Mathematics for physicists

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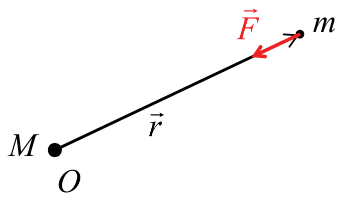
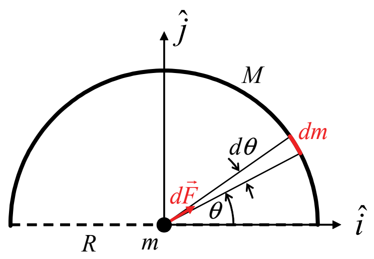
Application of Calculus on Mechanics

1. Universal gravity



$$\vec{F} = -\frac{GMm\hat{r}}{r^2} = -\frac{GMm\vec{r}}{r^3}$$

Example: Total force on m at the center by a uniform semicircular wire with mass M



$$\theta \text{ to } \theta + d\theta \implies dM = \frac{M}{\pi R} \cdot R \cdot d\theta = \frac{M}{\pi} d\theta$$

$$\implies d\vec{F} = dF\cos\theta \cdot \hat{\imath} + dF\sin\theta \cdot \hat{\jmath}$$

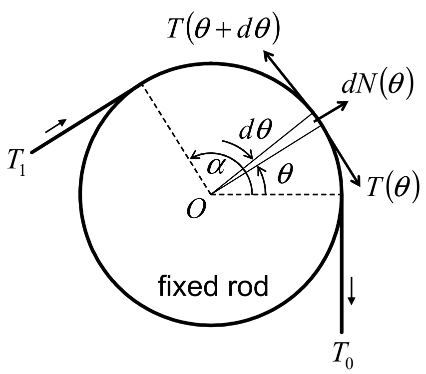
$$dF = \frac{GmdM}{R^2} = G\frac{mM}{\pi R^2} d\theta$$

$$\therefore \vec{F} = \int d\vec{F} = \hat{\imath}\frac{GMm}{\pi R^2}\underbrace{\int_0^\pi \cos\theta \cdot d\theta}_{=\sin\theta|_0^\pi = 0} + \hat{\jmath}\frac{GMm}{\pi R^2}\underbrace{\int_0^\pi \sin\theta \cdot d\theta}_{=-\cos\theta|_0^\pi = 2}$$

$$\vec{F} = \hat{\jmath}\frac{2GMm}{\pi R^2}$$

2. Friction force and Tension: $f_\mu = \mu N$, f_μ : sliding friction force, N : normal force, μ :coefficient of sliding friction

Example: Lift a weight by a string tossed over a fixed circular rod



$\theta \to T(\theta) \qquad \theta + d\theta \to T(\theta + d\theta) = T(\theta) + dT(\theta)$

Normal component:

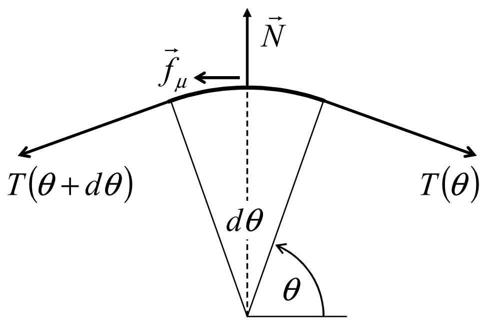
$(T + dT)\sin\frac{d\theta}{2} + T\sin\frac{d\theta}{2} = dN$ Normal force

Tangent component:

$(T + dT)\cos\frac{d\theta}{2} - T\cos\frac{d\theta}{2} = -\mu dN$ Friction force

$\because d\theta \ll 1 \therefore \sin\dfrac{d\theta}{2} \to \dfrac{d\theta}{2}; \quad \cos\dfrac{d\theta}{2} \to 1$

$\therefore dT \cdot d\theta \ll T \cdot d\theta$ the $dT \cdot d\theta$ term is neglected

