.8759\\ -0.6495 & -0.3139 & -0.3122 & -0.3122 & -0.3122 & -0.3122 & -0.3122 & -0.3122\\ 0.5947 & 1.2586 & 1.2586 & 1.2586 & 1.2586 & 1.2586 & 1.2586 & 1.2590\\ 1.3730 & 2.8006 & 2.8293 & 2.8294 & 2.8294 & 2.8294 & 2.8294 & 2.8294 \end{matrix} \tag{59}$$

this conclusion from the experimental results shown in Table II. We measured the minimal phase difference (denoted by d_N) among all the eigenvalues of the offset DFT for $N = 7 \sim 22$. (We fix $\{a, b\}$ to $\{0.1, 0.3\}$.) We found that d_N is never zero. When N is increased by 1, d_{N+1} becomes 7~9 times smaller than d_N. Thus, we can conclude that if $a + b \neq$ integer, no matter how much larger N is, the offset DFT has no repeated eigenvalues. The minimal-phase difference among all the eigenvalues becomes smaller and smaller when N grows larger; however, this difference is never zero.

The approximation of eigenvectors in (56) and (57) and the approximation of eigenvalues in (58) are well known when N is large since when N is large, the offset DFT is more and more similar to the continuous offset FT. In Table III, we show how many eigenvalues satisfying

$$\left|\arg(\lambda_q) - \left(-\frac{\pi q}{2} + \theta\right)\right| < 10^{-4} \tag{60}$$

when a, b are fixed to 0.1 and 0.3. When $N \geq 100$, more than 90% of the eigenvalues can be approximated well by (58).

Besides, we also find that when the value of $a+b$ is near some integer, and $a-b$ is near 0, then the approximated formula in (58) is even better.

There is an interesting thing to be noticed. In (52), we have stated that after doing the offset DFT for $V_{q,a,b}[n]$, the left-half part of $V_{q,a,b}[n]$ is multiplied by $s_1 = (-j)^q \exp[j\pi(a-b)^2/2N]$, and the right-half part of $V_{q,a,b}[n]$ is multiplied by $s_2 = (-j)^q \exp[j\pi(a-b)^2/2N]\exp[j2\pi(a+b)]$. Their geometric average is $\sqrt{s_1 s_2} = (-j)^q \exp[j\pi(a-b)^2/2N]\exp[j\pi(a+b)]$. It is just the approximated eigenvalue [see (58)] found by experiments.

2) When $a \approx 0$ and $b \approx 0$, there are only two eigenvalues that cannot be approximated by (58). Although the two eigenvalues cannot be approximated by (58), they can be approximated by

$$\lambda_0 \approx \exp[j\phi], \quad \lambda_2 \approx \exp\left[j(-\pi + \theta - \phi)\right] \tag{61}$$

where $\phi = \pi(a+b)(1 - N^{-1/2})/2$, and θ is defined as (58).

IV. Properties of the Eigenvectors/Eigenvalues of Offset DFTs

The following properties are satisfied for any a and b: The offset DFT matrix is unitary. Besides

$$\mathbf{F}_{a,b} = \mathbf{F}_{b,a}^T = \overline{\mathbf{IF}_{b,a}} = \mathbf{IF}_{a,b}^H \tag{62}$$

where $\mathbf{F}_{a,b}$ and $\mathbf{IF}_{a,b}$ mean the transform matrices of the offset DFT and offset IDFT. Thus, we have the following.

a) All the eigenvalues have unity amplitude.

b) If $E[n]$ is an eigenvector of $\mathbf{F}_{a,b}$ with eigenvalue λ, then
   ① $conj(E[n])$ is an eigenvector of $\mathbf{F}_{b,a}$ with eigenvalue λ.
   ② $E[n]$ is an eigenvector of $\mathbf{IF}_{a,b}$ with eigenvalue $conj(\lambda)$.
   ③ $conj(E[n])$ is an eigenvector of $\mathbf{IF}_{b,a}$ with eigenvalue $conj(\lambda)$.

TABLE II
MINIMAL-PHASE DIFFERENCE (DENOTED BY d_N) AMONG ALL THE EIGENVALUES OF THE OFFSET DFT WHEN $a = 0.1, b = 0.3$, AND $N = 7$ TO 22

N	7	8	9	10	11
d_N	12.93	1.767	0.224	0.030	4.10E-03
N	12	13	14	15	16
d_N	5.50E-04	7.32E-05	9.58E-06	1.24E-06	1.57E-07
N	17	18	19	20	21
d_N	1.97E-08	2.44E-09	2.99E-10	3.61E-11	4.41E-12

c) The offset DFT has at least one *complete orthogonal eigenvectors set*.
   ① When $a+b$ is not an integer, since all the N eigenvalues are different (N is the number of points), the eigenvectors belonging to each of the eigenvalue construct an orthogonal eigenvectors set. However, when N is larger, many eigenvalues are nearly the same. Due to the round-off error, the two eigenvectors obtained from computer may not be orthogonal if their corresponding eigenvalues are almost the same. In these conditions, we should use Gram–Schmidt method [15] to convert them into an orthogonal eigenvectors set.
   ② When $a+b$ is an integer, the eigenvectors belonging to different eigenvalues are orthogonal, but the eigenvectors belonging to the same eigenvalue may not be orthogonal. We can use the Gram–Schmidt method to convert the eigenvectors belonging to the same eigenvalue into an orthogonal eigenvectors set.

