Mathematics for physicists

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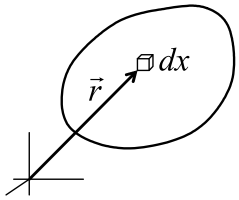
Application of Calculus

1. mass, density of a body

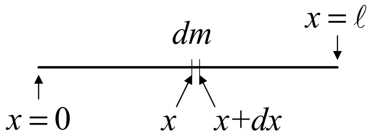


$\vec{r} \to$ space element $d\tau \implies dm = \rho(\vec{r})\, d\tau$

$\rho(\vec{r})$: mass density

$m = \int dm = \int \rho(\vec{r})\, d\tau$

linear mass distribution



x to $x + dx \implies dx \implies dm = \rho(x)\, dx$

$\rho(x) = dm/dx$ line mass density

$m = \int_0^\ell \rho(x)\, dx$

total gravity (weight)

$$F = \int dF = \int dm \cdot g = \int \rho(x) \cdot g \cdot dx$$

g : acceleration of gravity

center of gravity

$$x_{CG} = x_{CM} = \frac{\int x \cdot dm}{m} = \frac{\int \rho(x) \cdot x \cdot dx}{m} \;, \quad x_{CM} : \text{coordinate of the center of mass}$$

total torque of the gravity with respect to $x = 0$

$$\Gamma_0 = \int d\Gamma = \int x \cdot dF = \int x \cdot dm \cdot g = \int x \cdot \rho(x) \cdot g \cdot dx$$
$$= F \cdot x_{CG}$$

$$\therefore x_{CG} = \frac{\int x \cdot \rho \cdot g \cdot dx}{\int \rho \cdot g \cdot dx} = x_{CM}$$

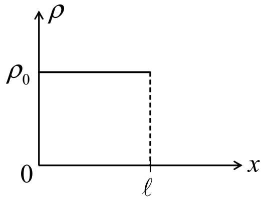
moment of inertia or rotational inertia w. r. t. $x = 0$

$$I_0 = \int dI = \int x^2 dm = \int x^2 \cdot \rho \cdot dx$$

Example:

case 1. $\rho = \rho_0 = const.$

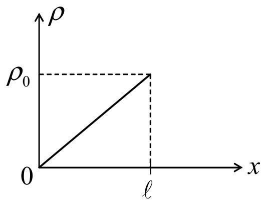
$$m = \int_0^\ell \rho_0 \cdot dx = \rho_0 \cdot \ell$$
$$F = m \cdot g$$



$$\Gamma_0 = \int_0^\ell x \cdot \rho_0 \cdot g \cdot dx = \frac{1}{2}\rho_0 \cdot g \cdot \ell^2 = \frac{1}{2} m \cdot g \cdot \ell = mg \cdot x_{CM}$$
$$\therefore x_{CM} = \frac{\ell}{2}$$
$$I_0 = \int_0^\ell x^2 \cdot \rho_0 \cdot dx = \frac{1}{3}\rho_0 \cdot \ell^3 = \frac{1}{3} m \cdot \ell^2$$

case 2. $\rho\left(x\right) = \dfrac{\rho_0}{\ell} \cdot x$

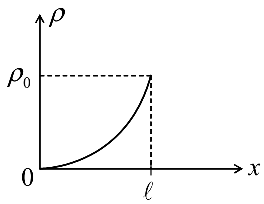
$$m = \int_0^\ell \frac{\rho_0}{\ell} \cdot x \cdot dx = \frac{1}{2}\rho_0 \cdot \ell$$
$$F = \frac{1}{2}\rho_0 \cdot \ell \cdot g$$



$$\Gamma_0 = \int x \cdot \frac{\rho_0}{\ell} \cdot x \cdot dx \cdot g = \frac{1}{3}\rho_0 \cdot g \cdot \ell^2 = F \cdot x_{CM}$$
$$\therefore x_{CM} = \frac{2}{3}\ell$$
$$I_0 = \int_0^\ell x^2 \cdot \frac{\rho_0}{\ell} \cdot x dx = \frac{1}{4}\rho_0 \cdot \ell^3 = \frac{1}{2} m \cdot \ell^2$$

case 3. $\rho\left(x\right) = \rho_0 \cdot \left(\dfrac{x}{\ell}\right)^n$

$$m = \int_0^\ell \cdot \left(\frac{x}{\ell}\right)^n dx = \frac{\rho_0 \cdot \ell}{n+1}$$
$$F = m \cdot g$$



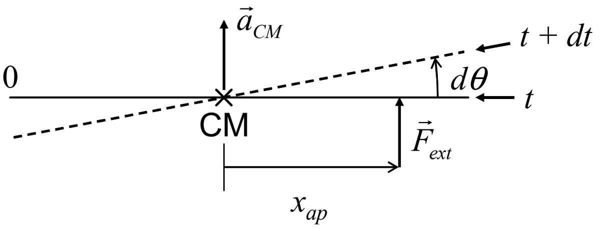
$$\Gamma_0 = \int x \cdot \rho_0 \cdot g \cdot \left(\frac{x}{\ell}\right)^n dx = \frac{\rho_0 \cdot g \cdot \ell^2}{n+1} = \frac{n+1}{n+2} m \cdot g \cdot \ell$$
$$\therefore x_{CM} = \frac{n+1}{n+2}\ell$$
$$I_0 = \int_0^\ell x^2 \cdot \rho_0 \cdot \left(\frac{x}{\ell}\right)^n dx = \frac{\rho_0 \cdot \ell^3}{n+3} = \frac{n+1}{n+3} m \cdot \ell^2$$

Newton's equation of translational motion

$$m \cdot a_{CM} = \sum \vec{F} \ , \ \ a_{CM} = \frac{d^2 x_{CM}}{dt^2} \ , \ \ \sum \vec{F} : \text{total external force}$$

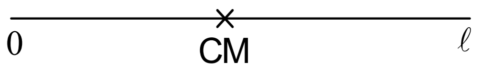
Newton's equation of rotational motion

$$I \cdot \vec{\alpha} = \sum \vec{\Gamma} \ , \ \ \alpha = \frac{d^2\theta}{dt^2} \ , \ \alpha \rightarrow \text{angular acceleration} \ , \ \ \sum \vec{\Gamma} : \text{total external torque}$$



$$t \text{ to } t + dt \implies d\theta = \theta\,(t + dt) - \theta\,(t)$$
$$\omega = d\theta/dt \quad \text{angular veloctiy}$$
$$\alpha = d^2\theta/dt^2$$

moment of inertia or rotational inertia with respect to the center of mass



$$\rho = \rho_0 = const.$$

$$\begin{aligned} I_{CM} &= \int_0^\ell \rho_0 \cdot (x - x_{CM})^2 \, dx \quad , \quad x_{CM} = \frac{\ell}{2} \\ &= \frac{1}{3}\rho_0 \cdot \left(x - \frac{\ell}{2}\right)^3 \Bigg|_0^\ell \\ &= \frac{1}{12}\rho_0 \cdot \ell^3 = \frac{1}{12} m \cdot \ell^2 \end{aligned}$$

torque of the external force F_{ext} with respect to the CM

$$\Gamma_{CM} = F_{ext} \cdot x_{ap} \qquad F_{ext} : \text{external force}$$