Mathematics for physicists

Lecturer: Prof. Ven-Chung Lee

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1. Forced damped oscillation

1. **a)** Free oscillation

$$m\ddot{x} = -k \cdot x \quad \rightarrow \quad x(t) = A \cdot \cos(\omega_0 t + \theta_0) \ , \ \omega_0^2 = \frac{k}{m} : \text{natural frequency}$$

b) Damped oscillation

$$m\ddot{x} = -k \cdot x - \gamma \cdot \dot{x} \quad \rightarrow \quad x_H(t) = A_1 \cdot e^{\beta_1 \cdot t} + A_2 \cdot e^{\beta_2 \cdot t}$$

$$m\beta^2 + \gamma\beta + k = 0$$

$x_H(t)$ is the transient state solution $x_H(t) \rightarrow 0$ as $t \rightarrow \infty$

c) Forced damped oscillation

$$m\ddot{x} = -k \cdot x - \gamma \cdot \dot{x} + F_{ext}$$

$$F_{ext}(t) = F_0 \cdot \cos\omega \cdot t \quad \rightarrow \quad x_s(t) = B_0(\omega)\cos(\omega t + \delta_0) \quad \text{stable state solution}$$

complete solution $x(t) = x_H(t) + x_s(t)$. $x_s(t)$ is the harmonic response.

Approach (1) Comparison of coefficients: set $x_s(t) = B_1 \cos\omega t + B_2 \sin\omega t$

$$\rightarrow \begin{cases} (-m\omega^2 + k) B_1 + \gamma\omega B_2 = F_0 \\ -\gamma\omega B_1 + (-m\omega^2 + k) B_2 = 0 \end{cases} \Longrightarrow B_1, B_2 \Longrightarrow B_0, \delta_0$$

Approach (2) Phasor, complex domain

$$\because \cos\omega t = \text{Re}\left(e^{i\omega t}\right) \quad , \quad \text{Re : Real part of}$$

$$\therefore F_{ext}(t) = F_0 \cdot \cos\omega t = \text{Re}\left(F_0 \cdot e^{i\omega t}\right) = \text{Re}\left(\tilde{F}_{ext}\right)$$

$$x_s(t) = B_0 \cdot \cos(\omega t + \delta) = \text{Re}\left(B_0 e^{i\omega t} \cdot e^{i\delta}\right) = \text{Re}\left(\tilde{x}_s(t)\right)$$

$$\therefore m\ddot{x} + \gamma\dot{x} + kx = F_0 \cdot \cos\omega t \rightarrow \text{Re}\left(m\ddot{\tilde{x}} + \gamma\dot{\tilde{x}} + k\tilde{x} = F_0 e^{i\omega t}\right)$$

set $\tilde{x}_s(t) = B_0 e^{i(\omega t+\delta)}$

$$\rightarrow \left(-m\omega^2 + i\omega\gamma + k\right) \cdot B_0 \cdot e^{i(\omega t+\delta)} = F_0 \cdot e^{i\omega t}$$

$$\therefore B_0 e^{i(\omega t+\delta)} = \frac{F_0 \cdot e^{i\omega t}}{(k - m\omega^2) + i\omega\gamma}$$

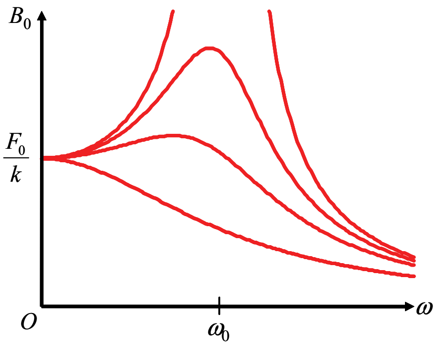
set $\tilde{z} = (k - m\omega^2) + i\omega\gamma = \left((k - m\omega^2)^2 + \omega^2\gamma^2\right)^{\frac{1}{2}} e^{i\varphi} = z e^{i\varphi}$

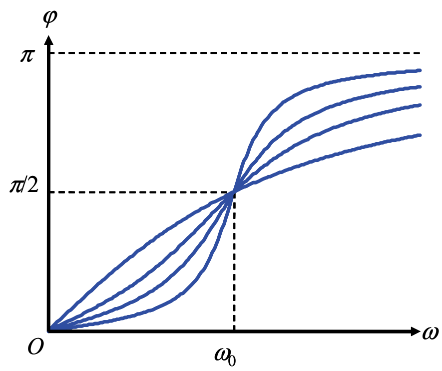
$$\tan\varphi = \frac{\omega\gamma}{k - m\omega^2}$$

$$\therefore B_0 = \frac{F_0}{z} \quad \text{and} \quad \delta = -\varphi$$

$$B_0 = \frac{F_0}{\left((k - m\omega^2)^2 + \omega^2\gamma^2\right)^{\frac{1}{2}}} \ , \quad B_0 \rightarrow \frac{F_0}{k} \text{ as } \omega \rightarrow 0 \text{ static}$$

$$B_0 \rightarrow 0 \text{ as } \omega \rightarrow \infty \ , \quad B_0 \rightarrow \frac{F_0}{\gamma \cdot \omega} \text{ as } \omega = \left(\frac{k}{m}\right)^{\frac{1}{2}} = \omega_0 \text{ Resonance}$$