The following properties are satisfied when $a+b$ is an integer: Suppose that $a_1 + b_1$ is some integer, and $G_{a1,b1}[n]$ is any eigenvector of the offset DFT with parameters a_1, b_1. Then, from (32), we can prove the following.

1) If $a_2 + b_2 = a_1 + b_1$, we can always find an eigenvector of the offset DFT with parameters a_2, b_2 (denoted by $G_{a2,b2}[n]$) that satisfies

$$|G_{a2,b2}[n]| = |G_{a1,b1}[n]|. \tag{63}$$

2) If $a_2 = a_1 + B$ and $b_2 = b_1 + B$, where B is some integer, then we can always find an eigenvector of the offset DFT with parameters a_2, b_2 (denoted by $G_{a2,b2}[n]$) that satisfies

$$|G_{a2,b2}[n \oplus B]| = |G_{a1,b1}[n]| \qquad \oplus: \text{ cyclic shift.} \tag{64}$$

The following properties are satisfied when $a = b = M/2$ (M is some integer):

1) In this case, the offset DFT matrix is symmetric. Therefore, the offset DFT has *real* eigenvectors sets. In other conditions, the offset DFT has no real eigenvector set.
2) The eigenvalues are 1, -1, j, or $-j$. This is all the same as the original DFT.

TABLE III
NUMBER OF THE EIGENVALUES THAT CAN BE APPROXIMATED BY (58) WHEN $a = 0.1$ AND $b = 0.3$ (DENOTED BY M) AND ITS RATIO TO N

N	7	10	20	32	50	100	200	300
M	2	4	13	24	41	90	188	288
M/N	28.6%	40%	65%	75%	82%	90%	94%	96%

3) In this case, the eigenvectors are *symmetric* or *asymmetric* with *respect to* $\boldsymbol{a}$ *or* $\boldsymbol{N}+\boldsymbol{a}$:

$$E[n] = \pm\widetilde{E}\left[((2a-n))_{2N}\right], \qquad ((\,)) : \text{modulus} \tag{65}$$

where

$$\begin{aligned} \text{length}\left(\widetilde{E}[n]\right) &= 2N, \quad \widetilde{E}[n] = E[n] \\ &\text{when} \quad 0 \le n \le N-1 \\ \widetilde{E}[n] &= E[n-N], \quad \text{when} \quad N \le n \le 2N-1 \\ &a \text{ is an integer} \\ \widetilde{E}[n] &= -E[n-N], \quad \text{when} \quad N \le n \le 2N-1 \\ &a = \text{integer} + \frac{1}{2}. \end{aligned} \tag{66}$$

V. EIGENVECTORS/EIGENVALUES OF DCTS/DSTS OF TYPES 4 AND 8 AND DHTS OF TYPE 4

There are eight types of discrete cosine transforms (DCT-I, DCT-II, ..., DCT-VIII) [16]–[18], eight types of discrete sine transforms (DST-I, DST-II, ..., DST-VIII) [16]–[18], and four types of discrete Hartley transforms (DHT-I ~ DHT-IV) [19]. Since the DCT-I, DCT-V, DST-I, DST-V, and DHT-I have very close relations with the original DFT, we can use the eigenvectors/eigenvalues of the original DFT to derive their eigenvectors and eigenvalues.

Although the DCTs, DSTs, and DHTs of other types have no obvious relations with the original DFT, they have close relations with the offset DFT. We can use these relations together with the eigenvectors/eigenvalues of the offset DFT to derive the eigenvectors/eigenvalues of the DCT-IV, DCT-VIII, DST-IV, DST-VIII, and DHT-V.

A. Eigenvectors and Eigenvalues of the DCT-IV and the DCT-VIII

There are eight types of DCTs [16]–[18]. We list the DCT-IV and DCT-VIII as follows:

$$\begin{aligned} &\text{DCT-IV}: \\ &X_{C4}[m] = \text{DCT}_4(x[n]) \\ &\quad = \sqrt{\frac{2}{N}}\sum_{n=0}^{N-1}\cos\left(\frac{\pi\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}{N}\right)x[n] \end{aligned} \tag{67}$$

$$\begin{aligned} &\text{DCT-VIII}: \\ &X_{C8}[m] = \text{DCT}_8(x[n]) \\ &\quad = \sqrt{\frac{2}{N+\frac{1}{2}}}\sum_{n=0}^{N-1}\cos\left(\frac{\pi\left(m-\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}{N+\frac{1}{2}}\right)x[n]. \end{aligned} \tag{68}$$

In fact, the N-point DCT-IV can be viewed as the special case of the $2N$-point offset DFT, where $a = b = -1/2$. If

$$\begin{aligned} \hat{X}_{-\frac{1}{2},-\frac{1}{2}}[m] &= \text{DFT}_{-\frac{1}{2},-\frac{1}{2}}\left(\hat{x}[n]\right) \\ &= \sqrt{(2N)^{-1}}\sum_{n=0}^{2N-1} e^{-j\frac{\pi}{N}\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}\hat{x}[n] \end{aligned} \tag{69}$$

where

$$\hat{x}[n] = -\hat{x}[2N-1-n], \quad n = 0,1,2,\ldots\ldots,N-1 \tag{70}$$

we can show that

$$\begin{aligned} \hat{X}_{-\frac{1}{2},-\frac{1}{2}}[m] &= \sqrt{\frac{2}{N}}\sum_{n=0}^{N-1}\cos\left(\frac{\pi\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}{N}\right)\hat{x}[n] \\ &= \hat{X}_{C4}[m] \\ \text{where} \quad \hat{X}_{C4}[m] &= \text{DCT}_4(\hat{x}[n]) \\ m &= 0,1,2,\ldots\ldots,N-1. \end{aligned} \tag{71}$$

