Isometric path numbers of graphs*

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Abstract

An isometric path between two vertices in a graph G is a shortest path joining them. The isometric path number of G, denoted by $\mathrm{ip}(G)$, is the minimum number of isometric paths needed to cover all vertices of G. In this paper, we determine exact values of isometric path numbers of complete r-partite graphs and Cartesian products of 2 or 3 complete graphs.

Keywords. Isometric path, complete r-partite graph, Hamming graphs

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1 Introduction

An *isometric path* between two vertices in a graph G is a shortest path joining them. The *isometric path number* of G, denoted by $\mathrm{ip}(G)$, is the minimum number of isometric paths required to cover all vertices of G. This concept has a close relationship with the game of cops and robbers described as follows.

The game is played by two players, the *cop* and the *robber*, on a graph. The two players move alternatively, starting with the cop. Each player's first move consists of choosing a vertex at which to start. At each subsequent move, a player may choose either to stay at the same vertex or to move to an adjacent vertex. The object for the cop is to catch the robber, and for the robber is to prevent this from happening. Nowakowski and Winkler [7] and Quilliot [9] independently proved that the cop wins if and only if the graph can be reduced to a single vertex by successively removing pitfalls, where a *pitfall* is a vertex whose closed neighborhood is a subset of the closed neighborhood of another vertex.

As not all graphs are cop-win graphs, Aigner and Fromme [1] introduced the concept of *cop-number* of a general graph G, denoted by $c(G)$, which is the minimum number of cops needed to put into the graph in order to catch the robber . On the way to give an upper bound for the cop-numbers of planar graphs, they showed that a single cop moving on an isometric path P guarantee that after a finite number of moves the robber will be immediately caught if he moves onto P. Observing this fact, Fitzpatrick [3] then introduced the concept of isometric path cover and pointed out that $c(G) \leq \mathrm{ip}(G)$.

The isometric path number of the Cartesian product $P_{n_1}\Box P_{n_2}\Box \ldots \Box P_{n_r}$ has been studied in the literature. Fitzpatrick [4] gave bounds for the case when $n_1 = n_2 = \ldots = n_r$. Fisher and Fitzpatrick [2] gave exact values for the case $r = 2$. Fitzpatrick et al [5] gave a lower bound, which is in fact the exact value if $r + 1$ is a power of 2, for the case when $n_1 = n_2 = \ldots = n_r = 2$. Pan and Chang [8] gave a linear-time algorithm to solve the isometric path problem on block graphs.

In this paper we determine exact values of isometric path numbers of all complete r-partite graphs and Cartesian products of 2 or 3 complete graphs. Recall that a *complete r-partite graph* is a graph whose vertex set can be partitioned into disjoint union of r nonempty parts, and two vertices are adjacent if and only if they are in different parts. We use $K_{n_1,n_2,\ldots,n_r}$ to denote a complete r-partite graph whose parts are of sizes $n_1, n_2, \ldots, n_r$, respectively. A *Hamming graph* is the Cartesian product of complete graphs, which is the graph $K_{n_1}\Box K_{n_2}\Box \ldots \Box K_{n_r}$ with vertex set

$$V(K_{n_1}\Box K_{n_2}\Box \ldots \Box K_{n_r}) = \{(x_1, x_2, \ldots, x_r) : 0 \leq x_i < n_i \text{ for } 1 \leq i \leq r\}$$

and edge set $E(K_{n_1}\Box K_{n_2}\Box \ldots \Box K_{n_r})$ is

$$\{(x_1, x_2, \ldots, x_r)(y_1, y_2, \ldots, y_r) : x_i = y_i \in V(K_i) \text{ for all } i \text{ except just one } x_j \neq y_j\}.$$

2 Complete r-partite graphs

The purpose of this section is to determine exact values of the isometric path numbers of all complete r-partite graphs.

