Mathematics for physicists

Lecturer: Prof. Ven-Chung Lee

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Application of Calculus

1. Newton's law of motion in time domain

$$\vec{p}\,(t) = m \cdot \vec{v}\,(t) \qquad \text{(momentum)}$$

$$\vec{v}\,(t) = \frac{d\vec{r}\,(t)}{dt} \qquad \text{(velocity)}$$

$$t \rightarrow \vec{p}\,(t)\ \ ;\ \ t + dt \rightarrow \vec{p}\,(t + dt) = \vec{p}\,(t) + d\vec{p}\,(t)$$

$$d\vec{p}\,(t) = \vec{F}\,(t)\,dt \qquad \text{(Newton's law of motion)}$$

$$\vec{F}\,(t) \rightarrow \text{Force}$$

in one-dimension case

$$dp_x\,(t) = F_x\,(t)\,dt \quad \text{or} \quad \frac{dp_x\,(t)}{dt} = F_x\,(t) \implies m\frac{dv_x\,(t)}{dt} = m\frac{d^2x\,(t)}{dt^2} = F_x\,(t)$$

$$t \rightarrow \vec{r}\,(t)\ \ ;\ \ t + dt \rightarrow \vec{r}\,(t + dt) = \vec{r}\,(t) + d\vec{r}\,(t)$$

$d\vec{r}\,(t) =?$

$$d\vec{r}\,(t) = \frac{d\vec{r}\,(t)}{dt} \cdot dt = \vec{v}\,(t) \cdot dt$$

in one-dim. case

$$dx\,(t) = \frac{dx\,(t)}{dt} \cdot dt = v_x\,(t) \cdot dt$$

$$\frac{d\vec{p}\,(t)}{dt} = m\frac{d\vec{v}\,(t)}{dt} = m\vec{a}\,(t) \rightarrow m\vec{a} = \vec{F}\,(t) \qquad \vec{a} : \text{acceleration}$$

in general

$$\vec{F} = \vec{F}\,(\vec{r}, \vec{v}) = \vec{F}\,(\vec{r}\,(t)\,, \vec{v}\,(t)\,, \cdots)$$

(a) $\vec{F} = -k \cdot \vec{r}$ Hooke's force

one-dim. case $F_x = -k \cdot x$

$$\rightarrow m\frac{d^2x\left(t\right)}{dt^2} = -k \cdot x\left(t\right)$$

$$\rightarrow x\left(t\right) = A \cdot \cos\left(2\pi f \cdot t + \delta\right) \qquad \text{(SHM)}$$

$$\left(2\pi f\right)^2 = \frac{k}{m} \because \frac{d^2\cos\left(\omega t + \delta\right)}{dt^2} = -\omega^2\cos\left(\omega t + \delta\right)$$

$$\therefore f = \frac{1}{2\pi} \cdot \left(\frac{k}{m}\right)^{1/2} \quad , \quad x\left(0\right) = A \cdot \cos\delta \quad , \quad v\left(0\right) = -2\pi f \cdot A\sin\delta$$

$$\therefore \forall\ x\left(0\right), v\left(0\right) \implies A, \delta \qquad (1)$$

$\therefore$ A and δ are two constants of integration and determined by the initial conditions.

(b) $\vec{F} = -\gamma \cdot \vec{v}$

one-dim. case $F_x = -\gamma \cdot v_x$

$$m\frac{dv_x\left(t\right)}{dt} = -\gamma \cdot v_x\left(t\right)$$

$$\therefore v_x\left(t\right) = B \cdot e^{-\frac{\gamma}{m}\cdot t} \quad , \quad B = v_x\left(0\right) \qquad \because \frac{de^{\beta\cdot t}}{dt} = \beta \cdot e^{\beta\cdot t}$$

$$\frac{dx\left(t\right)}{dt} = v_x\left(0\right) \cdot e^{-\gamma\cdot\frac{t}{m}}$$

$$x\left(t\right) = C - \frac{m \cdot v_x\left(0\right)}{\gamma}e^{-\gamma\cdot\frac{t}{m}} \quad , \quad x\left(0\right) = C - \frac{m \cdot v_x\left(0\right)}{\gamma}$$

Note: B and C are two constants of integration and determined by $v_x\left(0\right)$ and $x\left(0\right)$.

$$\therefore x\left(t\right) = x\left(0\right) + \frac{m \cdot v_x\left(0\right)}{\gamma}\left(1 - e^{-\gamma\cdot\frac{t}{m}}\right)$$

$$x\left(\infty\right) = x\left(0\right) + \frac{m \cdot v_x\left(0\right)}{\gamma}$$