Notice that when $a = b = -1/2$ and q is even, the close form Hermite-like eigenvectors of the $2N$-point offset DFT [see (32), where N is replaced by $2N$] satisfy the symmetric relation in (65). Thus, if $V_{q,C4}[n]$ is the former half part of $V_{2q,-1/2,-1/2}[n]$:

$$\begin{aligned} &V_{\substack{q,C4 \\ N \text{ points}}}[n] = V_{\substack{2q,-\frac{1}{2},-\frac{1}{2} \\ 2N \text{ points}}}[n] \\ &\quad\text{when} \quad n = 0,1,2,\ldots\ldots\ldots\ldots,N-1 \end{aligned} \tag{72}$$

and $q = 0,1,2,\ldots\ldots,N-1$, then

$$\begin{aligned} \text{DCT}_4\left(V_{q,C4}[n]\right) &= \text{DFT}_{-\frac{1}{2},-\frac{1}{2}}\left(V_{2q,-\frac{1}{2},-\frac{1}{2}}[n]\right) \\ &= (-1)^q V_{2q,-\frac{1}{2},-\frac{1}{2}}[m] \\ &= (-1)^q V_{q,C4}[m]. \end{aligned} \tag{73}$$

Thus, $V_{q,C4}[n]$ is the eigenvector of the N-point DCT-IV. From the above discussion, we can conclude that *the close form Hermite-like eigenvectors set of the* $\boldsymbol{N}$*-point DCT-IV* is

$$\begin{aligned} V_{q,C4}[n] = &\sum_{p=-\infty}^{\infty}(-1)^p e^{-\frac{\pi\left(n+2pN+\frac{1}{2}\right)^2}{2N}} \\ &\times H_{2q}\left(\left(n+2pN+\frac{1}{2}\right)\sqrt{\frac{\pi}{N}}\right) \\ \text{where} \quad &n = 0 \sim N-1, \quad q = 0 \sim N-1 \end{aligned} \tag{74}$$

and the corresponding *eigenvalues* are ± 1:

$$\mathrm{DCT}_4\left(V_{q,C4}[n]\right) = (-1)^q \cdot V_{q,C4}[n]. \tag{75}$$

The eigenspace belonging to the eigenvalue 1 has the rank of $N - [N/2]_G$ ($[\,]_G$ is Gaussian symbol), and the eigenspace belonging to the eigenvalue -1 has the rank of $[N/2]_G$.

Besides, since $V_{q,C4}[n], V_{q+2,C4}[n], V_{q+4,C4}[n], \ldots\ldots\ldots\ldots$ belongs to the same eigenspace, we can conclude that *the general form of the eigenvectors of the* $\boldsymbol{N}$*-point DCT-IV* is

$$E_{d,C4}[n] = \sum_{p=-\infty}^{\infty} (-1)^p e^{-\frac{\pi\left(n+2pN+\frac{1}{2}\right)^2}{2N}} \times \sum_{L=0}^{L_d} c_L H_{2d+4L}\left(\left(n+2pN+\frac{1}{2}\right)\sqrt{\frac{\pi}{N}}\right) \tag{76}$$

where $d=0$ or 1, c_L's are any constants, and $L_d = \mathrm{Max}\{L|d+2L \le N-1\}$. The corresponding eigenvalue for $E_{d,C4}[n]$ is $(-1)^d$.

The eigenvectors of the DCT-VIII can also be derived similarly. Since the close form Hermite-like eigenvectors of the $2N+1$-point offset DFT with parameters $a=b=-1/2$ satisfies the symmetric relations as follows:

$$V_{2q,-\frac{1}{2},-\frac{1}{2}}[n] = -V_{2q,-\frac{1}{2},-\frac{1}{2}}[2N-n], \quad V_{2q,-\frac{1}{2},-\frac{1}{2}}[N] = 0 \tag{77}$$

and the corresponding eigenvalue for $V_{2q,-1/2,-1/2}[n]$ is $(-1)^q$, then

$$\begin{aligned}(-1)^q V_{2q,-\frac{1}{2},-\frac{1}{2}}[m] &= \sqrt{\frac{1}{2N+1}} \cdot \sum_{n=0}^{2N} e^{-j\frac{2\pi}{2N+1}\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)} V_{2q,-\frac{1}{2},-\frac{1}{2}}[n] \\ &= \sqrt{\frac{2}{N+\frac{1}{2}}} \sum_{n=0}^{N-1} \cos\left(\frac{2\pi\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}{2N+1}\right) \cdot V_{2q,-\frac{1}{2},-\frac{1}{2}}[n].\end{aligned} \tag{78}$$

Therefore, *the close form Hermite-like eigenvectors set of the* $\boldsymbol{N}$*-point DCT-VIII* is

$$V_{q,C8}[n] = \sum_{p=-\infty}^{\infty} (-1)^p e^{-\frac{\pi\left(n+p(2N+1)+\frac{1}{2}\right)^2}{2N+1}} \times H_{2q}\left(\left(n+p(2N+1)+\frac{1}{2}\right)\sqrt{\frac{\pi}{2N+1}}\right) \tag{79}$$

where $q = 0, 1, 2, \ldots\ldots, N-1$

and the corresponding eigenvalues are $(-1)^q$

$$\mathrm{DCT}_8\left(V_{q,C8}[n]\right) = (-1)^q \cdot V_{q,C8}[n]. \tag{80}$$

The general form of the eigenvectors of the DCT-VIII can be obtained by the linear combination of $V_{d,C8}[n]$ ($d=0$ or 1), $V_{d+2,C8}[n], V_{d+4,C8}[n], V_{d+6,C8}[n], \ldots\ldots\ldots\ldots\ldots\ldots$.

