Mathematics for physicists

Lecturer: Prof. Ven-Chung Lee

(Dated: May 27, 2006)

From SHM to ellipse and Planck's quantization condition

1.

$$x = A \cdot \cos(\omega \cdot t + \delta) = A \cdot \cos\theta$$

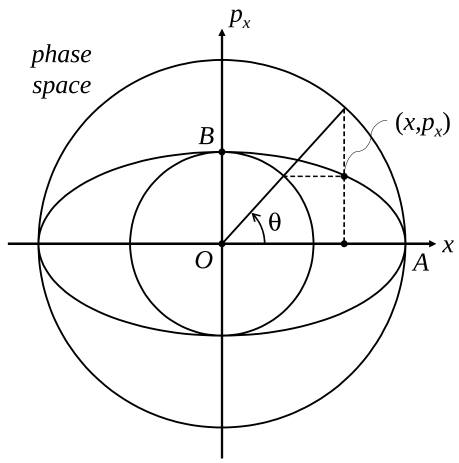
$$mv_x = p_x = -m\omega \cdot A \cdot \sin(\omega \cdot t + \delta) = B \cdot \sin\theta \quad , \quad B \equiv m\omega A$$

$$\left(\frac{x}{A}\right)^2 + \left(\frac{p_x}{m\omega A}\right)^2 = 1$$

area enclosed by the ellipse

$$\text{Area} = \pi \cdot A \cdot m\omega A = \int p_x dx \quad \text{for a cycle}$$

As the time increase, the point corresponding to the state of the motion (x, p) moves along the ellipse clockwise in the phase space.



2. Calculation of the area enclosed by the ellipse with equation $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$

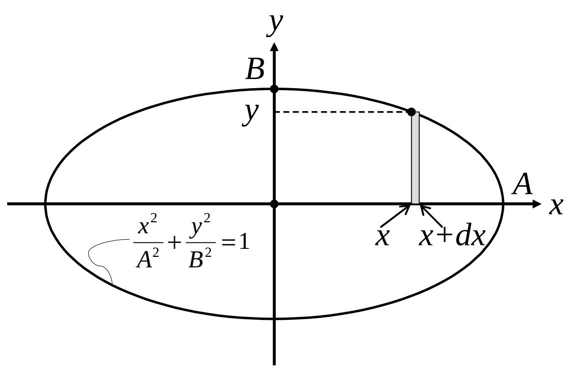
$\text{Area} = 4\int_0^y y \cdot dx$

$y = B\sin\theta$

$x = A\cos\theta \rightarrow dx = A\sin\theta \cdot d\theta$

$x : 0 \rightarrow A$

$\therefore \theta : \frac{\pi}{2} \rightarrow 0$



$$\text{Area} = -4\int_{\pi/2}^{0} B\sin\theta\cdot A\sin\theta\cdot d\theta$$
$$= -4AB\int_{\frac{\pi}{2}}^{0}\sin^2\theta\cdot d\theta$$
$$= -4AB\cdot\frac{1}{2}\int_{\pi/2}^{0}(1-\cos 2\theta)\,d\theta = -2AB\cdot\left(\theta-\frac{1}{2}\sin 2\theta\right)\Big|_{\frac{\pi}{2}}^{0} = \pi\cdot AB$$

3. Total mechanical energy:

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2m}(m\omega A)^2 = \frac{1}{2}kA^2$$

$$\therefore \omega^2 = \frac{k}{m} \qquad \omega = 2\pi\nu \qquad ,\ \nu : \text{frequency}$$

$$\text{Area} = \oint p_x dx = \pi m\omega A^2 = \frac{E\cdot 2\pi}{\omega} = \frac{E}{\nu}$$

(1) area = any value ≥ 0 (Newtonian mechanics E = any value ≥ 0)

(2) area = $n\cdot h$, $n = 0, 1, 2, \cdots$, h = Planck const. (Planck's quantum theory $E = nh\nu$)

(3) area = $\left(n+\frac{1}{2}\right)h$ (Heisenberg's quantum theory $E = \left(n+\frac{1}{2}\right)h\nu$)

Heisenberg's result is exact.

4. area of an ellipse with equation $\frac{x^2}{A^2}+\frac{y^2}{B^2} = 1$

$$\text{Area} = \int dA = \int y(x)\,dx = \int\frac{B}{A}\left(A^2-x^2\right)^{\frac{1}{2}}dx$$

integration by change of the variable:

$$x = A\cos\theta\ ,\ \ dx = -A\sin\theta d\theta\ ;\ \ x = 0\to\theta = \frac{\pi}{2}\ ,\ \ x = a\to\theta = 0$$

$$\left(A^2-x^2\right)^{\frac{1}{2}} = A\sin\theta$$

$$\text{Area} = -4\int_{\frac{\pi}{2}}^{0} B\sin\theta\cdot A\sin\theta d\theta = -4AB\int_{\frac{\pi}{2}}^{0}\sin^2\theta d\theta = 2AB\int_{0}^{\frac{\pi}{2}}(1-\cos 2\theta)\,d\theta$$
$$= 2AB\left(\theta-\frac{1}{2}\sin 2\theta\right)\Big|_{0}^{\frac{\pi}{2}} = \pi AB$$