Suppose G is the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ of n vertices, where $r \geq 2$, $n_1 \geq n_2 \geq \ldots \geq n_r$ and $n = n_1 + n_2 + \ldots + n_r$. Let G has α parts of odd sizes. We notice that every isometric path in G has at most 3 vertices. Consequently,

$$\mathrm{ip}(G) \geq \lceil n/3 \rceil.$$

Also, for any path of 3 vertices in an isometric path cover $\mathcal{C}$, two end vertices of the path is in a part of G and the center vertex in another part. In case when two paths of 3 vertices in $\mathcal{C}$ have a common end vertex, we may replace one by a path of 2 vertices. And, a path of 1 vertex can be replaced by a path of 2 vertices. So, without loss of generality, *we may only consider isometric path covers in which every path is of* 2 *or* 3 *vertices, and two* 3*-vertices paths have different end vertices.*

Lemma 1 *If* $3n_1 > 2n$*, then* $\mathrm{ip}(G) = \lceil n_1/2 \rceil$.

Proof. First, $\mathrm{ip}(G) \geq \lceil n_1/2 \rceil$ since every isometric path contains at most two vertices in the first part.

On the other hand, we use an induction on $n - n_1$ to prove that $\mathrm{ip}(G) \leq \lceil n_1/2 \rceil$. When $n - n_1 = 1$, we have $G = K_{n-1,1}$. In this case, it is clear that $\mathrm{ip}(G) \leq \lceil n_1/2 \rceil$. Suppose $n - n_1 \geq 2$ and the claim holds for $n' - n_1' < n - n_1$. Then we remove two vertices from the first part and one vertex from the second part to form an isometric 3-path P. Since $3n_1 > 2n$, we have $n_1 - 2 > 2(n - n_1 - 1) > 0$ and so $n_1 - 2 > n_2$. Then, the remaining graph G' has $r' \geq 2$, $n_1' = n_1 - 2$ and $n' = n - 3$. It then still satisfies $3n_1' > 2n'$. As $n' - n_1' = n - n_1 - 1$, by the induction hypothesis, $\mathrm{ip}(G') \leq \lceil n_1'/2 \rceil$ and so $\mathrm{ip}(G) \leq \lceil n_1'/2 \rceil + 1 = \lceil n_1/2 \rceil$. ∎

Lemma 2 *If* $3\alpha > n$*, then* $\mathrm{ip}(G) = \lceil (n + \alpha)/4 \rceil$.

Proof. Suppose $\mathcal{C}$ is an optimum isometric path cover with p_2 paths of 2 vertices and p_3 paths of 3 vertices. Then

$$2p_2 + 3p_3 \geq n.$$

Notice that there are at most $n - \alpha$ vertices in G can be paired up as the end vertices of the 3-paths in $\mathcal{P}$. Hence $p_3 \leq (n - \alpha)/2$ and so

$$2p_2 + 2p_3 \geq n - (n - \alpha)/2 = (n + \alpha)/2 \text{ or } \mathrm{ip}(G) = p_2 + p_3 \geq \lceil (n + \alpha)/4 \rceil.$$

On the other hand, we use an induction on $n - \alpha$ to prove that $\mathrm{ip}(G) \leq \lceil (n + \alpha)/4 \rceil$. When $n - \alpha \leq 1$, we have $n = \alpha$ and G is the complete graph of order n. So, $\mathrm{ip}(G) = \lceil n/2 \rceil = \lceil (n + \alpha)/4 \rceil$. Suppose $n - \alpha \geq 2$ and the claim holds for $n' - \alpha' < n - \alpha$. In this case, $3\alpha > n \geq \alpha + 2$ which implies $\alpha > 1$ and $n > 3$. Then we may remove two vertices from the first part of and one vertex form an odd part other than the first part to form a isometric 3-path P of G. The remaining graph G' has $n' = n - 3$ and $\alpha' = \alpha - 1$. It then satisfies $3\alpha' > n'$. Notice that $r' \geq 2$ unless $G = K_{2,1,1}$ in which $n = 4$ and $\alpha = 2$ imply $\mathrm{ip}(G) = 2 = \lceil (n+\alpha)/4 \rceil$. By the induction hypothesis, $\mathrm{ip}(G') \leq \lceil (n' + \alpha')/4 \rceil$ and so $\mathrm{ip}(G) \leq \lceil (n' + \alpha')/4 \rceil + 1 = \lceil (n + \alpha)/4 \rceil$. ∎