Although it seems that DCT-II, DCT-III, DCT-VI, and DCT-VII have some relations with the offset DFT, it is still very hard to derive their eigenvectors and eigenvalues.

B. Eigenvectors/Eigenvalues of DST-IV, DST-VIII, and DHT-IV

DST-IV, DST-VIII, [16]–[18] and DHT-IV [19] are defined as follows:

DST-IV:

$$X_{S4}[m] = \mathrm{DST}_4\left(x[n]\right) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \sin\left(\frac{\pi\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}{N}\right) x[n] \tag{81}$$

DST-VIII:

$$X_{S8}[m] = \mathrm{DST}_8(x[n]) = \sqrt{\frac{2}{N-\frac{1}{2}}} \sum_{n=0}^{N-1} B(m)B(n) \times \sin\left(\frac{\pi\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}{N-\frac{1}{2}}\right) x[n] \tag{82}$$

where $B(n) = \frac{1}{\sqrt{2}}$ when $n = N-1$, $B(n) = 1$ otherwise

DHT-IV:

$$X_{H4}[m] = \mathrm{DHT}_4(x[n]) = \sqrt{\frac{1}{N}} \sum_{n=0}^{N-1} \mathrm{cas}\left(\frac{2\pi\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}{N}\right) x[n] \tag{83}$$

where $\mathrm{cas}[n] = \cos[n] + \sin[n]$. (84)

Following the similar process as in Section V-A, we can derive their eigenvectors and eigenvalues from those of the offset DFT. The results are shown as follows:

- *The close form Hermite-like eigenvectors set of the DST-IV*:

$$V_{q,S4}[n] = \sum_{p=-\infty}^{\infty} (-1)^p e^{-\frac{\pi\left(n+2pN+\frac{1}{2}\right)^2}{2N}} \times H_{2q+1}\left(\left(n+2pN+\frac{1}{2}\right)\sqrt{\frac{\pi}{N}}\right). \tag{85}$$

where $q = 0, 1, 2, \ldots\ldots, N-1$

and the corresponding eigenvalues are $(-1)^q$:

$$\mathrm{DST}_4\left(V_{q,S4}[n]\right) = (-1)^q V_{q,S4}[n]. \tag{86}$$

- *The close form Hermite-like eigenvectors set of the DST-VIII*

$$V_{q,S8}[n] = S[n] \sum_{p=-\infty}^{\infty} (-1)^p e^{-\frac{\pi\left(n+p(2N-1)+\frac{1}{2}\right)^2}{2N-1}} \times H_{2q+1}\left(\left(n+p(2N-1)+\frac{1}{2}\right)\sqrt{\frac{2\pi}{2N-1}}\right) \tag{87}$$

$$\text{where}\quad q=0,1,2,\ldots\ldots,N-1,\quad S[N-1]=1/\sqrt{2}$$
$$S[n]=1,\quad \text{when } n\neq N-1$$

and the eigenvalues are $(-1)^q$.

- *The close form Hermite-like eigenvectors set of the DHT-VI*

$$V_{q,H4}[n]=e^{j\pi\frac{b-a}{N}n}\sum_{p=-\infty}^{\infty}(-1)^{Ap}e^{-\frac{\pi\left(n+pN-\frac{A}{2}\right)^2}{N}}\times H_q\left(\left(n+pN-\frac{A}{2}\right)\sqrt{\frac{2\pi}{N}}\right)\quad(88)$$

where $A=a+b$, $q=0,1,2,\ldots,N-1$ for even N, and $q=0,1,2,\ldots,N-2,N$ for odd N. Notice that the eigenvectors of the N-point DHT-IV are all the same as those of the N-point offset DFT with parameters $a=b=-1/2$. However, the their eigenvalues are different:

$$\begin{aligned}\mathrm{DHT}_4\left(V_{q,H4}[n]\right)&=V_{q,H4}[n]\\ &\quad\text{when } q=4L,4L+1\\ \mathrm{DHT}_4\left(V_{q,H4}[n]\right)&=-V_{q,H4}[n]\\ &\quad\text{when } q=4L+2,4L+3.\end{aligned}\quad(89)$$

The eigenvectors sets shown as the above are not the only eigenvectors set for the DST-IV, DST-VIII, and DHT-VI. The general form of the eigenvectors can be obtained by a linear combination of the eigenvectors with the same eigenvalue.

VI. Fractional Operations and Applications

After the eigenvectors/eigenvalues of the offset DFT, DCT-IV, DCT-VIII, DST-IV, DST-VIII, and DHT-IV were derived, we can use them to discuss the properties and performance of these operations. Besides, we can use them to derive the fractional operations.

