A Theoretical Investigation of Low-Dimensional Bose-Einstein Condensations

Ching-Fuh Lin*

Department of Electrical Engineering and Graduate Institute of Electronics Engineering, National Taiwan University, Taipei 106, Taiwan

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Low-dimensional Bose-Einstein condensations (BEC) for g(ε) (the density of states) = $C_d\varepsilon^{d-1}$ with $d \leq 1$, corresponding to a dimensionality of not larger than 2, are theoretically analyzed. When the energy quantization of the first excited state is taken into account, the boson number in the excited states is finite even for $d \leq 1$ with the density of states remaining proportional to ε^{d-1}. The condensation temperature increases with the particle number faster for $d \leq 1$ than for $d > 1$. The condensation depends highly on the quantized energy ε_1 of the first excited state. When ε_1 just deviates slightly from zero, the condensation temperature quickly increases with ε_1. Theoretical estimation shows that a low-dimensional BEC can occur at a temperature above 1 K for excitons in semiconductor quantum wells even without the harmonic-oscillator potential.

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I. INTRODUCTION

The Bose-Einstein condensation (BEC) has attracted significant attention recently. It has been observed in atoms [1–4] and molecules [5, 6]. A BEC of excitons in semiconductors [7–9] is also expected. In the past, a low-dimensional BEC was deemed to be impossible unless the potential is modified, and so the density of states $g(\varepsilon)$ $(= C_d\varepsilon^{d-1})$ is changed to have $d > 1$. The conditions with modified potentials are like the harmonic oscillator potential [10–13], power-law traps [14], optical lattices [15], and so on. In this work, we derive the low dimensional BEC for the density of states $g(\varepsilon)$ $(= C_d\varepsilon^{d-1})$ with $d \leq 1$ and compare it with the situation when $d > 1$. In contradiction to conventional concepts, the derivation shows that a low dimensional BEC could occur at a higher temperature for $d \leq 1$ than for $d > 1$.

The density of states plays a crucial role for explaining the existence of a BEC. The density of states for the general 3-dimensional case has the form of $g(\varepsilon) = C_{3D}\varepsilon^{1/2}$, so the integral $\int \frac{g(\varepsilon)d\varepsilon}{Z^{-1}e^{\beta\varepsilon}-1}$ converges, giving rise to a finite number of bosons in the excited states. For the 2-dimensional case or lower dimensionality without special types of potentials, $g(\varepsilon) = C_d\varepsilon^{d-1}$ with $d \leq 1$. A low dimensional BEC is supposed to be impossible because the integral for excited states diverges [16] as long as the density of states has the form $g(\varepsilon) = C_d\varepsilon^{d-1}$ with $d \leq 1$. It would lead to a near-to-zero number of bosons in the ground state, so there is no condensation. However, the divergence is actually caused by the singular point at $\varepsilon = 0$. In a practical physical system, the excited states cannot start from $\varepsilon = 0$. In particular, when the quantization of energy is taken into account, there is always an energy difference $\Delta\varepsilon$ between the ground state and the first excited state. If the integral $\int \frac{g(\varepsilon)d\varepsilon}{Z^{-1}e^{\beta\varepsilon}-1}$ excludes the ground state, the integration converges for $\Delta\varepsilon > 0$. Then, even though the density of states $g(\varepsilon)$ $(= C_d\varepsilon^{d-1})$ of low dimensionality retains its original form, i.e., $d \leq 1$, a BEC still occurs.

II. CONVERGENCE OR DIVERGENCE OF THE INTEGRAL $\int_0^\infty \frac{g(\varepsilon)d\varepsilon}{Z^{-1}e^{\beta\varepsilon}-1}$

For a boson system consisting of many energy states ε_0, ε_1, ε_2, ..., the total number of bosons is given by

$$N = \sum_{i=0} \frac{1}{Z^{-1}e^{\beta\varepsilon_i} - 1}\,, \tag{1}$$

where $Z = e^{\beta\mu}$, $\beta = 1/\mathrm{kT}$. The total number can be further divided into two terms: the number N_0 in the ground state with energy ε_0 and the number N_{ex} in the excited states with energies ε_1, ε_2, ... ε_0 is usually treated as zero for the simplicity of discussion, $\varepsilon_0 = 0$. Then we have

$$N_0 = \frac{Z}{1-Z}\,, \tag{2}$$

$$N_{ex} = \sum_{i=1} \frac{1}{Z^{-1}e^{\beta\varepsilon_i} - 1}\,. \tag{3}$$

When the energy levels are closely spaced, the summation of (3) is replaced with the integral