2. Newton's equation of motion in space domain

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

one-dim. case $F = m\dfrac{dv}{dt}$

if $F = F(x)$

then

$$F(x) = m\frac{dv(t)}{dt} = m\frac{dv(x)}{dx}\cdot\frac{dx}{dt} = m\cdot v(x)\cdot\frac{dv(x)}{dx}$$
$$= \frac{d\left(\frac{1}{2}mv^2(x)\right)}{dx}$$

$$\therefore \frac{1}{2}mv^2(x) - \frac{1}{2}mv^2(x_0) = \int_{x_0}^{x} F(x)\cdot dx = W \qquad \text{(work-energy theorem)}$$

when $x = x_0 \to v = v_0 \equiv v(x_0)$

$$\frac{1}{2}mv^2(x) = K \qquad \text{(kinetic energy)}$$

$$\int_{x_0}^{x} F(x)\cdot dx = W \qquad \text{(work done by } F \text{ from } x_0 \text{ to } x\text{)}$$

$$\to v = v(x) = \frac{dx}{dt} \therefore dt = \frac{dx}{v(x)} \to t = t(x)$$

(a) Hooke's force $F = -k\cdot x$ if $t = 0$, $x = x_0$, $v = v_0$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{x_0}^{x}(-k\cdot x)\,dx = -\frac{1}{2}k\cdot x^2 + \frac{1}{2}k\cdot x_0^2$$

or

$$\frac{1}{2}mv^2 + \frac{1}{2}k\cdot x^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}k\cdot x_0^2 = E = \text{total mechanical energy}$$

Note $\frac{1}{2}k\cdot x^2 \to$ elastic potential energy

$$\therefore v = \left(\frac{2}{m}\left(E - \frac{1}{2}k\cdot x^2\right)\right)^{\frac{1}{2}} = \frac{dx}{dt}$$
$$\therefore dt = \frac{dx}{\left(\frac{2}{m}\left(E - \frac{1}{2}k\cdot x^2\right)\right)^{\frac{1}{2}}} \implies t - 0 = \int_{x_0}^{x}\frac{dx}{\left(\frac{2}{m}\left(E - \frac{1}{2}k\cdot x^2\right)\right)^{\frac{1}{2}}}$$

change of variable: set $x^2 = \frac{2E}{k}\cdot\cos^2\theta$ i.e. $x = \left(\frac{2E}{k}\right)^{1/2}\cos\theta$, $x = x_0 \implies \theta = \theta_0$

then

$$dx = \left(\frac{2E}{k}\right)^{\frac{1}{2}}(-\sin\theta)\,d\theta$$

$$\left(\frac{2}{m}\left(E - \frac{1}{2}k\cdot x^2\right)\right)^{\frac{1}{2}} = \left(\frac{2E}{m}\right)^{\frac{1}{2}}\sin\theta$$

$$\therefore t - 0 = \int_{\theta_0}^{\theta} \left(\frac{m}{k}\right)^{\frac{1}{2}} (-d\theta) = -\left(\frac{m}{k}\right)^{\frac{1}{2}} (\theta - \theta_0)$$

$$\left[x_0 = \left(\frac{2E}{k}\right)^{\frac{1}{2}} \cos\theta_0 \ , \ \ t = 0 \implies \theta = \theta_0 \quad \text{(initial phase angle)}\right]$$

$$\therefore \theta - \theta_0 = -\left(\frac{k}{m}\right)^{\frac{1}{2}} \cdot t = -\omega t$$

$$\text{set } \omega \equiv \left(\frac{k}{m}\right)^{\frac{1}{2}} = 2\pi\nu = \text{angular frequency}$$

$$\therefore x = \left(\frac{2E}{k}\right)^{\frac{1}{2}} \cos\theta = \left(\frac{2E}{k}\right)^{\frac{1}{2}} \cos(\omega t - \theta_0)$$

Note: $\omega \cdot T = 2\pi$, $T = \frac{2\pi}{\omega} = \frac{1}{\nu}$; $T \to$ period, $\nu \to$ frequency