The fractional operation has been a popular topic in recent years. The continuous fractional operation (such as the fractional Fourier transform [20]) has been extensively explored. Some discrete fractional operations have also been developed. Several different types of the discrete fractional Fourier transform (DFRFT) were derived [8], [9], [21]. In [9] and [21], the authors used the eigenvector-decomposition method to derive the DFRFT. Besides, in [11], the similar eigenvector-decomposition method was used to obtain the discrete fractional cosine transforms of type 2. In [5], Tseng also used the eigenvector-decomposition method to derive the fractional operation of the offset DFT when $a=b=1/2$. In this section, we use eigenvector-decomposition method to derive the *fractional operations* of the *general offset DFT* (*when* $\boldsymbol{a+b}$ *is an integer), DCT-IV, DCT-VIII, DST-IV, DST-VIII, and DHT-IV* (see Fig. 1).

If a discrete transform can be expressed as the matrix operation as follows:

$$\mathbf{X}=\mathbf{A}\mathbf{x},\qquad \text{where } \mathbf{A} \text{ is the transform matrix}\quad(90)$$

we can define its corresponding fractional transform as

$$\mathbf{X}_\alpha=\mathbf{A}^\alpha\mathbf{x}.\quad(91)$$

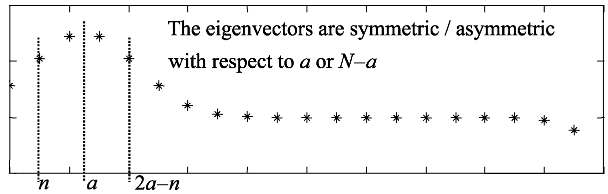

Fig. 1. Symmetry property of eigenvectors when $a=b=M/2$ (M is some integer).

If we know the eigenvectors/eigenvalues of $\mathbf{A}$, we can calculate $\mathbf{A}^\alpha$ as

$$\mathbf{A}^\alpha=\mathbf{E}\mathbf{D}^\alpha\mathbf{E}^{-1}\quad(92)$$

$$\text{where}\quad \mathbf{E}=\left\lfloor\mathbf{v_0^T}\quad\mathbf{v_1^T}\quad\cdots\quad\cdots\quad\mathbf{v_{N-1}^T}\right\rfloor$$
$$\mathbf{v_n}\text{'s: the eigenvectors of } \mathbf{A}\quad(93)$$

and $\mathbf{D}$ is a diagonal matrix where

$$\begin{aligned}D^\alpha(n,n)&=\lambda_n^\alpha,\quad \lambda_n\text{'s are the corresponding}\\ &\qquad\text{eigenvalues of } \mathbf{v_n}\text{'s}\\ D^\alpha(m,n)&=0,\quad \text{if } m\neq n.\end{aligned}\quad(94)$$

Since the eigenvectors/eigenvalues of the offset DFT (when $a+b$ is an integer), DCT-IV, DCT-VIII, DST-IV, DST-VIII, and DHT-IV have been derived, we can use the above method to derive the fractional operations of these transforms. For example, to define the *FRODFT*, we can choose the eigenvectors set as follows [from (32)]

$$\mathbf{v_m}[n]=e^{j\pi\frac{b-a}{N}n}\sum_{p=-\infty}^{\infty}(-1)^{Ap}e^{-\frac{\pi\left(n+pN-\frac{A}{2}\right)^2}{N}}\times H_m\left(\left(n+pN-\frac{A}{2}\right)\sqrt{\frac{2\pi}{N}}\right)$$
$$\begin{aligned}\text{where}\quad & m=0,1,2,\ldots\ldots,N-2,N_M\\ & N_M=N-1 \text{ when } N+a+b \text{ is odd}\\ & N_M=N \text{ when } N+a+b \text{ is even.}\end{aligned}\quad(95)$$

Besides, $\lambda_m=(-j)^m\cdot\exp[j(a-b)^2\pi/2N]$ [from (22)]. We can set the fractional powers of λ_m's as

$$\lambda_m^\alpha=\exp\left\{j\alpha\left[\frac{(a-b)^2}{2N}\pi-\frac{m}{2}\pi\right]\right\}.\quad(96)$$

After substituting (95) and (96) into (92)–(94), the FRODFTs are defined successfully.

Because of the following two reasons, the offset DFT, DCT-IV, DCT-VIII, DST-IV, DST-VIII, and DHT-IV have various definitions of fractional transforms.

1) The eigenvector set is not unique.

   From the discussion in Sections II and V, all the discrete transforms described above have infinite possible eigenvectors sets (due to the fact that all the eigenvalues of these transforms are not distinct). Therefore, in (93), there are many possible choices for $\mathbf{v_n}$'s.

2) The fractional power of the eigenvalue is not unique.

   From (94), to calculate the fractional operation, we must calculate the fractional power of λ_n's. Nevertheless,

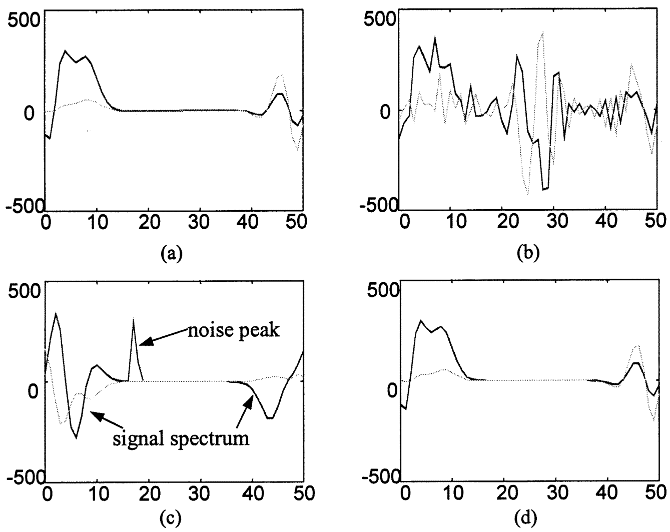

Fig. 2. Example for using the fractional offset DFT (FRODFT) with parameters $a = 0.3, b = 0.7$, and $\alpha = 0.76$ to remove the noise. (a) Input signal. (b) Input signal + noise. (c) Results of FRODFT. (d) Recovered signal.

