# Mathematics for physicists

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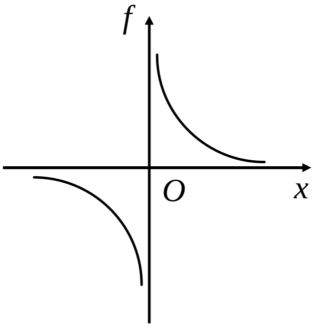
## 11. Continuity

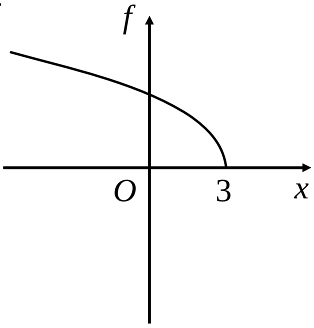
The function $f(x)$ is continuous on an interval $(a, b)$. If for any $c \in (a, b)$, $f(c)$ is defined, $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) = f(c)$.

**(1).** $f(x) = \dfrac{1}{x}$



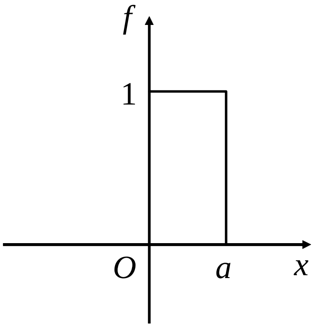

$f$ is continuous on $(-\infty, 0)$ and $(0, \infty)$.

**(2).** $f(x) = \sqrt{3 - x}$



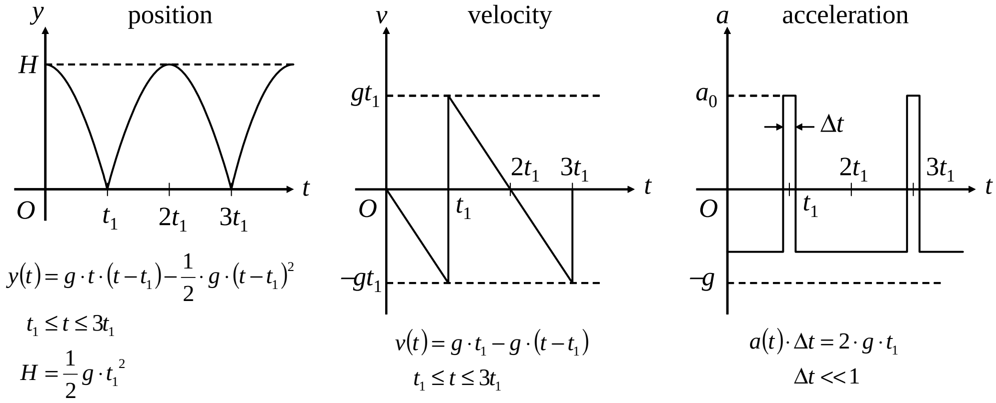

$f$ is continuous on $(-\infty, 3]$.

**(3).** $f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$



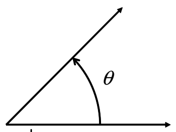

$f$ is not continuous on $x = 0$ and $x = a$.

## 12. Motion of a falling elastic bead

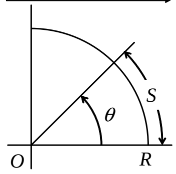



## New Topics.

## 1. Trigonometric function

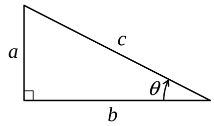



angle $\theta$




$$S = R \cdot \theta \quad \theta \text{ in radian}$$

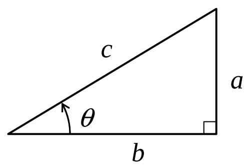
$$\therefore 2\pi \text{ radian} = 360^\circ$$

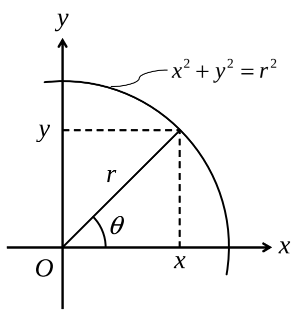



$a^2 + b^2 = c^2$ Pythagorean theorem

## 2. Definitions:

| | |
|---|---|
| sine | $\sin\theta = \frac{a}{c} = \frac{y}{r}$ |
| cosine | $\cos\theta = \frac{b}{c} = \frac{x}{r}$ |
| tangent | $\tan\theta = \frac{a}{b} = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$ |
| cotangen | $\cot\theta = \frac{b}{a} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}$ |
| secant | $\sec\theta = \frac{c}{b} = \frac{1}{\cos\theta} = \frac{r}{x}$ |
| cosecant | $\csc\theta = \frac{c}{a} = \frac{1}{\sin\theta} = \frac{r}{y}$ |