Lemma 3 *If* $3n_1 \leq 2n$ *and* $3\alpha \leq n$, *then* $\mathrm{ip}(G) = \lceil n/3 \rceil$.

Proof. Since every isometric path in G has at most 3 vertices, $\mathrm{ip}(G) \geq \lceil n/3 \rceil$.

On the other hand, we use an induction on n to prove that $\mathrm{ip}(G) \leq \lceil n/3 \rceil$. When $n \leq 8$, by the assumptions that $3n_1 \leq 2n$ and $3\alpha \leq n$ we have $G \in \{K_{2,1}, K_{2,2}, K_{3,2}, K_{2,2,1}, K_{4,2}, K_{4,1,1}, K_{3,3}, K_{3,2,1}, K_{2,2,2}, K_{2,2,1,1}, K_{4,3}, K_{4,2,1}, K_{3,2,2}, K_{2,2,2,1}, K_{5,3}, K_{5,2,1}, K_{4,4}, K_{4,3,1}, K_{4,2,2}, K_{4,2,1,1}, K_{3,3,2}, K_{3,2,2,1}, K_{2,2,2,2}, K_{2,2,2,1,1}\}$. It is straightforward to check that $\mathrm{ip}(G) \leq \lceil n/3 \rceil$.

Suppose $n \geq 9$ and the claim holds for $n' < n$. We remove two vertices from the first part and one vertex from the jth part to form an isometric 3-path P for G, where j is the largest index such that $j \geq 2$ and n_j is odd (when n_i are even for all $i \geq 2$, we choose $j = r$). Then, the remaining subgraph G' has $n' = n - 3$ and $\alpha' = \alpha - 1$ or $\alpha' \leq 2$. Therefore, $3\alpha \leq n$ and $n \geq 9$ imply that $3\alpha' \leq n'$ in any case. We shall prove that $3n'_1 \leq 2n'$ according to the following cases.

Case 1. $n_1 \geq n_2 + 2$.

In this case, $n_1 - 2 \geq n_2 \geq n_i$ for all $i \geq 2$ and so $n'_1 = n_1 - 2$. Therefore, $3n'_1 = 3(n_1 - 2) \leq 2(n - 3) = 2n'$.

Case 2. $n_1 \leq n_2 + 1$ and $n_2 \leq 4$.

In this case, $n'_1 \leq n_2 \leq 4$ and $n' \geq 6$. Then, $3n'_1 \leq 12 \leq 2n'$.

Case 3. $n_1 \leq n_2 + 1$ and $n_2 \geq 5$ and $r = 2$.

In this case, $n'_1 \leq n_2 - 1$ and $n' = n - 3 = n_1 + n_2 - 3 \geq 2n_2 - 3$. Then, $3n'_1 \leq 3n_2 - 3 \leq 4n_2 - 8 < 2n'$.

Case 4. $n_1 \leq n_2 + 1$ and $n_2 \geq 5$ and $r \geq 3$.

In this case, $n'_1 \leq n_2$ and $n' = n - 3 \geq n_1 + n_2 + 1 - 3 \geq 2n_2 - 2$. Then, $3n'_1 \leq 3n_2 \leq 4n_2 - 5 < 2n'$. ∎

According to Lemma 1, 2 and 3, we have the following theorem.

