# Mathematics for physicists

Lecturer: Prof. Ven-Chung Lee

(Dated: December 17, 2005)

**Applications of trigonometry: wave and interference**

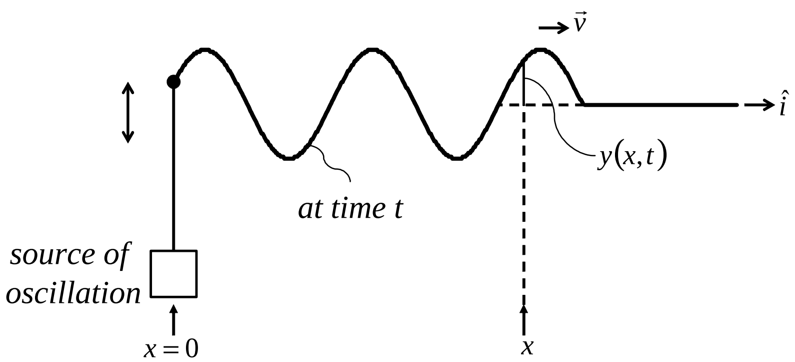
**1. Oscillation function of time**

$$y\left(t\right) = A \cdot \cos 2\pi\nu \cdot t \qquad , \quad A : \text{Amplitude}$$

$$\text{period} : \tau \to y\left(t + \tau\right) = y\left(t\right)$$

$$\therefore \nu \cdot \tau = 1 \ , \quad \therefore \nu = \frac{1}{\tau} \to \text{frequency}$$

**2. Wave propagates along $x$-axis with frequency $\nu$ and velocity $v$**




$$t_p = \frac{x}{v}$$

$$\therefore y\left(x,t\right) = y\left(0, t - t_p\right)$$

at $x = 0$

$$y\left(0,t\right) = A \cdot \cos 2\pi\nu t$$

$$\therefore y\left(x,t\right) = A \cdot \cos\left(2\pi\nu t - \frac{2\pi\nu \cdot x}{v}\right) \to \text{wave function of position } x \text{ and time } t$$

$v$ : velocity of simple harmonic wave

$$2\pi\nu \cdot t - 2\pi\nu \cdot \frac{x}{v} = \phi = 2\pi\nu t - 2\pi\frac{x}{\lambda} \qquad \phi : \text{phase of the wave function}$$

for a point with constant phase $\phi$ on the wave

$$2\pi\nu \cdot t - 2\pi\frac{x}{\lambda} = 2\pi\nu \cdot (t + \Delta t) - 2\pi \cdot \frac{(x + \Delta x)}{\lambda}$$

$$\therefore \frac{\Delta x}{\Delta t} = \lambda \cdot \nu = v$$

$\rightarrow$ constant phase velocity or the velocity of the constant phase point on the wave

$$\text{and} \qquad y(x,t) = y(x+\lambda, t) = y(x, t+\tau)$$

$\therefore \lambda \rightarrow$ wave length

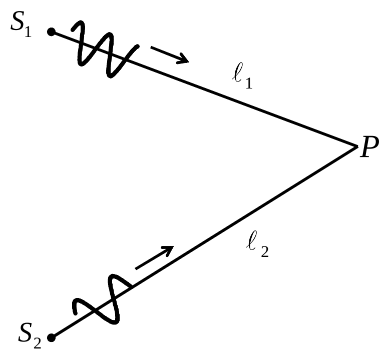
### 3. Interference of waves

wave sources $S_1$ and $S_2$ generate wave with:

$y(t) = A \cdot \cos 2\pi\nu t = A \cdot \cos\omega \cdot t$

$\omega \equiv 2\pi\nu$ angular frequency

$\rightarrow$ change of phase per unit time




total wave at $P \rightarrow$

$$y_{\text{total}} = y_1 + y_2 \qquad \text{(Principle of superposition)}$$
$$= A \cdot \cos(\omega t - k \cdot \ell_1) + A \cdot \cos(\omega t - k \cdot \ell_2)$$