**3. Identities:**

**(1).** $\cos^2\theta + \sin^2\theta = 1$

$$\because \frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{(a^2+b^2)}{c^2} = 1$$

$$\text{or} \quad \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{(x^2+y^2)}{r^2} = 1$$

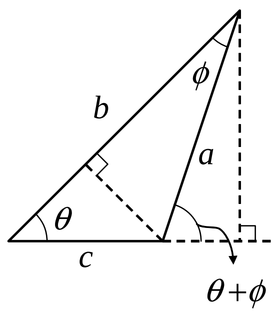
**(2).** $\tan^2 + 1 = \sec^2\theta$

$$\because \frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \left(\frac{1}{\cos\theta}\right)^2$$

**(3).** $\cot^2\theta + 1 = \csc^2\theta$

**(4).** $\sin(-\theta) = -\sin\theta$ , $\cos(-\theta) = \cos\theta$ , $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

**(5).** $\cos(\theta \pm \phi) = \cos\theta \cdot \cos\phi \mp \sin\theta \cdot \sin\phi$




$$\because a \cdot \cos(\theta+\phi) = b \cdot \cos\theta - c$$

$$= (a \cdot \cos\phi + c \cdot \cos\theta) \cdot \cos\theta - c$$

$$= a \cdot \cos\theta \cdot \cos\phi - c \cdot \sin^2\theta \qquad \curvearrowleft \because \cos^2\theta - 1 = -\sin^2\theta$$

$$= a \cdot (\cos\theta \cdot \cos\phi - \sin\theta \cdot \sin\phi) \qquad \curvearrowleft \because c \cdot \sin\theta = a \cdot \sin\phi$$

**(6).** $\sin(\theta \pm \phi) = \sin\theta \cdot \cos\phi \pm \cos\theta \cdot \sin\phi$

$$\because a \cdot \sin(\theta+\phi) = b \cdot \sin\theta = (a \cdot \cos\phi + c \cdot \cos\theta) \cdot \sin\theta$$

$$= a \cdot \sin\theta \cdot \cos\phi + c \cdot \sin\theta \cdot \cos\theta$$

$$= a \cdot (\sin\theta \cdot \cos\phi + \cos\theta \cdot \sin\phi) \qquad \curvearrowleft \because c \cdot \sin\theta = a \cdot \sin\phi$$

or

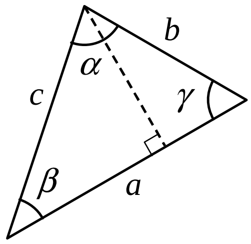
$$\because \sin(\theta+\phi) = \cos\left(\frac{\pi}{2} - (\theta+\phi)\right) = \cos\left(\frac{\pi}{2} - \theta\right) \cdot \cos\phi + \sin\left(\frac{\pi}{2} - \theta\right) \cdot \sin\phi$$

$$= \sin\theta \cdot \cos\phi + \cos\theta \cdot \sin\phi$$

**(7).** $a^2 = b^2 + c^2 - 2b \cdot c \cdot \cos\theta$

$$\because a^2 = (b \cdot \sin\theta)^2 + (b \cdot \cos\theta - c)^2 = b^2 \cdot \left(\sin^2\theta + \cos^2\theta\right) - 2 \cdot b \cdot c \cdot \cos\theta + c^2$$

**(8).** $a/\sin\alpha = b/\sin\beta = c/\sin\gamma$




$\because b\cdot\sin\gamma = c\cdot\sin\beta$

$\therefore \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$ etc.

**(9).** $\sin 2\theta = 2\sin\theta\cdot\cos\theta$

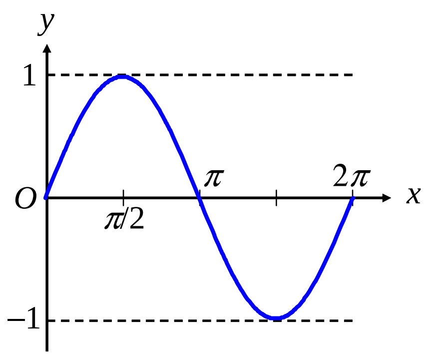
**(10).** $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\cdot\sin^2\theta = 2\cdot\cos^2\theta - 1$