Theorem 4 *Suppose* G *is the complete* r*-partite graph* $K_{n_1,n_2,\ldots,n_r}$ *of* n *vertices with* $r \geq 2$, $n_1 \geq n_2 \geq \ldots \geq n_r$ *and* $n = n_1 + n_2 + \ldots + n_r$. *If there are exactly* α *indices* i *with* n_i *odd, then*

$$\mathrm{ip}(G) = \begin{cases} \lceil n_1/2 \rceil, & \text{if } 3n_1 > 2n; \\ \lceil (n+\alpha)/4 \rceil, & \text{if } 3\alpha > n; \\ \lceil n/3 \rceil, & \text{if } 3\alpha \leq n \text{ and } 3n_1 \leq 2n. \end{cases}$$

In the proofs of the lemmas above, the essential points for the arguments is not the fact that each partite set of the complete r-partite graph is trivial. If we add some edges into the graph but still keep that each partite set can be partitioned into $\lfloor n_i/2 \rfloor$ pairs of two nonadjacent vertices and $n_i - 2\lfloor n_i/2 \rfloor$ vertex, then the same result still holds.

Corollary 5 *Suppose G is the graph obtained from the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ of n vertices by adding edges such that each i-th part can be partitioned into $\lfloor n_i/2\rfloor$ pairs of two nonadjacent vertices and $n_i - 2\lfloor n_i/2\rfloor$ vertex, where $r \geq 2$, $n_1 \geq n_2 \geq \ldots \geq n_r$ and $n = n_1 + n_2 + \ldots + n_r$. If there are exactly α indices i with n_i odd, then*

$$\mathrm{ip}(G) = \begin{cases} \lceil n_1/2\rceil, & \text{if } 3n_1 > 2n; \\ \lceil (n+\alpha)/4\rceil, & \text{if } 3\alpha > n; \\ \lceil n/3\rceil, & \text{if } 3\alpha \leq n \text{ and } 3n_1 \leq 2n. \end{cases}$$

3 Hamming graphs

This section establishes isometric path numbers of Cartesian products of 2 or 3 complete graphs.

Suppose G is the Hamming graph $K_{n_1} \Box K_{n_2} \Box \ldots \Box K_{n_r}$ of n vertices, where $n = n_1 n_2 \ldots n_r$ and $n_i \geq 2$ for $1 \leq i \leq r$. We notice that every isometric path in G has at most $r+1$ vertices. Consequently,

$$\mathrm{ip}(G) \geq \lceil n/(r+1)\rceil.$$

Recall that the vertex set of $K_{n_1} \Box K_{n_2} \Box \ldots \Box K_{n_r}$ is

$$V(K_{n_1} \Box K_{n_2} \Box \ldots \Box K_{n_r}) = \{(x_1, x_2, \ldots, x_r) : 0 \leq x_i < n_i \text{ for } 1 \leq i \leq r\}.$$

We first consider the case when $r = 2$

Theorem 6 *If $n_1 \geq 2$ and $n_2 \geq 2$, then* $\mathrm{ip}(K_{n_1} \Box K_{n_2}) = \lceil n_1 n_2/3\rceil$.

Proof. We only need to prove that $\mathrm{ip}(K_{n_1} \Box K_{n_2}) \leq \lceil n_1 n_2/3\rceil$. We shall prove this assertion by induction on $n_1 + n_2$. For the case when $n_1 + n_2 \leq 6$, the isometric path covers

$$\begin{aligned} \mathcal{C}_{2,2} &= \{(0,0)(0,1), (1,0)(1,1)\}, \\ \mathcal{C}_{2,3} &= \{(0,0)(0,1)(1,1), (0,2)(1,2)(1,0)\}, \\ \mathcal{C}_{2,4} &= \{(0,0)(0,1)(1,1), (0,2)(1,2)(1,0), (0,3)(1,3)\} \text{ and} \\ \mathcal{C}_{3,3} &= \{(0,0)(2,0)(2,2), (0,1)(0,2)(1,2), (1,0)(1,1)(2,1)\} \end{aligned}$$

for $K_2 \Box K_2$, $K_2 \Box K_3$, $K_2 \Box K_4$ and $K_3 \Box K_3$ respectively, gives the assertion.