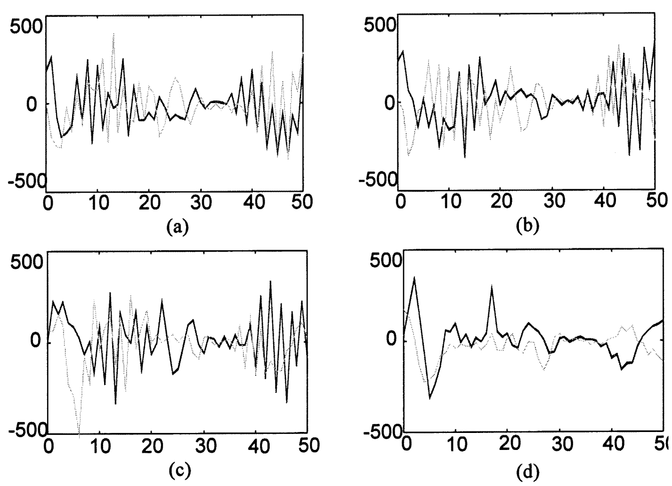

Fig. 3. Transform results of the interfered signal in Fig. 2(b). (a) Original DFT. (b) Original offset DFT ($a = 0.3, b = 0.7, \alpha = 1$). (c) FRODFT with $a = 0.3$, $b = 0.7, \alpha = 0.5$. (d) FRODFT with $a = 0.5, b = 0.5, \alpha = 0.67$.

the fractional powers of λ_n's have many possible choices. In fact, if

$$\lambda_n = \exp(j\theta_n) \tag{97}$$

the general form of ${\lambda_n}^\alpha$ is

$$\lambda_n^\alpha = \exp\left[j\alpha(\theta_n + 2\pi M_n)\right], \quad \text{where } M_n\text{'s are any integer.} \tag{98}$$

Although there are infinite possible fractional operations, all of them have the following properties.

A) When $\alpha = 1$, the fractional operation becomes the original operation. When $\alpha = 0$, the fractional operation becomes the identity operation.

B) If the eigenvectors set and the values of M_n's in (98) we choose are fixed for α_1 and α_2, the fractional operation has the following additivity property:

$$\mathbf{A}^{\alpha_1}\mathbf{A}^{\alpha_2} = \mathbf{A}^{\alpha_1+\alpha_2} \tag{99}$$

especially $\mathbf{A}^\alpha\mathbf{A}^{-\alpha} = \mathbf{I}$, which means that the inverse of the fractional operation with parameter α is just the fractional operation with parameter $-\alpha$.

We can use the fractional operations of the offset DFT, DCT-IV, DCT-VIII, DST-IV, DST-VIII, and DHT-IV for some applications.

1) Replace the original operations in some digital signal processing applications:

Since the fractional operation has a parameter α, it is more flexible than the original transform. We can adjust the parameter α to control the performance. It can happen that for some α, the performance of the fractional operation is better than that of the original one, and we can use the fractional operation instead of the original one to improve the performance.

For example, the original offset DFT can be used for filter design and signal compression. We can use the fractional offset DFT instead of the original offset DFT to improve the performance of filter design and signal compression. The DCT-IV can be used for data compression. We can also use the fractional DCT-IV instead of the original DCT-IV to improve the performance of data compression.

We give an example in Fig. 2. We use the FRODFT, defined as in (95) and (96), instead of the original offset DFT for filter design. We use Fig. 2(a) as the input signal (we use two lines to show the real part and the imaginary part). It is interfered with by shifted-modulated chirp-like noise, as in Fig. 2(b). Then, we try to use the FRODFT to remove the noise. We find that when we do the FRODFT with parameters $a = 0.3$, $b = 0.7$, and $\alpha = 0.76$, the spectrums of the noise part (around the peak at $m = 18$) and signal part (in the range of [0, 15] and [35, 50]) are well-separated, as in Fig. 2(c). Therefore, we can remove the spectrum of noise part and do the inverse transform (i.e., FRODFT with parameters $a = 0.3$, $b = 0.7$, and $\alpha = -0.76$). Then, we find that in the time domain, the noise is removed perfectly [Fig. 2(d)].

In contrast, if we use the original DFT [Fig. 3(a)] or the original offset DFT with parameters $a = 0.3$, $b = 0.7$ [Fig. 3(b)] or the FRODFT with wrong parameters [Fig. 3(c) and (d)], the spectrums of the noise and the signal are not separated well, and the noise cannot be removed.

2) Encryption and watermarking:

The block diagrams of encryption and watermarking are shown in Figs. 4 and 5. In convention, we choose O_T and O_T^{-1} in Figs. 4 and 5 as the DFT and IDFT, respectively. Then, in Fig. 4, the key is $E_1[n]$, and in Fig. 5, the key is the embedded algorithm. If we do not know the key, we cannot recover the original data $X[n]$ in Fig. 4 and cannot recover the watermark in Fig. 5. However, for the applications of encryption and watermarking, using just one key is not safe enough. To protect the hidden data, it is better to use more keys.

