Mathematics for physicists

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Differentiation

1.

$$
\begin{aligned}
y(x) &= a \cdot x^n \\
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{a \cdot (x+\Delta x)^n - a \cdot x^n}{\Delta x} \\
(x+\Delta x)^n &= x^n + n \cdot x^{n-1} \cdot \Delta x + \frac{n \cdot (n-1)}{1 \cdot 2} \cdot x^{n-2} \cdot (\Delta x)^2 + \cdots + (\Delta x)^n \\
\therefore \frac{dy(x)}{dx} &= a \cdot n \cdot x^{n-1} \quad , \quad (n \neq 0)
\end{aligned}
$$

2.

$$
\begin{aligned}
y(x) &= \sum_{i=0}^{n} a_i \cdot x^i = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n \\
\frac{dy(x)}{dx} &= a_1 + 2 \cdot a_2 \cdot x^2 + \cdots + n \cdot a_n \cdot x^{n-1} \\
&= \sum_{i=1}^{n} i \cdot a_i \cdot x^{i-1}
\end{aligned}
$$

3.

$$
\begin{aligned}
y(x) &= a \cdot \cos \beta x \\
\frac{dy(x)}{dx} &= \lim_{\Delta x \to 0} \frac{a \cdot \cos[\beta \cdot (x+\Delta x)] - a \cdot \cos \beta}{\Delta x} \\
\cos[\beta \cdot (x+\Delta x)] &= \cos \beta x \cdot \cos \beta \Delta x - \sin \beta x \cdot \sin \beta \Delta x \\
\frac{dy(x)}{dx} &= \lim_{\Delta x \to 0} \frac{a \cdot (\cos \beta \Delta x - 1)}{\Delta x} - \lim_{\Delta x \to 0} \sin \beta x \frac{\sin \beta \Delta x}{\Delta x}
\end{aligned}
$$

$$
\begin{aligned}
\because \lim_{\Delta x \to 0} \frac{\sin \beta}{\Delta x} &= \beta \qquad \text{OR} \qquad \sin \beta \Delta x \stackrel{\Delta x \to 0}{\longrightarrow} \beta \cdot \Delta x \\
\lim_{\Delta x \to 0} \frac{\cos \beta \Delta x - 1}{\Delta x} &= 0 \qquad \text{OR} \qquad \cos \beta \Delta x \stackrel{\Delta x \to 0}{\longrightarrow} 1 \\
\therefore \frac{dy(x)}{dx} &= -a \cdot \beta \cdot \sin \beta x
\end{aligned}
$$

4.

$$y(x) = a \cdot \sin \beta x$$
$$\frac{dy(x)}{dx} = a \cdot \beta \cdot \cos \beta x$$

5. Series expansion of a function

$$\text{set} \quad y(x) = \sum_{n=0}^{\infty} a_n \cdot x^n = a_0 + a_1 \cdot x + \cdots + a_n \cdot x^n + \cdots$$
$$y(0) = a_0$$
$$\frac{dy(x)}{dx} = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} = a_1 + 2 \cdot a_2 \cdot x + \cdots + n \cdot a_n \cdot x^{n-1} + \cdots$$
$$\therefore \left.\frac{dy(x)}{dx}\right|_{x=0} = a_1 = y'(0) \qquad y'(x) \equiv \frac{dy(x)}{dx}$$
$$\therefore \frac{d^n y(x)}{dx^n} \equiv \left(\frac{d}{dx}\right)^n \cdot y(x) = n! \cdot a_n + \frac{(n+1) \cdot n \cdot \cdots \cdot 2}{1} \cdot x^1 + \cdots$$
$$\therefore a_n = \frac{1}{n!} \cdot \left.\frac{d^n y(x)}{dx^n}\right|_{x=0} \quad , \quad n! \equiv 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$$
$$\rightarrow y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left.\frac{d^n y(x)}{dx^n}\right|_{x=0} \cdot x^n \qquad \text{Maclaurin series expansion}$$

6.

$$y(x) = \cos x = \sum_{n=0}^{\infty} a_n \cdot x^n$$
$$a_n = \ ?$$

$$a_n = \left.\frac{d^n \cos x}{dx^n}\right|_{x=0}$$
$$a_0 = 1 \ , \quad a_1 = 0 \ , \quad a_2 = -1 \ , \quad a_3 = 0 \ , \quad a_4 = a_0$$
$$\therefore \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

7.

$$y(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$$

try it!

Integration

1. $y = x^n \rightarrow \dfrac{dy}{dx} = n \cdot x^{n-1}$

$$\text{set} \quad f(x) = x^n \quad \text{then} \quad \int_a^b f(x)\,dx = \int_a^b x^n \cdot dx = \frac{1}{n+1} \cdot x^{n+1}\bigg|_a^b$$
$$= \frac{b^{n+1} - a^{n+1}}{n+1}$$
$$\because \frac{dx^{n+1}}{dx} = (n+1) \cdot x^n$$

2. $\int \cos \beta x dx = ?$

$$\because \frac{d \sin \beta x}{dx} = \beta \cdot \cos \beta x$$
$$\therefore \int \cos \beta x dx = \frac{1}{\beta} \int d(\sin \beta x) = \frac{1}{\beta} \sin \beta x + C \quad , \quad C = \text{constant.}$$