**(11).** $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

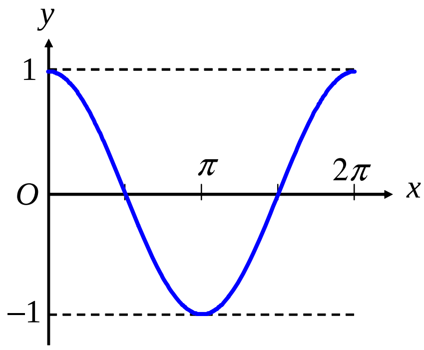
**(12).** $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

## 4. Graph

**(1).** $y = \sin x$



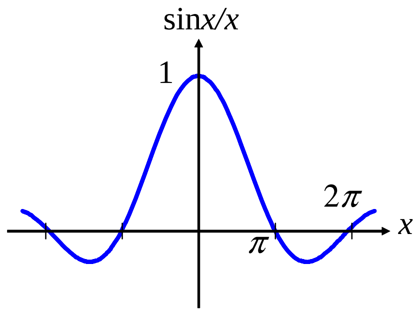

**(2).** $y = \cos x$



## 5. Limit

**(1).** $\lim_{x \to 0} \frac{\sin x}{x} = 1$

**(2).** $\lim_{x \to 0} \frac{\sin 4x}{x} = 4$




For reference

power series expansion of $\sin\theta$ and $\cos\theta$; useful approximation for $\theta \ll 1$

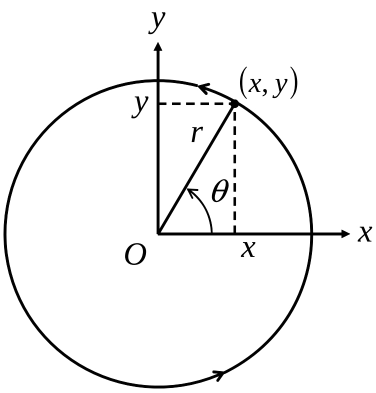
$$\sin\theta = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\theta^{2n+1}}{(2n+1)!} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

$$\cos\theta = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\theta^{2n}}{(2n)!} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots$$

$$n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$$

## 6. Applications

**(1). uniform circular motion** coordinates: $(x, y)$




$$x = r \cdot \cos\theta$$

$$y = r \cdot \sin\theta$$

$$\theta = \omega \cdot t + \theta_0$$

$t \to$ time

$\omega = \frac{\Delta\theta}{\Delta t} \to$ angular velocity

$\theta_0 \to$ initial angle

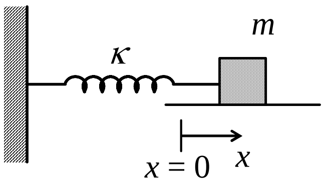
$$\therefore x(t) = r \cdot \cos(\omega t + \theta_0)$$

$$y(t) = r \cdot \sin(\omega t + \theta_0)$$

$$x(0) = r \cdot \cos\theta_0 \quad , \quad y(0) = r \cdot \sin\theta_0 \rightarrow \text{initial position}$$

projection on $x$-axis: $x(t) = r \cdot \cos(\omega t + \theta_0) \rightarrow$ simple harmonic motion-like

**(2). Simple harmonic motion of the mass-spring system**



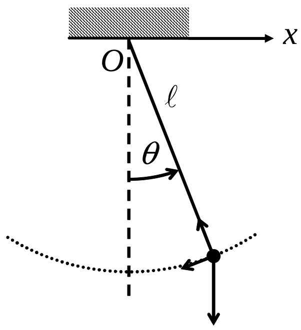

$x(t) = A \cdot \cos(\omega t + \theta_0)$

$\omega = (\kappa/m)^{\frac{1}{2}}$ , $k$ : force const. of the spring, $m$ : mass

$A \rightarrow$ Amplitude, $\theta_0 \rightarrow$ initial phase angle

$\omega \rightarrow$ angular frequency

**(3). Simple pendulum**




$x = \ell \cdot \sin\theta$

$\because \theta \ll 1 \qquad \sin\theta \approx \theta$

$\therefore x = \ell \cdot \theta$

$\theta = \theta(t) = \theta_A \cdot \cos(\omega t + \delta_0)$

$\therefore x = A \cdot \cos(\omega t + \delta_0)$ , $A = \ell \cdot \theta_A \rightarrow$ amplitude

$x(t)$ and $\theta(t)$ are both simple harmonic functions of time.

$\omega = \left(\frac{g}{\ell}\right)^{\frac{1}{2}}$ , $\delta_0$ : initial phase angle , $g$ : acceleration due to gravity