Note: when $x \to$ max i.e. $x = A =$ Amplitude, then $v = 0$

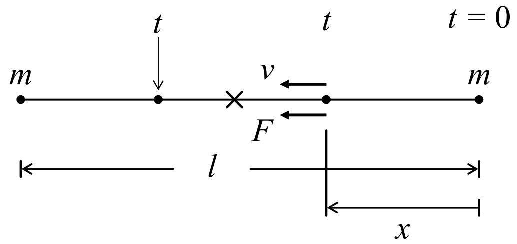
$$\therefore E = \frac{1}{2}k \cdot A^2 \qquad A = \left(\frac{2E}{k}\right)^{\frac{1}{2}} \to x = A \cdot \cos(\omega t - \theta_0)$$

$$\therefore x = A \cdot \cos\left(\sqrt{\frac{k}{m}} \cdot t - \theta_0\right)$$

or

$$t = \left(\frac{m}{k}\right)^{\frac{1}{2}} \left(\cos^{-1}\frac{x_0}{A} - \cos^{-1}\frac{x}{A}\right)$$

(b) Gravitational force



gravitational force acting on the particle at time t :

$$F = \frac{Gm^2}{(\ell - 2x)^2} = F(x)$$

Find the time required for the two initially rest particles to collide.

at time t , $v = \dfrac{dx}{dt}$, $v(0) = 0$

$$\frac{1}{2}mv^2 - 0 = \int_0^x F(x) \cdot dx = \int_0^x \frac{Gm^2}{(\ell - 2x)^2} \cdot dx$$

$$\therefore \frac{1}{2}mv^2 = \frac{1}{2}Gm^2 \cdot \left.\frac{1}{\ell - 2x}\right|_0^x = \frac{Gm^2}{2}\left(\frac{1}{\ell-2x} - \frac{1}{\ell}\right) = Gm^2 \cdot \frac{x}{\ell\left(\ell - 2x\right)}$$

$$\therefore v\left(x\right) = \left(\frac{2Gm}{\ell} \cdot \frac{x}{\ell - 2x}\right)^{\frac{1}{2}} = \frac{dx}{dt}$$

$$\therefore t_c - 0 = \int_0^{t_c} dt = \int_0^{\frac{\ell}{2}} \left(\frac{\ell}{2Gm} \cdot \frac{\ell - 2x}{x}\right)^{\frac{1}{2}} \cdot dx$$

change of variable: set $x = \frac{\ell}{2}\sin^2\theta$ $\quad \therefore \begin{array}{l} x = 0 \rightarrow \theta = 0 \rightarrow t = 0 \\ x = \frac{\ell}{2} \rightarrow \theta = \frac{\pi}{2} \rightarrow t = t_c \end{array}$

$$dx = \ell \cdot \sin\theta \cdot \cos\theta \cdot d\theta$$

$$\left(\frac{\ell - 2x}{x}\right)^{\frac{1}{2}} = \left(\frac{\ell \cdot \cos^2\theta}{\frac{\ell}{2}\sin^2\theta}\right)^{\frac{1}{2}} = \sqrt{2} \cdot \frac{\cos\theta}{\sin\theta}$$

$$\therefore t_c = \int_0^{\frac{\pi}{2}} \left(\frac{\ell}{Gm}\right)^{\frac{1}{2}} \cdot \ell\cos^2\theta d\theta = \left(\frac{\ell^3}{Gm}\right)^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$$
$$= \frac{\pi}{4} \cdot \left(\frac{\ell^3}{Gm}\right)^{\frac{1}{2}}$$

$$\because \int_0^{\pi/2} \cos^2\theta d\theta = \frac{1}{2}\int_0^{\pi/2} \left(\cos 2\theta + 1\right) d\theta = \frac{1}{2}\left.\left(\frac{1}{2}\sin 2\theta + \theta\right)\right|_0^{\pi/2} = \frac{\pi}{4}$$

Between $t = 0$ and $t = t_c$

$$t = t\left(x\right) = \left(\frac{\ell^3}{Gm}\right)^{\frac{1}{2}}\left(\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta\right) \quad \text{, where } \theta = \sin^{-1}\left(\frac{2x}{\ell}\right)^{\frac{1}{2}}$$

$$\sin 2\theta = 2 \cdot \left(\frac{2x}{\ell}\right)^{\frac{1}{2}} \cdot \left(1 - \frac{2x}{\ell}\right)^{\frac{1}{2}}$$

$$\therefore t\left(x\right) = \left(\frac{\ell^3}{Gm}\right)^{\frac{1}{2}} \cdot \frac{1}{2}\left[\left(\frac{2x}{\ell}\right)^{\frac{1}{2}}\left(1 - \frac{2x}{\ell}\right)^{\frac{1}{2}} + \sin^{-1}\left(\frac{2x}{\ell}\right)^{\frac{1}{2}}\right]$$