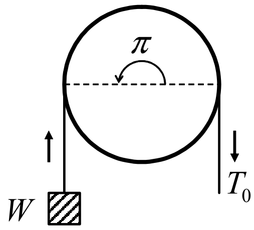


$$\therefore T \cdot d\theta \approx dN \quad \text{and} \quad dT \approx -\mu dN$$

$$\frac{dT}{T \cdot d\theta} \approx -\mu_k \quad \text{or} \quad \frac{dT}{d\theta} = -\mu \cdot T$$

$$\therefore T = A \cdot e^{-\mu \cdot \theta} \qquad T(0) = T_0 = A$$

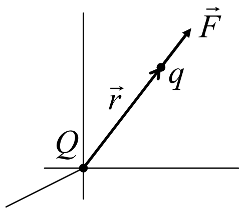
$$T(\alpha) = T_1 = T_0 e^{-\mu \cdot \alpha}$$



$$T_0 = W e^{\mu \pi} \qquad \therefore T_0 > W$$

Application of Calculus in Electrostatics

3. Coulomb's force

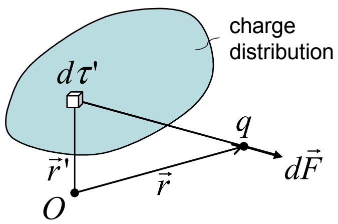


force on point charge q by Q at origin

$$\vec{F} = k \cdot \frac{Q \cdot q \cdot \hat{r}}{r^2}$$

$$\vec{r} = r \cdot \hat{r} \qquad \therefore \hat{r} = \vec{r}/r$$

$$\vec{F} = k \frac{Q \cdot q \cdot \vec{r}}{r^3}$$

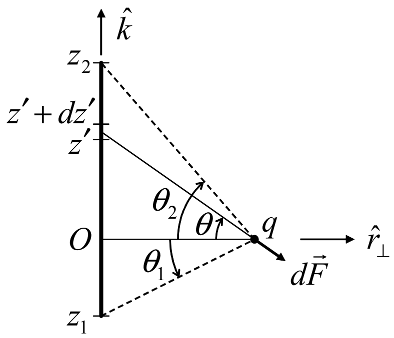


$dQ = \rho(\vec{r}')\, d\tau'$, $\rho(\vec{r}')$: charge density, $d\tau'$: space element

$$\vec{F} = \int d\vec{F}$$

$$d\vec{F} = k \cdot \frac{\rho(\vec{r}')\, d\tau' \cdot q \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

Example: Total Coulomb force on q by a uniformly charged line:



$\rho_L = \dfrac{dQ}{dz}$ linear charge density, $dQ = \rho_L \cdot dz'$

$$\vec{F} = \int d\vec{F}$$

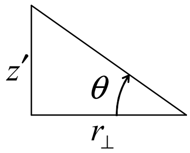
$$d\vec{F} = dF \cos\theta \cdot \hat{r}_\perp + dF \sin\theta \left(-\hat{k}\right)$$

$$\therefore \vec{F} = F_{r_\perp} \cdot \hat{r}_\perp + F_k \cdot \hat{k}$$

$$dF = k \cdot \frac{\rho_L dz' \cdot q}{z'^2 + r_\perp^2}$$

$$F_{r_\perp} = \int dF \cdot \cos\theta = \int k \cdot \frac{\rho_L dz' \cdot q}{z'^2 + r_\perp^2} \cdot \cos\theta$$

change of variable from z' to θ :



$$z' = r_\perp \cdot \tan\theta$$
$$\therefore z'^2 + r_\perp^2 = r_\perp^2 \cdot \sec^2\theta$$
$$dz' = r_\perp \cdot \sec^2\theta d\theta \qquad \left(\because \frac{d\tan\theta}{d\theta} = \sec^2\theta\right)$$

$$\therefore F_{r_\perp} = k \cdot \rho_L \cdot q \frac{r_\perp}{r_\perp^2} \int \frac{\sec^2 d\theta}{\sec^2\theta} \cdot \cos\theta = \frac{k\rho_L q}{r_\perp} \int_{\theta_1}^{\theta_2} \cos\theta d\theta = \frac{k\rho_L q}{r_\perp} (\sin\theta_2 - \sin\theta_1)$$

Similarly:

$$F_z = -\frac{k\rho_L q}{r_\perp} \int_{\theta_1}^{\theta_2} \sin\theta d\theta = \frac{k\rho_L q}{r_\perp} (\cos\theta_2 - \cos\theta_1) = k\rho_L q \cdot \left(\frac{1}{(z_2^2 + r_\perp^2)^{1/2}} - \frac{1}{(z_1^2 + r_\perp^2)^{1/2}} \right)$$

(1) If $\theta_1 \to -\frac{\pi}{2}$, $\theta_2 \to +\frac{\pi}{2} \implies \infty$-long line charge:

$$F_{r_\perp} = \frac{2\pi\rho_L q}{r_\perp} \ , \ \ F_z = 0$$

(2) If $\theta_1 \to 0$, $\theta_2 \to +\frac{\pi}{2} \implies$ semi-∞-long line charge:

$$F_{r_\perp} = \frac{k\rho_L q}{r_\perp} \ , \ \ F_z = -\frac{k\rho_L q}{r_\perp}$$