$$N_{ex} = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{Z^{-1}e^{\beta\varepsilon} - 1}\,. \tag{4}$$

where $g(\varepsilon)$ represents the density of states. The above integral is obtained under the assumption that $\varepsilon_1 \to 0$. In reality, the calculation of the excited states with the integral should start from ε_1 instead of 0, i.e.,

$$N_{ex} = \int_{\varepsilon_1}^\infty \frac{g(\varepsilon)d\varepsilon}{Z^{-1}e^{\beta\varepsilon} - 1}\,. \tag{5}$$

The above integral (5) can be written as two terms

$$N_{ex} = \int_{\varepsilon_1}^\infty \frac{g(\varepsilon)d\varepsilon}{Z^{-1}e^{\beta\varepsilon} - 1} = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{Z^{-1}e^{\beta\varepsilon} - 1} - \int_0^{\varepsilon_1} \frac{g(\varepsilon)d\varepsilon}{Z^{-1}e^{\beta\varepsilon} - 1}\,. \tag{6}$$

If $g(\varepsilon)$ $(= C_d\varepsilon^{d-1})$ has $d > 1$, corresponding to the dimension larger than 2, the second term on the right hand side (RHS) of (6) approaches zero for $\varepsilon_1 \to 0$. Therefore, it makes no

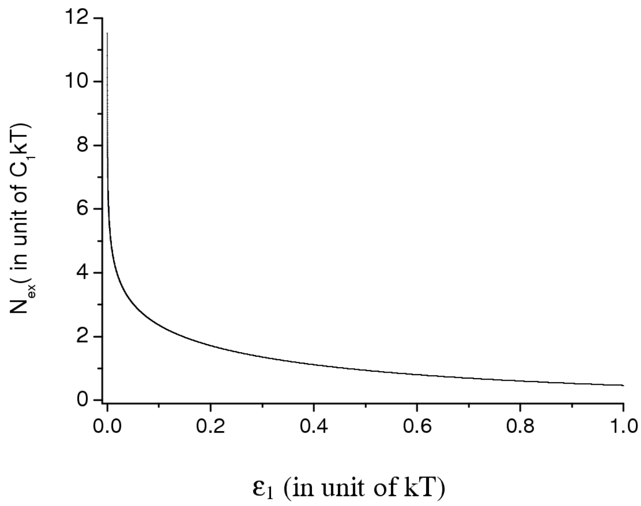

FIG. 1: Variation of boson number in the excited states N_{ex}) with the first excited energy ε_1.

difference whether the integration for the excited states starts from $\varepsilon_1 = 0$ or not. However, when $d \leq 1$, this term diverges. Therefore, it is important to do the integration excluding $\varepsilon_1 = 0$. In the following, we will first study the situation for $d = 1$, which represents the 2-dimensional case. That is, the particles are confined along the axis (called the third axis) perpendicular to the plane along which they can easily move. Because of the confinement along the third axis, the density of states $g(\varepsilon)$ $(= C_d \varepsilon^{d-1})$ has d=1. Then the number N_{ex} in the excited states is given by

$$N_{ex} = \int_{\varepsilon_1}^{\infty} \frac{C_1 d\varepsilon}{Z^{-1} e^{\beta\varepsilon} - 1} \,. \tag{7}$$

The RHS of the above equation can be integrated to obtain

$$N_{ex} = -C_1 kT \ln(1 - Ze^{-\varepsilon_1/kT}) = -C_1 kT \ln[1 - e^{-(\varepsilon_1 - \mu)/kT}] \,. \tag{8}$$

Because $Z \leq 1$, N_{ex} is always less than the value for $Z = 1$. As $Z \to 1$, the above equation becomes

$$N_{ex} = -C_1 kT \ln(1 - e^{-\varepsilon_1/kT}) \tag{9}$$

Fig. 1 shows the variation of N_{ex} with ε_1 for $Z = 1$. When ε_1 approaches zero, N_{ex} becomes infinite. This clearly indicates that the reason for the divergence of the number N_{ex} in the excited states is due to the approximation of ε_1 by zero. If $\varepsilon_1 \neq 0$, N_{ex} is finite even when $Z \to 1$.

