Mathematics for physicists

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What we can do with the power series expansion

1.
$$f(x)=\sum_{n=0}^{\infty}a_n\cdot x^n \quad , \quad a_n=\frac{f^{(n)}(0)}{n!} \qquad \text{(Maclaurin expansion)}$$

by the similar way

2.
$$f(x)=\sum_{n=0}^{\infty}b_n\cdot (x-x_0)^n \quad , \quad b_n=\frac{f^{(n)}(x_0)}{n!} \qquad \text{(Taylor expansion)}$$

3.
$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n}}{(2n)!} \quad ; \quad \sin x=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n+1}}{(2n+1)!}$$

4. If $f(x_0)=0$, $g(x_0)=0$ then $\lim_{x\to x_0}\frac{f(x_0)}{g(x_0)}=\,?$

$$\because f(x)=f'(x_0)\cdot(x-x_0)+f''(x_0)\cdot\frac{(x-x_0)^2}{2!}+\cdots$$
$$g(x)=g'(x_0)\cdot(x-x_0)+g''(x_0)\cdot\frac{(x-x_0)^2}{2!}+\cdots$$

$\therefore$ if $f'(x_0)$ or $g'(x_0)\neq 0$

$$\text{then }\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{f'(x_0)}{g'(x_0)}$$

or if $f'(x_0)$, $g'(x_0)=0$

$$\text{then }\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{f''(x_0)}{g''(x_0)} \qquad \text{and so on}$$

Application:

$$\lim_{x\to 0}\frac{\sin\beta x}{x}=\lim_{x\to 0}\frac{\beta\cdot\cos\beta x}{1}=\beta$$

5. If $f = f(t)$ & $\dfrac{df}{dt} = \beta \cdot t$, then $f(t) = ?$

   assume

$$f(t) = \sum_{n=0}^{\infty} a_n \cdot t^n = a_0 + a_1 \cdot t + a_2 \cdot t^2 + \cdots \qquad \text{(power series expansion)}$$

$$\to \frac{df}{dt} = a_1 + 2a_2 \cdot t + \cdots + na_n \cdot t^{n-1}$$
$$= \beta \cdot \left(a_0 + a_1 \cdot t + a_2 \cdot t^2 + \cdots\right)$$

$$\therefore a_1 = \beta \cdot a_0 \quad , \quad a_2 = \beta \cdot \frac{1}{2} a_1 \quad , \quad a_3 = \beta \cdot \frac{1}{3} \cdot a_2 \quad , \quad \cdots$$

   OR

$$a_n = \frac{\beta}{n} \cdot a_{n-1} = \beta^2 \cdot \frac{1}{n \cdot (n-1)} \cdot a_{n-2} = \cdots = \beta^n \cdot \frac{1}{n!} a_0$$

$$\therefore f(t) = \sum_{n=0}^{\infty} a_n \cdot t^n = a_0 \cdot \sum_n \frac{\beta^n}{n!} t^n = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} (\beta \cdot t)^n$$

   define:

$$g(t) = e^{\beta \cdot t} = \sum_{n=0}^{\infty} \frac{(\beta \cdot t)^n}{n!} \to \text{why?}$$

$$\because g(t_1 + t_2) = g(t_1) \cdot g(t_2) \ \ \& \ g(0) = 1$$

$$e^{x+y} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_n \frac{1}{n!} \left(x^n + nx^{n-1} \cdot y + \cdots + \tbinom{n}{m} x^{n-m} y^m + \cdots + y^n\right)$$
$$\tbinom{n}{m} = \frac{n \cdot (n-1) \cdots (n-m+1)}{m!}$$
$$e^x \cdot e^y = \left(\sum_n \frac{x^n}{n!}\right) \cdot \left(\sum_m \frac{y^m}{m!}\right)$$

   check $x^{n-m} \cdot y^m$ term

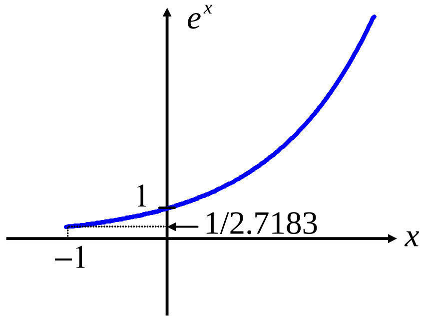
$$\to \frac{1}{(n-m)!} x^{n-m} \cdot y^m \cdot \frac{1}{m!} \Leftrightarrow \frac{n \cdot (n-1) \cdots (n-m+1)}{n! m!} = \frac{1}{m! (n-m)!}$$