Thus, in Figs. 4 and 5, we can choose O_T as the fractional operations of the DFT, DCT-IV, DCT-VIII,

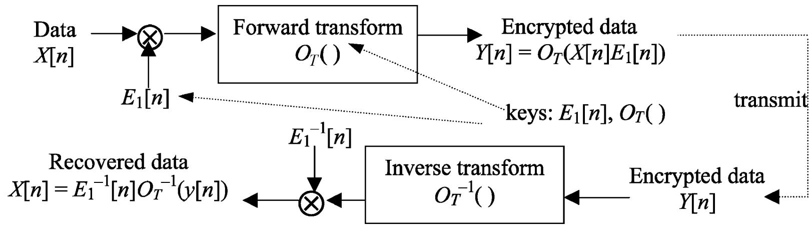

Fig. 4. Block diagram of encryption.

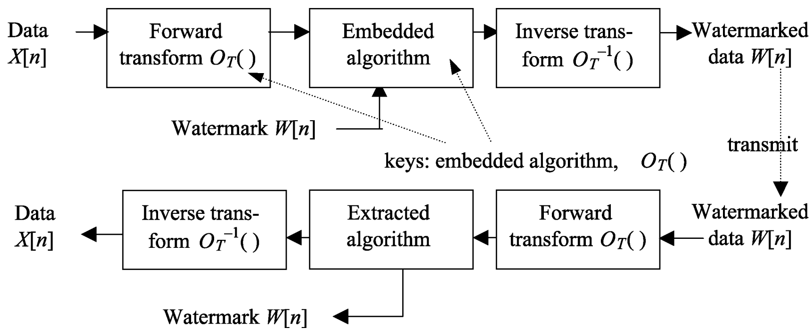

Fig. 5. Block diagram of watermarking.

DST-IV, DST-VIII, or DHT-IV instead of the DFT. Then, many components of these fractional operations can be treated as the keys for encryption and watermarking.

A) parameter α;

B) eigenvectors $\mathbf{v_n}$'s in (93) we choose

C) definitions of the fractional powers of eigenvalues [i.e., the choices of M_n's in (98)];

D) if the original operation is the offset DFT, then there are two extra parameters a, b.

Even if someone knows the value of $E_1[n]$ in Fig. 4 and the embedded algorithm in Fig. 5, if they do not know one of the parameters or components of the fractional operations described above, they cannot recover the original data or watermark. Thus, the safety is improved a lot.

3) Digital implementation of their continuous counterpart: For example, we can use the FRODFT to implement the continuous fractional offset Fourier transform [6] digitally.

VII. Conclusion

In this paper, we derive the close and the general forms of eigenvectors of the offset DFT when $a+b$ is an integer (a and b are the parameters of the offset DFT) and their corresponding eigenvalues. In the case where $a+b$ is not an integer, although the eigenvectors and eigenvalues are rather complicated, we also found some of their regularities. We also use the eigenvectors/eigenvalues of the offset DFT to derive the eigenvectors/eigenvalues of the DCT-IV, DCT-VIII, DST-IV, DST-VIII, and DHT-IV.

Besides, since the eigenvectors and eigenvalues have been derived, we can use eigenvector-decomposition method to derive the fractional operations of the offset DFT, DCT-IV, DCT-VIII, DST-IV, DST-VIII, and DHT-IV. These fractional operations are more flexible than the original ones. They can replace the original operations in digital signal processing applications, such as filter design. Besides, they are also useful for encryption and watermarking.

Appendix
Proof of Theorem 1

We prove the case where q is even. In this case, the corresponding eigenvalue of $V_{q+4h,a,b}[n]$ is $\exp(j\phi)$ (when $q=0$) or $-\exp(j\phi)$ (when $q=2$), $\phi=\pi(a-b)^2/2N$.

In (21), $V_{q,a,b}[n]$ is obtained from Hermite functions. Besides, the Hermite function $H_q(x)$ is the linear combination of $x^{2t}(t=0,1,\ldots,q/2)$ if q is even. Thus, $V_{q,a,b}[n]$ can be expressed as

$$V_{q,a,b}[n]=e^{j\pi\frac{b-a}{N}n}\cdot\sum_{t=0}^{\frac{q}{2}}a_t d_t[n]$$

$$\text{where } q \text{ is even, } a_t\text{'s are constants} \tag{100}$$

$$d_t[n]=\sum_{p=-\infty}^{\infty}(-1)^{(a+b)p}\exp\left(-\frac{x_{n+pN}^2}{2}\right)\cdot x_{n+pN}^{2t}$$

$$x_n=\left(n-\frac{a+b}{2}\right)\sqrt{\frac{2\pi}{N}}. \tag{101}$$

We then apply the theorem of Chebyshev set [22]. A continuous functions set $\{\varphi_1(x)\ \varphi_2(x),\ldots\ldots,\varphi_M(x)\}$ is called a Chebyshev set on $[\alpha,\beta]$ if and only if any linear combination of $\{\varphi_1(x)\ \varphi_2(x),\ldots\ldots,\varphi_M(x)\}$ has at most $M-1$ zeros in $[\alpha,\beta]$.