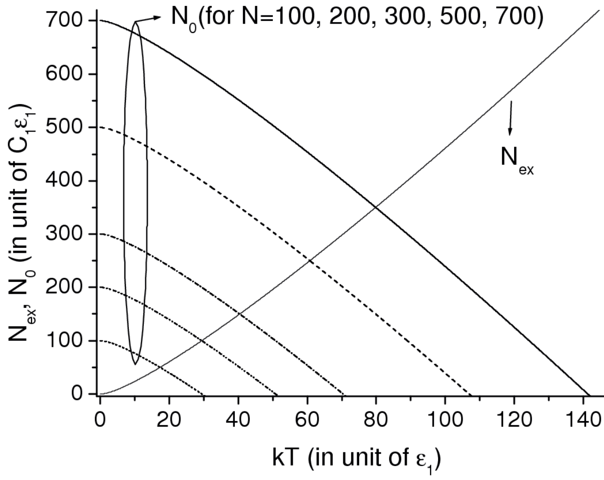

FIG. 2: N_{ex} and N_0 vs. temperature for a different total number density of bosons. (Temperature is represented as kT in units of ε_1. The boson number is in units of $C_1\varepsilon_1$.)

III. DISCUSSION OF THE BOSE-EINSTEIN CONDENSATION IN THE LOW-DIMENSIONAL CASE

When the total number N of bosons is larger than the number N_{ex} in the excited states, a BEC occurs. The number condensed in the ground state is given by

$$N_0 = N - N_{ex} = N + C_1 kT \ln(1 - Ze^{-\varepsilon_1/kT}). \tag{10}$$

As the condensation number N_0 is large, $Z \to 1$, so

$$N_0 = N - N_{ex} = N + C_1 kT \ln(1 - e^{-\varepsilon_1/kT}). \tag{11}$$

Fig. 2 shows N_{ex} and N_0 vs. temperature for a different numbers density n of bosons, where n_{ex} and n_0 are number densities, defined as N_{ex} and N_0 divided by the volume V, respectively, ad $n = N/V$. It is clear that condensation occurs at temperatures above 0 K and the condensation temperature increases with the total number of bosons. For the three-dimensional case ($d = 3/2$) and the three-dimensional harmonic-oscillator potential ($d = 3$), the condensation temperature is analytically related to the total number of bosons, $T_c \propto N^{1/d}$. In the two dimensional case with $d = 1$, there is no analytical formula to describe the condensation temperature. However, it can still be obtained from the intersection of a series of curves with the abscissa shown in Fig. 2. The condensation temperature T_c vs. the total numbers of bosons N for $d = 1$ is shown by Curve (a) in Fig. 3. The number density n is defined as the total number of bosons N divided by the colume V. Curves (b) and (c) are also shown in Fig. 3 for comparison. Curve (b) represents the relation $T_c \propto N^{2/3}$, which is the three-dimensional case ($d = 3/2$), and Curve (c) shows the relation $T_c \propto N^{1/3}$ for the

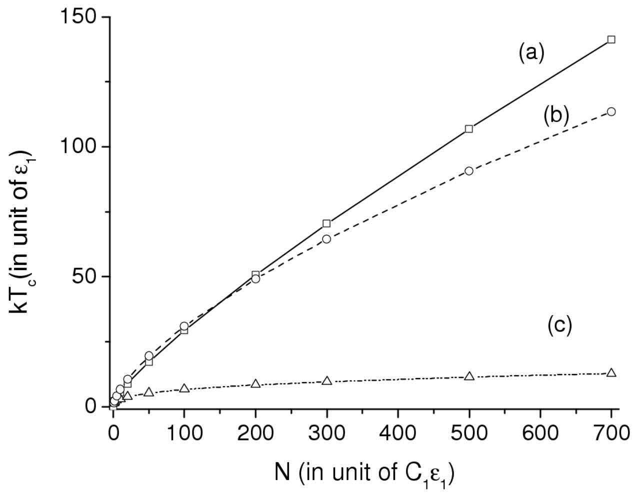

FIG. 3: Condensation temperature T_c vs. the total numbers of bosons N for $d = 1$. Curve (b) shows the relation $T_c \propto N^{2/3}$ for the case of $d = 3/2$. Curve (c) shows the relation $T_c \propto N^{1/3}$ for the case of $d = 3$. Curves (b) and (c) are multiplied by different factors for easy comparison.

case of the three-dimensional harmonic oscillator potential ($d = 3$). Curves (b) and (c) are multiplied by different factors from Curve (a) for easy comparison. Fig. 3 shows that the condensation temperature increases with N faster in the two dimensional case with $d = 1$ than the other two situations ($d > 1$). It indicates that a smaller d value results in more states at low energy, so a BEC should more easily occur.