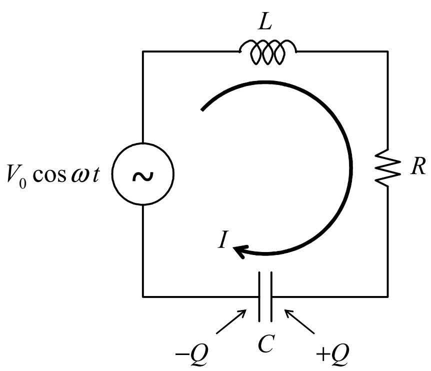


Note: $z \rightarrow \min.$ when $dz/d\omega = 0 \rightarrow \omega_r = \left(\frac{k}{m} - \frac{\gamma^2}{2m^2}\right)^{\frac{1}{2}}$ resonance frequency

$$\therefore \omega_r \rightarrow \omega_0 \text{ as } \gamma^2 \ll 2m^2$$

$$x_s(t) = \frac{F_0}{z}\cos(\omega t - \varphi)$$

2. LCR series circuit driven by ACV



$V_L = L\frac{dI}{dt}$ voltage of inductance L

$V_R = R \cdot I$ Ohm's law

$V_C = \frac{Q}{C}$ C : capacitance

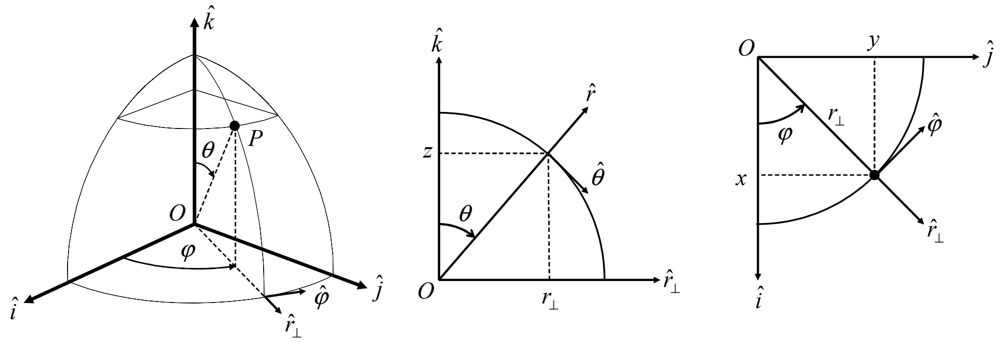
$$V_0 \cos\omega t = V_L + V_C + V_R = L\frac{dI}{dt} + \frac{Q}{C} + R \cdot I$$

as $I = \dfrac{dQ}{dt}$

$$\therefore V_0 \cos\omega t = L\ddot{Q} + R \cdot \dot{Q} + \frac{1}{C}Q$$

Similar math. as that of the forced damped oscillation.

3. Vector calculus coordinates of $P : (x, y, z)\,,\,(r, \theta, \varphi)\,,\,(r_\perp, \varphi, z)$



$$\overrightarrow{OP} = \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = r \cdot \hat{r} = r_\perp \cdot \hat{r}_\perp + z \cdot \hat{k}$$

velocity $\vec{v} = \dfrac{d\vec{r}}{dt}$

$$d\vec{r} = d\left(r \cdot \hat{r}\right) = dr \cdot \hat{r} + r \cdot d\hat{r}$$

$$d\hat{r} = \hat{r}\left(r + dr, \theta + d\theta, \varphi + d\varphi\right) - \hat{r}\left(r, \theta, \varphi\right)$$

$$= d\theta \cdot \hat{\theta} + \sin\theta \cdot d\varphi \cdot \hat{\varphi}$$

$$\therefore \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r \cdot \sin\theta \cdot \dot{\varphi}\hat{\varphi} = v_r \cdot \hat{r} + v_\theta \cdot \hat{\theta} + v_\varphi \cdot \hat{\varphi}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} + \dot{r} \cdot \dot{\varphi}\sin\theta\hat{\varphi} + r \cdot \ddot{\varphi}\sin\theta \cdot \hat{\varphi}$$
$$+ r \cdot \dot{\varphi} \cdot \dot{\theta}\cos\theta \cdot \hat{\varphi} + r \cdot \dot{\varphi}\sin\theta\frac{d\hat{\varphi}}{dt}$$

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} + \sin\theta \cdot \dot{\varphi}\hat{\varphi}$$
$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r} + \dot{\varphi}\cos\theta \cdot \hat{\varphi}$$
$$\frac{d\hat{\varphi}}{dt} = -\dot{\varphi}\hat{r}_\perp = -\dot{\varphi}\left(\cos\theta\hat{\theta} + \sin\theta\hat{r}\right)$$
$$\rightarrow \vec{a} = a_r \cdot \hat{r} + a_\theta \cdot \hat{\theta} + a_\varphi \cdot \hat{\varphi} \quad , \quad a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta \quad \text{etc.}$$

a) central force problem

$$\vec{F} = F(r) \cdot \hat{r}$$

gravitational force

$$F(r) = -\frac{GMm}{r^2}$$

then

$$ma_r = F(r) \quad \text{and} \quad a_\theta = a_\varphi = 0.$$

Note:

$$a_\theta \neq \frac{dv_\theta}{dt} \quad \because \frac{d\hat{\theta}}{dt} \neq 0 \ , \quad \frac{d\hat{r}}{dt} \neq 0 \ , \quad \text{etc.}$$