Suppose $n_1 + n_2 \geq 7$ and the assertion holds for $n_1' + n_2' < n_1 + n_2$. For the case when all $n_i \leq 4$, without loss of generality we may assume that $n_1 = 4$ and $3 \leq n_2 \leq 4$. As we can partition the vertex set of $K_{n_1} \Box K_{n_2}$ into the vertex sets of two copies of distance invariant induced subgraphs $K_2 \Box K_{n_2}$,

$$\mathrm{ip}(K_{n_1} \Box K_{n_2}) \leq 2\mathrm{ip}(K_2 \Box K_{n_2}) \leq 2\lceil 2n_2/3\rceil = \lceil n_1 n_2/3\rceil.$$

For the case when there is at least one $n_i \geq 5$, say $n_1 \geq 5$, again we can partition the vertex set of $K_{n_1} \square K_{n_2}$ into the vertex sets of two distance invariant induced subgraphs $K_3 \square K_{n_2}$ and $K_{n_1-3} \square K_{n_2}$. Then,

$$\mathrm{ip}(K_{n_1} \square K_{n_2}) \leq \mathrm{ip}(K_3 \square K_{n_2}) + \mathrm{ip}(K_{n_1-3} \square K_{n_2}) \leq \lceil 3n_2/3 \rceil + \lceil (n_1-3)n_2/3 \rceil = \lceil n_1 n_2/3 \rceil.$$

∎

Lemma 7 *If n_1, n_2 and n_3 are positive even integers, then*

$$\mathrm{ip}(K_{n_1} \square K_{n_2} \square K_{n_3}) = n_1 n_2 n_3/4.$$

Proof. We only need to prove that $\mathrm{ip}(K_{n_1} \square K_{n_2} \square K_{n_3}) \leq n_1 n_2 n_3/4$. First, the isometric path cover $\mathcal{C}_{2,2,2} = \{(0,0,0)(0,0,1)(0,1,1)(1,1,1), (1,0,1)(1,0,0)(1,1,0)(0,1,0)\}$ for $K_2 \square K_2 \square K_2$ proves the assertion for the case when $n_1 = n_2 = n_3 = 2$. For the general case, as the vertex set of $K_{n_1} \square K_{n_2} \square K_{n_3}$ can be partitioned into the vertex sets of $n_1 n_2 n_3/8$ copies of distance invariant induced subgraphs $K_2 \square K_2 \square K_2$,

$$\mathrm{ip}(K_{n_1} \square K_{n_2} \square K_{n_3}) \leq (n_1 n_2 n_3/8)\mathrm{ip}(K_2 \square K_2 \square K_2) \leq n_1 n_2 n_3/4.$$

∎

Lemma 8 *If $n_3 \geq 3$ is odd, then* $\mathrm{ip}(K_2 \square K_2 \square K_{n_3}) = n_3 + 1$.

Proof. First, we claim that $\mathrm{ip}(K_2 \square K_2 \square K_{n_3}) \geq n_3 + 1$. Suppose to the contrary that the graph can be covered by n_3 isometric paths

$$P_i : (x_{i1}, x_{i2}, x_{i3})(y_{i1}, y_{i2}, y_{i3})(z_{i1}, z_{i2}, z_{i3})(w_{i1}, w_{i2}, w_{i3}),$$

$i = 1, 2, \ldots, n_3$. These paths are in fact vertex-disjoint paths of 4 vertices, each contains exactly one type-j edge for $j = 1, 2, 3$, where an edge $(x_1, x_2, x_3)(y_1, y_2, y_3)$ is type-j if $x_j \neq y_j$. For each P_i we then have $x_{i1} = 1 - w_{i1}$ and $x_{i2} = 1 - w_{i2}$, which imply that $x_{i1} + x_{i2}$ has the same parity with $w_{i1} + w_{i2}$. We call the path P_i *even* or *odd* when $x_{i1} + x_{i2}$ is even or odd, respectively. Also, as P_i has just one type-3 edge, by symmetric, we may assume either $x_{i3} \neq y_{i3} = z_{i3} = w_{i3}$ or $x_{i3} = y_{i3} \neq z_{i3} = w_{i3}$, for which we call P_i type 1-3 or type 2-2 respectively. For a type 2-2 path P_i we may further assume that $x_{i1} \neq y_{i1} = z_{i1} = w_{i1}$.