The BEC for the case of $d = 1$ highly depends on the energy ε_1. Fig. 4 shows the variation of the condensation number density n_0 with temperature for various ε_1 under a fixed number density of n. ε_1 is in units of ε, which is an arbitrary energy unit used as a parameter for comparison. From these curves, we are able to obtain the variation of the condensation temperature with ε_1, as shown in Fig. 5. For small ε_1, the condensation temperature quickly increases with ε_1. In other words, the nonzero value of the quantized energy ε_1 is very crucial for the existence of a BEC in the low-dimensional case.

In the curves shown above, kT is much larger than ε_1 for most of the ranges. Thus the energy levels are treated as closely spaced compared to kT, and the summation (3) can be replaced with an integral assuring the validity of the above calculation.

IV. BOSE-EINSTEIN CONDENSATION FOR $d < 1$

For $d < 1$, corresponding to a dimensionality lower than 2, we also calculate the integration for N_{ex} starting from ε_1.

$$N_{ex} = \int_{\varepsilon_1}^{\infty} \frac{C_d \varepsilon^{d-1}\, d\varepsilon}{Z^{-1} e^{\beta\varepsilon} - 1} \tag{12}$$

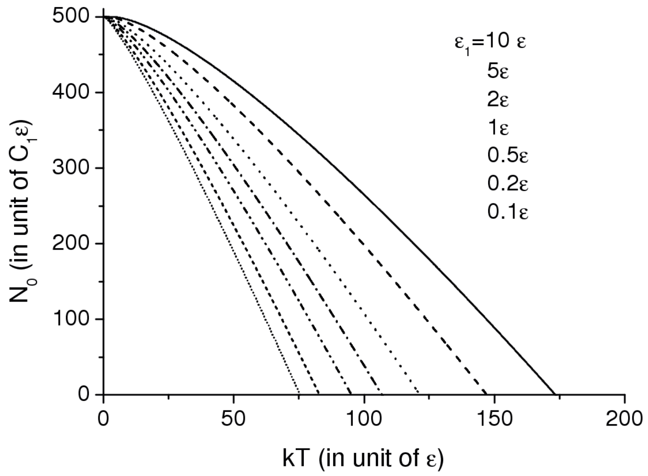

FIG. 4: Variation of the condensation number N_0 with temperature for various ε_1 under the condition of $N = 500C_1\varepsilon_1$. ε is an arbitrary unit of energy used as a parameter for comparison.

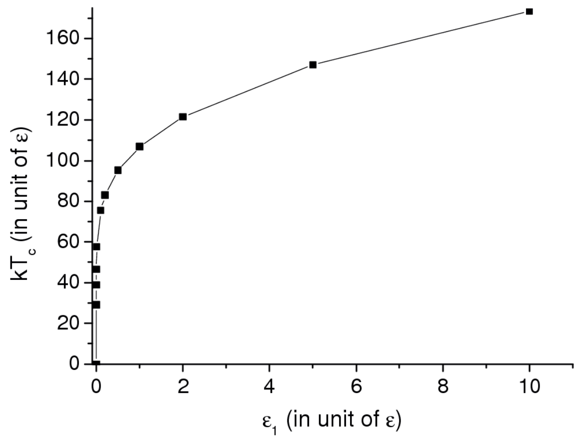

FIG. 5: Variation of the condensation temperature with ε_1. ε is an arbitrary unit of energy used as a parameter for comparison.

Rewrite the integral as

$$N_{ex} = \int_{\varepsilon_1}^{\infty} \frac{C_d \, d\varepsilon}{Z^{-1}\varepsilon^{1-d}e^{\beta\varepsilon} - 1} < \int_{\varepsilon_1}^{\infty} \frac{C_d \, d\varepsilon}{Z^{-1}\varepsilon_1^{1-d}e^{\beta\varepsilon} - 1} = \frac{C_d}{C_1}\frac{1}{\varepsilon_1^{1-d}} \int_{\varepsilon_1}^{\infty} \frac{C_1 d\varepsilon}{Z^{-1}e^{\beta\varepsilon} - 1} \tag{13}$$