Q.E.D.

   then

$$f(t) = a_0 \cdot e^{\beta \cdot t}$$
$$f(0) = a_0 \qquad \therefore f(t) = f(0) e^{\beta \cdot t} \Leftrightarrow \frac{df}{dt} = \beta \cdot f(t)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \therefore e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.7183\cdots$$



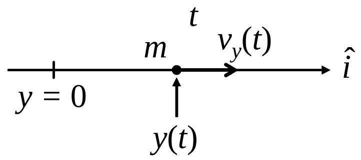
some rule
$$e^{x+y} = e^x \cdot e^y$$
$$e^{-y} = \frac{1}{e^y}$$

another definition: $\quad e = \lim_{x\to 0} (1+x)^{\frac{1}{x}}$

6. One-dimensional motion with drag force

$$m\frac{d\vec{v}_y(t)}{dt} = -\gamma \cdot \vec{v}_y(t) = \vec{F}$$

γ : coefficient of drag



$$\therefore v_y(t) = \alpha \cdot e^{\beta \cdot t}$$

$$\Longrightarrow \beta = -\frac{\gamma}{m} \quad \text{and} \quad \alpha = v_y(0) \quad \text{initial velocity}$$

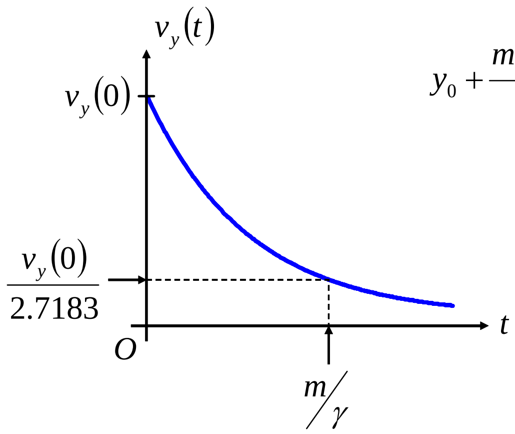
$$v_y(t) = v_y(0)\, e^{-\gamma \cdot t/m} = \frac{dy(t)}{dt}$$

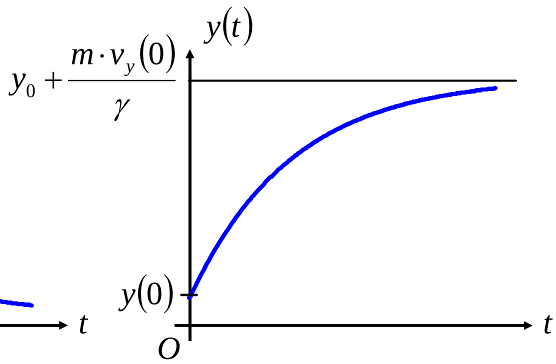
$$\therefore y(t) = k + \eta \cdot e^{\zeta \cdot t} \rightarrow \frac{dy(t)}{dt} = \zeta \cdot \eta \cdot e^{\zeta \cdot t} = v_y(0)\, e^{-\gamma \cdot t/m}$$

$$\therefore \eta \cdot \zeta = v_y(0) \quad \text{and} \quad \zeta = -\frac{\gamma}{m}\ , \ \ \eta = -\frac{m \cdot v_y(0)}{\gamma}$$

$$y(0) = k - \frac{m \cdot v_y(0)}{\gamma} \qquad \therefore k = y(0) + \frac{m \cdot v_y(0)}{\gamma}$$

$$\therefore y(t) = y(0) + \frac{m \cdot v_y(0)}{\gamma}\left(1 - e^{-\gamma \cdot t/m}\right) \xrightarrow[t\to\infty]{} y(0) + \frac{m \cdot v_y(0)}{\gamma}$$





7. Euler relation

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad , \qquad \begin{aligned} \cos x &= \sum (-1)^n \cdot \frac{x^{2n}}{(2n)!} \\ \sin x &= \sum (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

How to link these three functions?

$$\text{Define: } \; i \equiv \sqrt{-1}$$

$$i^2 = -1 \; , \; \; i^3 = -i \; , \; \; i^4 = 1$$

$$\therefore e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \underbrace{\sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!}}_{\text{odd terms}}$$

$$i^{2n} = \left(i^2\right)^n = (-1)^n$$

$$i^{2n+1} = i^{2n} \cdot i = i \cdot (-1)^n$$

$$\begin{aligned} \therefore e^{ix} &= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!} + i \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} \\ &= \cos x + i \cdot \sin x \qquad \text{(Euler relation)} \end{aligned}$$