For $0 \leq x_3 < n_3$, the x_3-*square* is the set $S(x_3) = \{(0,0,x_3), (0,1,x_3), (1,0,x_3), (1,1,x_3)\}$. Notice that a type 1-3 path P_i contains 1 vertex in $S(x_{i3})$ and 3 vertices in $S(w_{i3})$, while a type 2-2 path P_i contains 2 vertices in $S(x_{i3})$ and 2 vertices in $S(w_{i3})$. We call a type 1-3 path P_i is *adjacent to* another type 1-3 path P_j if the last 3 vertices of P_i and the first vertex of P_j form a square. This defines a digraph D whose vertices are all type 1-3 paths, in which each vertex has out-degree one and in-degree at most one. In fact, each vertex then has in-degree one. In other words, the "adjacent to" is a bijection. Consequently, vertices of all type 1-3 paths together form p squares; and so vertices of all type 2-2 paths form the other $n_3 - p$ squares.

Since $x_{i1} \neq y_{i1} = z_{i1} = w_{i1}$ for a type 2-2 path P_i, the first two vertices of a type 2-2 path together with the first two vertices of another type 2-2 path form a square. This shows that there is an even number of type 2-2 paths. Therefore, there is an odd number of type 1-3 paths.

On the other hand, in a type 1-3 path P_i we have $x_{i_1} + x_{i_2} = y_{i_1} + y_{i_2}$ has the different parity with $z_{i_1} + z_{i_3}$, and the same parity with $w_{i_1} + w_{i_2}$. So it is adjacent to a type 1-3 path whose parity is the same as $z_{i_1} + z_{i_2}$. That is, a type 1-3 path is adjacent to a type 1-3 path of different parity. Therefore, the digraph D is the union of some even directed cycle. This is a contradiction to the fact that there is an odd number of type 1-3 paths.

The arguments above prove that $\mathrm{ip}(K_2 \Box K_2 \Box K_{n_3}) \geq n_3+1$. On the other hand, since the vertex set of $K_2 \Box K_2 \Box K_{n_3}$ is the union of the vertex sets of $(n_3+1)/2$ copies of $K_2 \Box K_2 \Box K_2$, by the cover $\mathcal{C}_{2,2,2}$ in the proof of Lemma 7, we have $\mathrm{ip}(K_2 \Box K_2 \Box K_{n_3}) \leq n_3 + 1$. ∎

Theorem 9 *If all $n_i \geq 2$, then $\mathrm{ip}(K_{n_1} \Box K_{n_2} \Box K_{n_3}) = \lceil n_1 n_2 n_3/4 \rceil$ except for the case when two n_i are 2 and the third is odd. In the exceptional case, $\mathrm{ip}(K_{n_1} \Box K_{n_2} \Box K_{n_3}) = n_1 n_2 n_3/4 + 1$.*

Proof. The exceptional case holds according to Lemma 8.

For the main case, by Lemma 7, we may assume that at least one n_i is odd. Again, we only need to prove that $\mathrm{ip}(K_{n_1} \Box K_{n_2} \Box K_{n_3}) \leq \lceil n_1 n_2 n_3/4 \rceil$. We shall prove the assertion by induction on $\sum_{i=1}^{3} n_i$. For the case when $\sum_{i=1}^{3} n_i \leq 10$, the following isometric path covers for $K_2 \Box K_3 \Box K_3$, $K_2 \Box K_3 \Box K_4$, $K_3 \Box K_3 \Box K_3$ and $K_3 \Box K_3 \Box K_4$, respectively, prove the assertion:

$$\begin{aligned} \mathcal{C}_{2,3,3} &= \{(0,1,1)(0,1,0)(0,0,0)(1,0,0),\ (0,2,2)(0,2,0)(1,2,0)(1,1,0), \\ &\quad (0,2,1)(1,2,1)(1,1,1),\ (0,0,2)(0,1,2)(1,1,2), \\ &\quad (0,0,1)(1,0,1)(1,0,2)(1,2,2)\}; \end{aligned}$$

$$\left(\begin{aligned} \text{Let } \mathcal{C}^*_{2,3,3} &= \mathcal{C}_{2,3,3} \backslash \{(0,2,1)(1,2,1)(1,1,1), (0,0,2)(0,1,2)(1,1,2)\} \cup \\ &\quad \{(0,2,1)(1,2,1)(1,1,1)(1,1,3), (0,0,2)(0,1,2)(1,1,2)(1,1,4)\}. \end{aligned} \right)$$

$$\begin{aligned} \mathcal{C}_{2,3,4} &= \{(0,1,1)(0,1,0)(0,0,0)(1,0,0),\ (0,2,1)(0,2,0)(1,2,0)(1,1,0), \\ &\quad (0,2,3)(0,2,2)(1,2,2)(1,1,2),\ (0,1,3)(0,1,2)(0,0,2)(1,0,2), \\ &\quad (0,0,1)(1,0,1)(1,1,1)(1,1,3),\ (1,2,1)(1,2,3)(1,0,3)(0,0,3)\}; \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{2,3,5} &= \mathcal{C}^*_{2,3,3} \cup \{(0,1,4)(0,1,3)(0,2,3)(1,2,3),\ (0,0,3)(0,0,4)(0,2,4)(1,2,4), \\ &\quad (1,0,3)(1,0,4)\}; \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{3,3,3} &= \{(0,0,0)(0,2,0)(1,2,0)(1,2,1),\ (1,1,0)(2,1,0)(2,2,0)(2,2,1), \\ &\quad (0,2,1)(0,1,1)(1,1,1)(1,1,2),\ (1,0,1)(2,0,1)(2,1,1)(2,1,2), \\ &\quad (0,1,0)(0,1,2)(0,2,2)(1,2,2),\ (0,0,1)(0,0,2)(2,0,2)(2,2,2), \\ &\quad (1,0,2)(1,0,0)(2,0,0)\}; \end{aligned}$$

$$
\begin{aligned}
\mathcal{C}_{3,3,4} \;=\; & \{(0,0,0)(0,2,0)(1,2,0)(1,2,1),\ (1,1,0)(2,1,0)(2,2,0)(2,2,1),\\
& (0,2,1)(0,1,1)(1,1,1)(1,1,2),\ (1,0,1)(2,0,1)(2,1,1)(2,1,2),\\
& (0,1,0)(0,1,2)(0,2,2)(1,2,2),\ (0,0,2)(2,0,2)(2,2,2)(2,2,3),\\
& (0,1,3)(1,1,3)(1,0,3)(1,0,2),\ (1,0,0)(2,0,0)(2,0,3)(2,1,3),\\
& (0,0,1)(0,0,3)(0,2,3)(1,2,3)\}.
\end{aligned}
$$

Suppose $\sum_{i=1}^{3} n_i \geq 11$ and the assertion holds for $\sum_{i=1}^{3} n'_i < \sum_{i=1}^{3} n_i$. We shall consider the following cases.

For the case when there is some i, say $i = 3$, such that $n_3 \geq 7$ or $n_3 = 6$ with all $n_j \geq 3$, we have $\mathrm{ip}(K_{n_1} \square K_{n_2} \square K_{n_3}) \leq \mathrm{ip}(K_{n_1} \square K_{n_2} \square K_4) + \mathrm{ip}(K_{n_1} \square K_{n_2} \square K_{n_3-4}) \leq \lceil n_1 n_2 4/4 \rceil + \lceil n_1 n_2 (n_3 - 4)/4 \rceil = \lceil n_1 n_2 n_3/4 \rceil$.