If we sample the Chebyshev set

$$\{\varphi_1(x)\ \varphi_2(x),\ldots\ldots,\varphi_M(x)\}$$

at J-point x_j's ($j=1\sim J$), where $J\leq M$, $\alpha\leq x_j\leq\beta$, $x_i\neq x_j$ if $i\neq j$, and x_j is not a zero for all $\varphi_m(x_j)$'s, then $\{\varphi_m(x_j)|m,j=1,2,\ldots\ldots,J\}$ form an independent discrete vectors set.

The following functions form a Chebyshev set in $\Omega = [0, \sqrt{\pi N/2}]$:

$$\psi_t(x) = \sum_{p=-\infty}^{\infty} (-1)^{(a+b)p} \times \exp\left[-\frac{(x+p\sqrt{2\pi N})^2}{2}\right] \cdot (x+p\sqrt{2\pi N})^{2t} \quad (102)$$
$$x \in \left[0, \sqrt{\frac{\pi N}{2}}\right], \quad t = 0, 1, 2, \ldots\ldots, T-1.$$

The value of T is hard to determine. However, it is always no less than $[N/2+1]_G$ ($[\,]_G$ is Gaussian symbol). In (101), we can see that $d_t[n]$ is in fact the uniform sampling of $\psi_t(x)$:

$$d_t[n] = \psi_t(x)|_{x=n'\Delta}$$
$$\text{where} \quad n' = n - \frac{(a+b)}{2}, \quad \Delta = \sqrt{\frac{2\pi}{N}}. \quad (103)$$

We then calculate how many sampling points are in the range of $[0, \sqrt{\pi N/2}]$. We find that

$$\begin{aligned}
&\textbf{Case 1}: N \text{ is even, } a+b \text{ is even,} \quad n'\Delta \in \Omega \\
&\quad \text{if } n' = 0, 1, \ldots\ldots, \frac{N}{2}, \quad \left(\frac{N}{2}+1 \text{ points}\right) \\
&\textbf{Case 2}: N \text{ is odd } a+b \text{ is even} \quad n'\Delta \in \Omega \\
&\quad \text{if } n' = 0, 1, \ldots\ldots, \frac{(N-1)}{2} \\
&\quad \left(\frac{(N+1)}{2}+1 \text{ points}\right) \\
&\textbf{Case 3}: N \text{ is even } a+b \text{ is odd} \quad n'\Delta \in \Omega \\
&\quad \text{if } n' = \frac{1}{2}, \frac{3}{2}, \ldots\ldots, \frac{(N-1)}{2} \\
&\quad \left(\frac{N}{2}+1 \text{ points}\right) \\
&\textbf{Case 4}: N \text{ is odd } a+b \text{ is odd} \quad n'\Delta \in \Omega \\
&\quad \text{if } n' = \frac{1}{2}, \frac{3}{2}, \ldots\ldots, \frac{(N-2)}{2} \\
&\quad \left(\frac{(N-1)}{2}+1 \text{ points}\right) \\
&\text{where} \quad \Omega = \left[0, \sqrt{\frac{\pi N}{2}}\right], \quad \Delta = \sqrt{\frac{2\pi}{N}}. \qquad (104)
\end{aligned}$$

In Case 4, we exclude the sampling point $n'\Delta$, where $n' = N/2$ because $\psi_t(N\Delta/2) = 0$ for all t's when $a+b$ is odd. From the above results and the theorem of the Chebyshev set, we obtain that

$$\{d_{2t}[n] | t = 0, 1, 2, \ldots, T_0 - 1\} \quad (105)$$

form an independent vector set, and $T_0 = N/2+1, (N+1)/2+1, N/2+1$, and $(N-1)/2+1$ for Cases 1–4, respectively. Since $\{d_t[n] | t = 0, 1, 2, \ldots, T_0 - 1\}$ are independent, and $V_{2q,a,b}[n]$ is the linear combination of $d_0[n], d_1[n], \ldots\ldots, d_{q-1}[n]$

$$\{V_{2q,a,b}[n] | q = 0, 1, 2, \ldots, T_0 - 1\} \quad (106)$$

also forms an independent vectors set. Among the above eigenvector sets

$$\begin{aligned}
&\{V_{4m,a,b}[n] | m = 0, 1, 2, \ldots, T_1 - 1\} \\
&\{V_{4m+2,a,b}[n] | m = 0, 1, 2, \ldots, T_2 - 1\} \qquad (107)
\end{aligned}$$

belonging to the eigenvalue of $\exp(j\phi)$, $-\exp(j\phi)$, respectively. The values of T_1 and T_2 can be calculated from T_0. After some computation, we find that

$$\begin{aligned}
T_1 &= m+e, \quad T_2 = m \\
&\text{if } N = 4m \text{ or } 4m+1, \quad e = 1 \text{ if } a+b \text{ is even} \\
&\quad e = 0 \text{ if } a+b \text{ is odd} \\
T_1 &= m+1, \quad T_2 = m+e \\
&\text{if } N = 4m+2 \text{ or } 4m+3. \qquad (108)
\end{aligned}$$

That is, in (107), the values of T_1 and T_2 match the multiplicities of the eigenvalues $\exp(j\phi)$ and $-\exp(j\phi)$ found in Table I. In other words, if the multiplicities of $(-j)^q \exp(j\phi)$ $(q = 0, 2)$ are k, $\{V_{q+4h,a,b}[n], |h = 0, 1, 2, \ldots\ldots, k-1\}$ forms a complete independent eigenvectors set of the eigenspace of $(-j)^q \exp(j\phi)$.

We can use the similar way to prove that Theorem 1 is also satisfied when $q = 1$ or 3.

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