Mathematics for physicists

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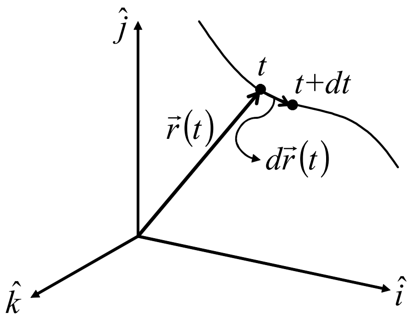
(Dated: March 4, 2006)

Kinematic analysis of motion in time domain

1. position vector $\vec{r} = \vec{r}\,(t)$ function of time



$$\vec{v}\,(t) = \frac{d\vec{r}\,(t)}{dt} \qquad \text{velocity}$$

$$\vec{a}\,(t) = \frac{d\vec{v}\,(t)}{dt} \qquad \text{acceleration}$$

$$\vec{v}\,(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\vec{a}\,(t) = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

2. constant velocity motion

$$\vec{r}\,(t) = \vec{r}_0 + \vec{v}_0 \cdot t \qquad \text{linear function of time}$$

$$\vec{r}\,(0) = \vec{r}_0 \qquad \text{initial position}$$

$$\vec{v}\,(t) = \vec{v}_0 = const. \qquad \rightarrow \text{ motion along a line}$$

$$\vec{a}\,(t) = 0$$

3. constant acceleration motion

$$\vec{r}\,(t) = \vec{r}_0 + \vec{v}_0 \cdot t + \frac{1}{2}\vec{a}_0 \cdot t^2 \qquad \text{quadratic function of time}$$

$$\vec{v}\,(t) = \vec{v}_0 + \vec{a}_0 \cdot t$$

$$\vec{v}\,(0) = \vec{v}_0$$

$$\vec{a}\,(t) = \vec{a}_0 = const.$$

if

$$\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}$$
$$\vec{a}_0 = -g \cdot \hat{j}$$

then

$$x(t) = x_0 + v_{0x} \cdot t$$
$$y(t) = y_0 + v_{0y} \cdot t - \frac{1}{2} g \cdot t^2$$

the trajectory is parabolic in x-y plane.

4. SHM.

$$x(t) = A \cdot \sin(2\pi f \cdot t + \delta_0) \qquad \text{Simple harmonic function of time}$$
$$x(t) = x(t+T) \qquad \therefore T = \frac{1}{f} \rightarrow \text{period}\ ,\ \ f \rightarrow \text{frequency}$$
$$x(0) = A \sin \delta_0$$
$$v(t) = 2\pi f \cdot A \cdot \cos(2\pi f \cdot t + \delta_0)$$
$$v(0) = 2\pi f \cdot A \cos \delta_0$$

$\therefore A$ and δ_0 are determined by $x(0)$ and $v(0)$.

$$a(t) = -(2\pi f)^2 A \cdot \sin(2\pi f t + \delta_0)$$
$$a(t) = -(2\pi f)^2 \cdot x(t)$$

$\therefore a(t)$ is proportional to $x(t)$ but in opposite direction.
Note: all $x(t)$, $v(t)$, $a(t)$ are simple harmonic function of time, but with different phase.

$$x(t) = A \cdot \sin(2\pi f t + \delta_0)$$
$$v(t) = 2\pi f A \cdot \sin\left(2\pi f t + \delta_0 + \frac{\pi}{2}\right)$$
$$a(t) = (2\pi f)^2 A \cdot \sin(2\pi f t + \delta_0 + \pi)$$

5. Damped motion

$$\vec{r}\,(t) = \vec{r}_0 + \frac{\vec{v}_0}{\beta} - \frac{\vec{v}_0}{\beta} e^{-\beta \cdot t}$$

$$\vec{r}\,(\infty) = \vec{r}_0 + \frac{\vec{v}_0}{\beta}$$

$$\vec{v}\,(t) = \vec{v}_0 e^{-\beta \cdot t} \qquad \text{exponentially decay function of time}$$

$$\vec{v}\,(\infty) = 0$$

$$\vec{a}\,(t) = -\beta \vec{v}_0 e^{-\beta \cdot t}$$

$$\vec{a}\,(t) = -\beta \cdot \vec{v}\,(t)$$

$\therefore \vec{a}\,(t)$ is proportional to $\vec{v}\,(t)$ but in opposite direction.

Newton's discovery: The cause of the motion is related to its real time acceleration, not to its velocity, as Aristotle supposed.