For the case when some n_i, say n_3, is equal to 4, we may assume $n_1 \geq n_2$ and so $n_1 \geq 4$. Then $\mathrm{ip}(K_{n_1} \square K_{n_2} \square K_4) \leq \mathrm{ip}(K_2 \square K_{n_2} \square K_4) + \mathrm{ip}(K_{n_1-2} \square K_{n_2} \square K_4) = \lceil 2n_2 4/4 \rceil + \lceil (n_1 - 2) n_2 4/4 \rceil = \lceil n_1 n_2 n_3/4 \rceil$.

There are 6 remaining cases. The following isometric path covers prove the assertion for $K_2 \square K_3 \square K_6$, $K_2 \square K_5 \square K_5$ and $K_3 \square K_5 \square K_5$, respectively:

$$
\begin{aligned}
\mathcal{C}_{2,3,6} \;=\; & \mathcal{C}^*_{2,3,3} \cup \{(0,0,4)(0,0,3)(1,0,3)(1,2,3),\ (0,1,3)(0,1,4)(0,2,4)(1,2,4),\\
& (0,2,3)(0,2,5)(1,2,5)(1,1,5),\ (0,1,5)(0,0,5)(1,0,5)(1,0,4)\};
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}_{2,5,5} \;=\; & \mathcal{C}_{2,3,5} \backslash \{(1,0,3)(1,0,4)\} \cup\\
& \{(0,4,1)(0,4,0)(0,3,0)(1,3,0),\ (1,4,0)(1,4,1)(1,3,1)(0,3,1),\\
& (0,4,3)(0,4,2)(0,3,2)(1,3,2),\ (1,4,2)(1,4,3)(1,3,3)(0,3,3),\\
& (1,0,3)(1,0,4)(1,4,4),\ (0,4,4)(0,3,4)(1,3,4)\};
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}_{3,5,5} \;=\; & \mathcal{C}_{2,3,5} \backslash \{(1,0,3)(1,0,4)\} \cup\\
& \{(0,4,0)(2,4,0)(2,0,0)(2,0,1),\ (0,3,0)(2,3,0)(2,1,0)(2,1,1),\\
& (0,4,1)(0,3,1)(1,3,1)(1,3,0),\ (1,4,0)(1,4,1)(2,4,1)(2,2,1),\\
& (1,0,3)(2,0,3)(2,2,3)(2,2,0),\ (1,0,4)(2,0,4)(2,3,4)(2,3,1),\\
& (0,3,2)(2,3,2)(2,1,2)(2,1,3),\ (0,4,4)(0,4,2)(2,4,2)(2,0,2),\\
& (0,4,3)(1,4,3)(1,3,3)(1,3,2),\ (0,3,3)(2,3,3)(2,4,3)(2,4,4),\\
& (0,3,4)(1,3,4)(1,4,4)(1,4,2),\ (2,2,2)(2,2,4)(2,1,4)\}.
\end{aligned}
$$

The other 3 cases follows from the following inequalities:

$$\mathrm{ip}(K_2 \square K_5 \square K_6) \leq \mathrm{ip}(K_2 \square K_3 \square K_6) + \mathrm{ip}(K_2 \square K_2 \square K_6) \leq 9 + 6 = 15,$$

$$\mathrm{ip}(K_3 \square K_3 \square K_5) \leq \mathrm{ip}(K_3 \square K_3 \square K_2) + \mathrm{ip}(K_3 \square K_3 \square K_3) \leq 5 + 7 = 12,$$

$$\mathrm{ip}(K_5 \square K_5 \square K_5) \leq \mathrm{ip}(K_5 \square K_5 \square K_3) + \mathrm{ip}(K_5 \square K_5 \square K_2) \leq 19 + 13 = 32.$$

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