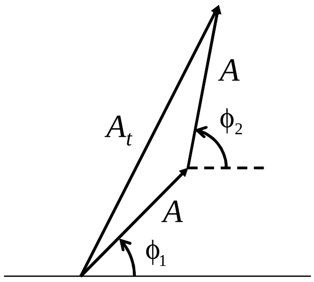
$$k \equiv \frac{2\pi}{\lambda} \quad \text{wave number} \implies \text{change of phase per unit length}$$

set

$$A \cdot \cos(\omega t - k \cdot \ell_1) + A \cdot \cos(\omega t - k \cdot \ell_2) = A_t \cos(\omega t - \quad )$$

⌜ Trigonometry:

$$\cos\phi_1 + \cos\phi_2 = A_t \cdot \cos\phi_3$$




by cosine theorem

$$A_t^2 = 2A^2 - 2A^2 \cos\left(\pi - (\phi_2 - \phi_1)\right)$$
$$= 2A^2 \cdot \left(1 + \cos\left(\phi_2 - \phi_1\right)\right)$$
$$= 4A^2 \cdot \cos^2\left(\frac{\phi_2 - \phi_1}{2}\right) \qquad \hookleftarrow \because \cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$\therefore A_t = 2A \cdot \cos\left(\frac{\phi_2 - \phi_1}{2}\right), \quad \phi_3 = \phi_1 + \frac{1}{2}\left(\phi_2 - \phi_1\right) = \frac{1}{2}\left(\phi_2 + \phi_1\right)$$

$$\therefore \cos\phi_1 + \cos\phi_2 = 2A \cdot \cos\left(\frac{\phi_2 - \phi_1}{2}\right) \cos\left(\frac{\phi_2 + \phi_1}{2}\right)$$

Similarly,

$$\sin\phi_1 + \sin\phi_2 = 2 \cdot \cos\left(\frac{\phi_2 - \phi_1}{2}\right) \sin\left(\frac{\phi_2 + \phi_1}{2}\right)$$

∟

Finally,

$$y_{\text{total}} = 2A\cos\left(k \cdot \frac{\ell_2 - \ell_1}{2}\right) \cdot \cos\left(\omega t - k \cdot \frac{\ell_1 + \ell_2}{2}\right)$$

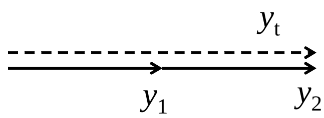
Time averaged intensity of the wave $\propto$ (Amplitude)$^2$ or $I = \alpha \cdot A^2$

Intensity: energy transferred per unit time by the wave.

$$\therefore \text{Intensity at } P \rightarrow I_P = \alpha \cdot A_t^2 = 4\alpha \cdot A^2 \cos^2\left(\frac{k \cdot \Delta\ell}{2}\right) = 4I_0 \cos^2\left(\frac{k \cdot \Delta\ell}{2}\right)$$

**case 1**

$$\Delta\ell = \ell_1 - \ell_2 = n \cdot \lambda\ , \quad n = 0, 1, 2, \cdots$$



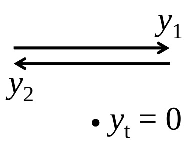

$$\rightarrow k \cdot \Delta\ell = 2n\pi$$

$\therefore I_P = 4 \cdot I_0 \quad , \quad I_0 = \alpha \cdot A^2 \quad , \quad I_0$ : Intensity of the single source $S_1$ and $S_2$ alone.

Constructive interference, $y_1$ and $y_2$ are in phase.

**case 2**

$$\Delta\ell = \left(n + \frac{1}{2}\right) \cdot \lambda\ , \quad n = 0, 1, 2, \cdots$$



$$k \cdot \Delta\ell = (2n+1)\,\pi$$

$$I_P = 0 \quad \text{destructive interference}$$

$y_1$ and $y_2$ are out of phase

**case 3**

$$\Delta\ell = \frac{\lambda}{4}$$
$$k \cdot \Delta\ell = \frac{\pi}{2} \qquad I_P = 2 \cdot I_0$$