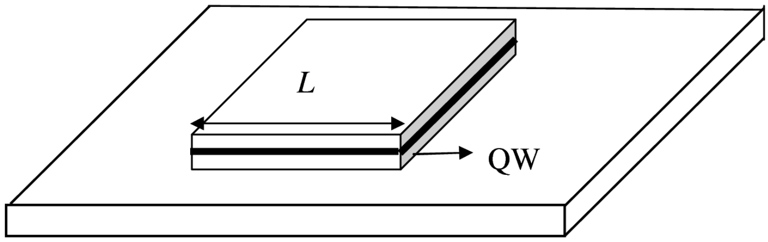

FIG. 6: A schematic of the example for excitons confined in the region with a QW. The lateral dimension is L.

where $\int_{\varepsilon_1}^{\infty} \frac{C_1 d\varepsilon}{Z^{-1}e^{\beta\varepsilon}-1} = -C_1 kT \ln(1 - e^{-\varepsilon_1/kT})$. Then we have $N_{ex}(d < 1) < -C_d \frac{1}{\varepsilon_1^{1-d}} kT \ln\left(1 - e^{-\varepsilon_1/kT}\right)$, so the condensation number $N_0(d < 1) = N - N_{ex} > N - C_d \frac{1}{\varepsilon_1^{1-d}} kT \ln\left(1 - e^{-\varepsilon_1/kT}\right)$. Therefore, the number of bosons in the ground state N_0 for $d < 1$ increases with the temperature faster than the situation of $d = 1$, indicating that the BEC occurs more easily for $d < 1$. It also highly depends on the quantized energy level ε_1. This is reasonable because a smaller d value means smaller dimensionality and hence more states in the low energy, so a BEC should more easily occur.

V. ATOMS OR EXCITONS CONFINED IN LOW DIMENSIONS

It might be difficult to create a potential with 1D or 2D confinement for atoms. However, it is relatively easy to confine excitons in a slab structure such as quantum wells (QWs) [9]. With a limited range of the QW structure, the energy in the momentum space can be easily treated as quantized values. The above derivation and calculation can thus be illustrated using such an example. For QWs, the energy is described as

$$E(n, k_x, k_y) = \frac{\pi^2 \hbar^2 n^2}{2m^* W^2} + \frac{\hbar^2 k_x^2}{2m^*} + \frac{\hbar^2 k_y^2}{2m^*} \tag{14}$$

Here the potential variation of the QW is along the z axis; W is the well width and n is an integer. When the excitons are limited in a region along the x and y axes with the size of L, $k_x = k_y = \frac{2\pi m}{L}$, where m is an integer, the first excited energy level in the momentum space along the x and y directions is

$$\varepsilon_1 = \frac{2\pi^2 \hbar^2}{m^* L^2} \tag{15}$$

.

A schematic of such an example is shown in Fig. 6. This structure can be easily fabricated using semiconductor processing techniques. The center region of the plateau with the lateral length L is created by etching down the substrate deep enough, so the QW (marked as the thick black lines) is above the etched bottom. The lateral dimension L is much larger than the width W of the QW. Within the plateau region with the size of L, the potential along the x and y axes is constant instead of a harmonic one. For $L = 20$ μm, ε_1 is around 10^{-8} eV. As the temperature is near 1 K, ε_1 is still much less than kT, so the summation (3) can be well approximated with an integration. However, such a small ε_1 already results in significant difference from $\varepsilon_1 = 0$. The density of states for the QW is given by $g(\varepsilon) = C_1 = \frac{m^*}{\pi\hbar^2}$ [17]. If the density of the excitons in the QW is 10^{12} cm^{-2}, the condensation temperature can be above 1 K. This might explain the observation of the BEC in Ref. [9], where the potential might not be that of a harmonic-oscillator upon careful examination of the reported data. According to the above theoretical derivation, a BEC could occur in QWs without the necessity of a harmonic-oscillator potential.

VI. CONCLUSION

We theoretically analyzed the BEC for $g(\varepsilon)$ $(= C_d\varepsilon^{d-1})$ with $d \leq 1$, which corresponds to a dimensionality of not larger than 2. When the energy quantization of the first excited state is taken into account, the condensation temperature increases with the particle number faster in the case of $d = 1$ than in the situation with $d > 1$. The condensation highly depends on the quantized energy ε_1 of the first excited state. The condensation temperature quickly increases when ε_1 deviates from zero. Our theoretical estimation shows that 2-dimensional excitons in semiconductors like QWs could have a BEC at temperatures above 1 K without a harmonic-oscillator potential.

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* Electronic address: `cflin@cc.ee.ntu.edu.tw`

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